International Journal of Mathematical Science
Vol. 2, No. 3-4, July-December 2003, pp. 237-247

ISSN: 0972-754X

SOME GENERALIZATIONS OF WAVELET FRAMES

M. K. Ahmad & Javid Iqbal

Abstract: The essential problem in signal analysis is to find a numerically stable algorithm for reconstruction of a signal from its atomic decomposition [4]. This leads to the notion of frames [6, 10] which is a main ingredient in the analysis and synthesis of signals. In this paper, we have obtained the frame bounds for wavelet packet frames which are more general than that of wavelet frames.

2000 Mathematics Subject Classification: 41A58, 42C15.

Keywords: Frame, translation, modulation, dilation.

1. INTRODUCTION AND PRELIMINARIES

Introduced by Duffin and Schaeffer [9] in the context of non-harmonic Fourier series, the theory of frames has been developed for Gabor and Wavelet transforms by many authors, see especially the papers by Daubechies [6], Heil and Walnut [10], Christensen [1], Sun and Zhou [13] and Shang and Zhou [12].

A system of elements $\{f_n\}_{n \in \Lambda}$ in a Hilbert space H is called a frame for H if there exists two +ve numbers A and B such that for any $f \in H$,

$$A\|f\|^2 \leq \sum_{n\in\Lambda} \left|\langle f, f_n\rangle\right|^2 \leq B\|f\|^2 .$$

The numbers A and B are called frame bounds. If $A = B$, the frame is said to be tight. The frame is called exact if it ceases to be a frame whenever any single element is deleted from the frame.

The continuous wavelet transformation of a L^2-function f with respect to the wavelet ψ, which satisfies admissibility condition, is defined as:

$$(T^{wav}f)(a, b) = |a|^{-1/2} \int_{-\infty}^{\infty} f(t)\overline{\psi\left(\frac{t-b}{a}\right)}, \; a, b \in \mathbb{R};\, a \neq 0.$$

The term wavelet denotes a family of functions of the form $\psi_{a,\bar{\bar{b}}}\, |a|^{-1/2}\, \psi\left(\frac{t-b}{a}\right)$, obtained from a single function ψ by the operation of dilation and translation.

Wavelet Packets: We have the following sequence of functions due to Wickerhauser [14]. For $l = 0, 1, 2, \ldots,$

$$\psi_{2l}(x) = \sqrt{2} \sum_{k \in \mathbb{Z}} a_k \psi_l(2x - k) \quad \text{and} \quad \psi_{2l+1}(x) = \sqrt{2} \sum_{k \in \mathbb{Z}} b_k \psi_l(2x - k), \qquad \text{(i)}$$

where a = $\{a_k\}$ is the filter such that $\Sigma_{n \in \mathbb{Z}} a_{n-2k} a_{n-2l} = \delta_{kl}$, $\Sigma_{n \in \mathbb{Z}} a_n = \sqrt{2}$ and $b_k = (-1)^k a_{1-k}$. For $l = 0$ in (i), we get

$$\psi_0(x) = \psi_0(2x) + \psi_0(2x - 1),\ \psi_1(x) = \psi_0(2x) - \psi_0(2x - 1),$$

where ψ_0 is a scaling function and may be taken as a characteristic function. If we increase l, we get the following

$$\psi_2(x) = \psi_1(2x) + \psi_1(2x - 1),\ \psi_3(x) = \psi_1(2x) - \psi_1(2x - 1)$$

$$\psi_4(x) = \psi_1(4x) + \psi_1(4x - 1) + \psi_1(4x - 2) + \psi_1(4x - 3)$$

and so on.

Here ψ_l's have a fixed scale but different frequencies. They are Walsh functions in [0, 1]. The functions $\psi_l(t - k)$, for integers k, l with $l \geq 0$, form an orthonormal basis of $L^2(\mathbb{R})$.

Theorem 1.1: For every partition P of the non negative integers into the sets of the form $I_{lj} = \{2^j l, \ldots, 2^j(l + 1) - 1\}$, the collection of functions $\psi_{l;j,k} = 2^{j/2}\psi_l(2^j x - k)$, $I_{lj} \in P$, $k \in \mathbb{Z}$, is an orthonormal basis of $L^2(\mathbb{R})$.

We have used the inner product of functions $f, g \in L^2(\mathbb{R})$ as $\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx$, the Fourier transform of $f \in L^2(\mathbb{R})$ as $\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega x} f(x)\,dx$ and the relationship between functions and their Fourier transform as $2\pi\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle$. For $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, the Fourier transform $\hat{f}$ of f is in $L^2(\mathbb{R})$ and satisfies the Parseval identity $\|\hat{f}\|_2^2 = 2\pi\|f\|_2^2$.

For a function $\psi \in L^2(\mathbb{R})$, we define the following operators as follows:

Translation: $T_a\psi(x) = \psi(x - a)$, $x \in \mathbb{R}$, $a > 0$.

Modulation: $E_a\psi(x) = e^{2\pi iax}\psi(x)$, $x \in \mathbb{R}$, $a > 0$.

Dilation: $D_a\psi(x) = |a|^{-1/2}\psi(x/a)$, for all $x \in \mathbb{R}$, $a > 0$.

Definition 1.2: A Sobolev space of order $s > 0$, denoted by $H^s(\mathbb{R})$, is a subspace of $L^2(\mathbb{R})$, given by

$$H^s(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : |\hat{f}(\xi)|(1 + |\xi|^2)^{s/2} \in L^2(\mathbb{R})\}.$$

Lemma 1.3: Suppose that the scaling function ϕ satisfies

$$\sup_{\xi \in \mathbb{R}} \sum_{k \in \mathbb{Z}} \left| \hat{\phi}(\xi + 2k\pi) \right|^{2-\sigma} < +\infty, \tag{1}$$

for some $\sigma > 0$, and

$$\sup_{\xi \in \mathbb{R}} (1 + |\xi|)^{\sigma} \left| \hat{\phi}(\xi) \right| < +\infty. \tag{2}$$

Then there exists a constant c such that, for all $f \in L^2(\mathbb{R})$,

$$\sum_{l} \sum_{j,k \in \mathbb{Z}} \left| \left\langle f, \psi_{l,jk} \right\rangle \right|^2 \leq c \| f \|_2^2. \tag{3}$$

Proof: Kindly see [3].

2. MAIN RESULTS

Theorem 2.1: Suppose that,

$$A = \inf_{|r| \in [1,2]} \left[\sum_{l} \sum_{j \in \mathbb{Z}} \left| \hat{\psi}_l(2^j r) \right|^2 - \sum_{m \neq 0} \sum_{l} \sum_{j \in \mathbb{Z}} \left| \hat{\psi}_l(2^j r) \hat{\psi}_l(2^j r + m) \right] > 0$$

$$B = \sup_{|r| \in [1,2]} \left[\sum_{l} \sum_{j \in \mathbb{Z}} \left| \hat{\psi}_l(2^j r) \right|^2 + \sum_{m \neq 0} \sum_{l} \sum_{j \in \mathbb{Z}} \left| \hat{\psi}_l(2^j r) \hat{\psi}_l(2^j r + m) \right] < \infty.$$

Then $\{ D_{2j} T_k \psi_l(x)_{j,k \in \mathbb{Z}, l = 1, 2, \ldots, k} \}$ is a wavelet packet frame for $L^2(\mathbb{R})$ with bounds A, B.

Proof: For a function $f \in L^2(\mathbb{R})$, we have

$$\begin{aligned}
\sum_{l} \sum_{j,k} \left| \left\langle f, D_{2^j} T_k \psi_l \right\rangle \right|^2 &= \sum_{l} \sum_{j,k \in \mathbb{Z}} \left| \left\langle \hat{f}, D_{2^{-j}} E_{-k} \hat{\psi}_l \right\rangle \right|^2 \\
&= \sum_{l} \sum_{j,k \in \mathbb{Z}} \left| \left\langle \hat{f}, E_{-k2^j} D_{2^{-j}} \hat{\psi}_l \right\rangle \right|^2 \\
&= \sum_{l} \sum_{j,k \in \mathbb{Z}} \left| \int_{\mathbb{R}} \hat{f}(r) \overline{E_{-k2^j} D_{2^{-j}} \hat{\psi}_l(r)} \, dr \right|^2 \\
&= \sum_{l} \sum_{j \in \mathbb{Z}} 2^{-j} \sum_{k} \left| \int_{\mathbb{R}} \hat{f}(r) \overline{\hat{\psi}_l(2^{-j} r)} e^{2\pi i k 2^{-j} r} dr \right|^2 \\
&= \sum_{l} \sum_{j \in \mathbb{Z}} 2^{-j} 2^j \int_0^{2^j} \left| \sum_{m} \hat{f}(r - 2^j m) \overline{\hat{\psi}_l(2^{-j} r - m)} \right|^2 dr
\end{aligned}$$

$$= \sum_{l} \sum_{j\in\mathbb{Z}} \int_0^{2^j} \sum_{h} \hat{f}(r-2^j h)\, \overline{\hat{\psi}_l(2^{-j}r-h)} \sum_{m} \overline{\hat{f}(r-2^j m)}\, \hat{\psi}_l(2^{-j}r-m)\, dr$$

$$= \sum_{l} \sum_{j\in\mathbb{Z}} \sum_{h} \int_0^{2^j} \hat{f}(r-2^j h)\, \overline{\hat{\psi}_l(2^{-j}r-h)} \sum_{m} \overline{\hat{f}(r-2^j m)}\, \hat{\psi}_l(2^{-j}r-m)\, dr$$

$$= \sum_{l} \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}} \hat{f}(r)\, \overline{\hat{\psi}_l(2^{-j}r)} \sum_{m} \overline{\hat{f}(r-2^j m)}\, \hat{\psi}_l(2^{-j}r-m)\, dr$$

$$= \sum_{l} \sum_{j\in\mathbb{Z}} \sum_{m} \int_{\mathbb{R}} \hat{f}(r)\, \overline{\hat{f}(r-2^j m)}\, \overline{\hat{\psi}_l(2^{-j}r)}\, \hat{\psi}_l(2^{-j}r-m)\, dr$$

$$= \int_{\mathbb{R}} \left|\hat{f}(r)\right|^2 \sum_{l} \sum_{j\in\mathbb{Z}} \left|\hat{\psi}_l(2^j r)\right|^2 dr$$

$$+ \sum_{m\neq 0} \sum_{l} \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}} \hat{f}(r)\, \overline{\hat{f}(r-2^j m)}\, \overline{\hat{\psi}_l(2^{-j}r)}\, \hat{\psi}_l(2^{-j}r-m)\, dr.$$

$$= (*).$$

Applying the Cauchy-Schwarz inequality twice, we have

$$(*) \le \int_{\mathbb{R}} \left|\hat{f}(r)\right|^2 \sum_{l} \sum_{j\in\mathbb{Z}} \left|\hat{\psi}_l(2^j r)\right|^2 dr$$

$$+ \sum_{m\neq 0} \sum_{l} \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}} \left|\hat{f}(r)\right| \left(\left|\hat{\psi}_l(2^{-j}r)\right| \left|\hat{\psi}_l(2^{-j}r-m)\right|\right)^{1/2}$$

$$\cdot \left|\hat{f}(r-2^j m)\right| \left(\left|\hat{\psi}_l(2^{-j}r)\right| \left|\hat{\psi}_l(2^{-j}r-m)\right|\right)^{1/2} dr$$

$$\le \int_{\mathbb{R}} \left|\hat{f}(r)\right|^2 \sum_{l} \sum_{j\in\mathbb{Z}} \left|\hat{\psi}_l(2^j r)\right|^2 dr$$

$$+ \sum_{m\neq 0} \sum_{l} \sum_{j\in\mathbb{Z}} \left(\int_{\mathbb{R}} \left|\hat{f}(r)\right|^2 \left|\hat{\psi}_l(2^{-j}r)\right| \left|\hat{\psi}_l(2^{-j}r-m)\right| dr\right)^{1/2}$$

$$\cdot \left(\int_{\mathbb{R}} \left|\hat{f}(r-2^j m)\right|^2 \left|\hat{\psi}_l(2^{-j}r)\right| \left|\hat{\psi}_l(2^{-j}r-m)\right| dr\right)^{1/2}$$

$$\le \int_{\mathbb{R}} \left|\hat{f}(r)\right|^2 \sum_{l} \sum_{j\in\mathbb{Z}} \left|\hat{\psi}_l(2^j r)\right|^2 dr + (a')\,(a'').$$

The terms (a') and (a'') are actually identical (use the change of variable $r \to r + 2^j m$ in (a'')), so by changing the summation index $j \to -j$, $m \to -m$, we have

$$(*) \leq \int_{\mathbb{R}} \left|\hat{f}(r)\right|^2 \sum_l \sum_{j\in\mathbb{Z}} \left|\hat{\psi}_l(2^j r)\right|^2 dr$$

$$+ \sum_{m\neq 0} \sum_l \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}} \left|\hat{f}(r)\right|^2 \left|\hat{\psi}_l(2^j r)\right| \left|\hat{\psi}_l(2^j r+m)\right| dr$$

$$\leq \int_{\mathbb{R}} \left|\hat{f}(r)\right|^2 \left(\sum_l \sum_{j\in\mathbb{Z}} \left|\hat{\psi}_l(2^j r)\right|^2 + \sum_{m\neq 0} \sum_l \sum_{j\in\mathbb{Z}} \left|\hat{\psi}_l(2^j r)\right| \left|\hat{\psi}_l(2^j r+m)\right|\right) dr .$$

Thus,

$$\sum_l \sum_{j,k} \left|\left\langle f, D_{2^j} T_k \psi_l \right\rangle\right|^2 \leq B \| f \|^2 .$$

A similar conclusion shows

$$(*) \geq \int_{\mathbb{R}} \left|\hat{f}(r)\right|^2 \left(\sum_l \sum_{j\in\mathbb{Z}} \left|\hat{\psi}_l(2^j r)\right|^2 - \sum_{m\neq 0} \sum_l \sum_{j\in\mathbb{Z}} \left|\hat{\psi}_l(2^j r)\right| \left|\hat{\psi}_l(2^j r+m)\right|\right) dr .$$

Thus result follows.

Theorem 2.2: Assume that the sequence $(\alpha_n)_n$ is given by $\phi(x) = \sum_n^{n_1} = n_0 \ \ \alpha_n \phi(2x-n)$ is finite and satisfies the condition

$$\sum_n \alpha_{n-2k}\alpha_{n-2l} = \delta_{lk} , \tag{4}$$

and assume that for some $\epsilon > 0$, we have

$$\left|\hat{\psi}(\xi)\right| < c\left(1+\left|\xi\right|^2\right)^{-\epsilon-1/4} , \tag{5}$$

where $\hat{\psi}$ denotes the Fourier transform of the mother wavelet ψ. Finally, assume that the scaling function $\phi \in H^s(\mathbb{R})$ for some positive number s. Then there exist two positive constants $c_1(s)$ and $c_2(s)$ such that

$$\forall\, f \in L^2(\mathbb{R}), \quad c_1(s) \| f \|_2^2 \leq \sum_l \sum_{j,k\in\mathbb{Z}} \left|\left\langle f, \psi_{l,jk} \right\rangle\right|^2 \leq c_2(s) \| f \|_2^2 . \tag{6}$$

Proof: To prove the upper bound of (6), it suffices to check that if $\phi \in H^s(\mathbb{R})$ for some $s > 0$, then conditions (1) and (2) of Lemma 1.3 are satisfied. To get (1) we first prove that there exists $0 < \alpha < 1$ such that

$$\int \left|\hat{\phi}(\xi)\right|^{2-2\alpha} d\xi < +\infty . \tag{7}$$

Since

$$\int \left|\hat{\phi}(\xi)\right|^{2-2\alpha} d\xi = \int \left[\left(1+|\xi|\right)^{2s} \left|\hat{\phi}(\xi)\right|^{2}\right]^{1-\alpha} \left(1+|\xi|\right)^{2s(\alpha-1)} d\xi,$$

and by using Hölder's inequality, one gets

$$\int \left|\hat{\phi}(\xi)\right|^{2-2\alpha} d\xi \leq \left[\int \left(1+|\xi|\right)^{2s} \left|\hat{\phi}(\xi)\right|^{2} d\xi\right]^{1-\alpha} \left[\int \left(1+|\xi|\right)^{2s(\alpha-1)/\alpha} d\xi\right]^{\alpha}$$

$$\leq c\left[\int \left(1+|\xi|\right)^{2s(\alpha-1)/\alpha} d\xi\right]^{\alpha}.$$

Hence, if $0 < \alpha < 1/(1 + 1/2s)$, then (7) holds. To get (1) it suffices to use the following inequalities which can be found in [3].

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \left|\hat{\phi}(\xi+2k\pi)\right|^{2-\alpha} &\leq \int \left|\frac{d}{d\xi}\left(\left|\hat{\phi}\right|^{2-\alpha}(\xi)\right)\right| d\xi \\ &\leq (2-\alpha)\int \left|\frac{d\hat{\phi}}{d\xi}\right| \left|\hat{\phi}(\xi)\right|^{1-\alpha} d\xi \\ &\leq \left[\int \left|\frac{d\hat{\phi}}{d\xi}\right|^{2} d\xi\right]^{1/2} \left[\int \left|\hat{\phi}(\xi)\right|^{2-2\alpha} d\xi\right]^{1/2}. \end{aligned} \tag{8}$$

Since $(\alpha_n)_n$ is a finite, it follows that the associated scaling function ϕ is compactly supported. Moreover, $\phi \in H^s(\mathbb{R})$ for some $s > 0$ implies that $\phi \in L^2(\mathbb{R})$. It becomes clear that the first factor of the last inequality is proportional to the L^2-norm of $x\phi(x)$ which is finite, and the second factor is finite whenever $0 < \alpha < 1/(1 + 1/(2s))$. To prove (2), we consider a point ξ such that $|\xi| \in [2^{n-1}\pi, 2^n\pi]$, $n \geq 1$. Then the techniques used to get (8) give us

$$\left|\hat{\phi}(\xi)\right|^{2} \leq c'\left[\int_{2^{n-1}\pi\leq|\xi|\leq 2^{n}\pi} \left|\hat{\phi}(\xi)\right|^{2} d\xi\right]^{1/2}. \tag{9}$$

Since,

$$\int_{2^{n-1}\pi\leq|\xi|\leq 2^{n}\pi} \left|\hat{\phi}(\xi)\right|^{2} d\xi = \int_{2^{n-1}\pi\leq|\xi|\leq 2^{n}\pi} \left[\left(\frac{1}{2}+|\xi|\right)^{2s} \left|\hat{\phi}(\xi)\right|^{2}\right]\left[\frac{1}{\left(1/2+|\xi|\right)^{2s}}\right] d\xi,$$

therefore,

$$\int_{2^{n-1}\pi \le |\xi| \le 2^n \pi} \left|\hat{\phi}(\xi)\right|^2 d\xi \le c_1(s)\left(\frac{1}{2} + 2^{n-1}\pi\right)^{-2s}$$

$$\le c_1(s)\left(\frac{1}{2} + \frac{|\xi|}{2}\right)^{-2s}$$

$$\le c_2(s)\left(1 + |\xi|\right)^{-2s}.$$

Consequently, there exists a constant $c_3(s)$ depending only on s such that

$$\left|\hat{\phi}(\xi)\right| \le c_3(s)\left(1 + |\xi|\right)^{-s}, \quad \forall \xi \in R.$$

Collecting everything together, one concludes that for any arbitrary real number α satisfying

$$0 < \alpha < \min\left(s, \frac{1}{1 + 1/(2s)}\right),$$

the scaling function ϕ satisfies condition (1) and (2). Consequently, the upper bound of (6) is proven. To prove the lower bound of (6), we first mention that under condition (4) and (6), the wavelet expansion of an L^2 function f converges in the L^2-sense, that is,

$$\forall f \in L^2(\mathbb{R}), f(x) = \sum_{l} \sum_{j,k \in \mathbb{Z}} \left\langle f, \psi_{l,jk} \right\rangle \psi_{l,jk}(x), \tag{10}$$

where the equality holds in the L^2-sense. For the proof of this result we refer to [5]. From (4) and (10), one concludes that, for all $f \in L^2(\mathbb{R})$,

$$\|f\|_2^2 = \lim_{N \to +\infty} \sum_{j=-N}^{N} \sum_{k \in \mathbb{Z}} \sum_{l} \left|\left\langle f, \psi_{l,jk} \right\rangle\right|^2$$

$$\le \sum_{l} \sum_{j,k \in \mathbb{Z}} \left|\left\langle f, \psi_{l,jk} \right\rangle\right|^2$$

$$\le \sqrt{c_2(s)}\, \|f\|_2 \left[\sum_{l} \sum_{j,k \in \mathbb{Z}} \left|\left\langle f, \psi_{l,jk} \right\rangle\right|^2\right]^{1/2}.$$

Hence,

$$\|f\|_2^2 \frac{1}{c_2(s)} \le \sum_{l} \sum_{j,k \in \mathbb{Z}} \left|\left\langle f, \psi_{l,jk} \right\rangle\right|^2,$$

which proves the lower bound of (6) and concludes the proof of the theorem.

Theorem 2.3: Let $\{\psi_{l;j,k} = 2^{j/2}\psi_l(2^j x - k)\}_{j,k\in\mathbb{Z}, l=1,2,\dots,k}$, is a wavelet packet frame of $L^2(\mathbb{R})$ with frame bounds A and B, i.e.,

$$A\|f\|^2 \le \sum_l \sum_{j,k\in\mathbb{Z}} \left|\left\langle f, \psi_{l;j,k}\right\rangle\right|^2 \le B\|f\|^2.$$

Then $\hat{\psi}$ satisfies

$$\mathrm{A} \le \sum_l \sum_{j\in\mathbb{Z}} \left|\left\langle \hat{\psi}_l(2^j\omega)\right\rangle\right|^2 \le B \qquad \text{a.e., ,}$$

for the same constants A and B.

Proof: Since $\psi_{l;j,k} = 2^{j/2}\psi_l(2^j x - k), j, k \in \mathbb{Z}, l = 1, 2, \dots, k$ and for any $f \in L^2(\mathbb{R})$, we have

$$\begin{aligned}\langle f, \psi_{l;j,k}\rangle &= 2^{j/2}\int_{-\infty}^{\infty} f(x)\,\overline{\psi_l(2^j x - k)}\,dx \\ &= \frac{1}{2\pi}2^{j/2}\int_{-\infty}^{\infty}\hat{f}(2^j\omega)\,\overline{\hat{\psi}_l(\omega)}\,e^{ik\omega}\,d\omega.\end{aligned}$$

Now, by setting

$$T = 2\pi, \tag{11}$$

we have,

$$\begin{aligned}\sum_l \sum_{j,k\in\mathbb{Z}}\left|\left\langle f, \psi_{l;j,k}\right\rangle\right|^2 &= \sum_l\sum_{j\in\mathbb{Z}}\frac{2^j}{4\pi^2}\sum_{k\in\mathbb{Z}}\left|\int_{-\infty}^{\infty}\hat{f}(2^j\omega)\,\overline{\hat{\psi}_l(\omega)}\,e^{ik\omega}d\omega\right|^2 \\ &= \sum_l\sum_{j\in\mathbb{Z}}\frac{2^jT^2}{4\pi^2}\sum_{k\in\mathbb{Z}}\left|\frac{1}{T}\int_0^T\left[\sum_{h\in\mathbb{Z}}\hat{f}(2^j(\omega\, hT))\,\overline{\hat{\psi}_l(\omega+hT)}\right]e^{ik\frac{2\pi}{T}\omega}d\omega\right|^2 \\ &= \sum_l\sum_{j\in\mathbb{Z}}2^j\int_0^T\left|\sum_{h\in\mathbb{Z}}\hat{f}(2^j(\omega+hT))\,\overline{\hat{\psi}_l(\omega+hT)}\right|^2 d\omega. \end{aligned}\tag{12}$$

Now, by the definition of wavelet packet frame and equation (12), we have

$$A\|\hat{f}\|^2 \le \sum_l\sum_{j\in\mathbb{Z}}2^j\int_0^T\left|\sum_{h\in\mathbb{Z}}\hat{f}(2^j(\omega+hT))\,\overline{\hat{\psi}_l(\omega+hT)}\right|^2 d\omega \le B\|\hat{f}\|^2. \tag{13}$$

If for any $M > 0$, $M \in \mathbb{Z}$, and $\omega_0 \in (-\infty, \infty)$, we have

$$\sum_l\sum_{j=-M}^{M}2^j\int_{2^{-j}\omega_0-\frac{T}{2}}^{2^{-j}\omega_0+\frac{T}{2}}\left|\sum_{h\in\mathbb{Z}}\hat{f}(2^j(\omega+hT))\,\overline{\hat{\psi}_l(\omega+hT)}\right|^2 d\omega \le B\|\hat{f}\|^2.$$

Now, consider $\hat{f} = \left(\frac{1}{\sqrt{2\epsilon}}\right)\chi_{[\omega_0 - \epsilon, \omega_0 + \epsilon]}, \epsilon > 0$. Then for sufficiently small ϵ, the above inequality becomes

$$\sum_l \sum_{j=-M}^{M} \frac{2^j}{2\epsilon} \int_{2^{-j}(\omega_0 - \epsilon)}^{2^{-j}(\omega_0 + \epsilon)} \left| \hat{\psi}_l(\omega) \right|^2 d\omega \leq B,$$

and thus, by taking $\epsilon \to 0$ and $M \to \infty$, we have

$$\sum_l \sum_{j=-M}^{M} \left| \hat{\psi}_l(2^j \omega) \right|^2 \leq B \qquad \text{a.e.,} \tag{14}$$

On the other hand, for any ω_0, $\eta > 0$, a positive integer M may be chosen so that

$$\int_{2^{M+1}\omega_0/3}^{\infty} \left| \hat{\psi}_l(\omega) \right|^2 \leq \eta. \tag{15}$$

Also, for

$$0 < \epsilon < \min\left\{\frac{\omega_0}{3}, \frac{T}{2}\right\},$$

the function $\hat{f} = \left(\frac{1}{\sqrt{2\epsilon}}\right)\chi_{[\omega_0 - \epsilon, \omega_0 + \epsilon]}$ satisfies

$$\hat{f}(2^j(\omega + hT)) = 0,$$

for all $h \in \mathbb{Z}$ with $|h| \geq \left(\frac{\epsilon}{2^j T}\right) + 1$ and all $\omega \in \left[2^{-j}\omega_0 - \frac{T}{2}, 2^{-j}\omega_0 + \frac{T}{2}\right]$. Hence, for this $\hat{f}$, we have

$$\sum_l \sum_{j=-\infty}^{-M} 2^j \int_{2^{-j}\omega_0 - \frac{T}{2}}^{2^{-j}\omega_0 + \frac{T}{2}} \left| \sum_{h \in \mathbb{Z}} \hat{f}(2^j(\omega + hT)) \overline{\hat{\psi}_l(\omega + hT)} \right|^2 d\omega$$

$$\leq \sum_l \sum_{j=-\infty}^{-M} \frac{2^j}{2\epsilon} \int_{2^{-j}\omega_0 - \frac{T}{2}}^{2^{-j}\omega_0 + \frac{T}{2}} \left[\sum_{h \in \mathbb{Z}} \left| \hat{\psi}_l(\omega + hT) \right|^2 \chi_{[\omega_0 - \epsilon, \omega_0 + \epsilon]}(2^j(\omega + hT)) \right] \left(\frac{\epsilon}{2^j T} + 1 \right)$$

$$\leq C \sum_l \sum_{j=-\infty}^{-M} \int_{2^{-j}(\omega_0 - \epsilon)}^{2^{-j}(\omega_0 + \epsilon)} \left\{ \left| \hat{\psi}_l(\omega) \right|^2 + \frac{2^j}{2\epsilon} \left| \hat{\psi}_l(\omega) \right|^2 \right\} d\omega. \tag{16}$$

Since $\epsilon < \frac{\omega_0}{3}$, the intervals,

$$[2^{-j}(\omega_0 - \epsilon), 2^{-j}(\omega_0 + \epsilon)], \quad j \in \mathbb{Z}$$

are mutually disjoint, and hence by equation (15), we have

$$\sum_l \sum_{j=-\infty}^{-M} \int_{2^{-j}(\omega_0 - \epsilon)}^{2^{-j}(\omega_0 + \epsilon)} \left| \hat{\psi}_l(\omega) \right|^2 \leq \int_{2^{M+1}\frac{\omega_0}{3}}^{\infty} \left| \hat{\psi}_l(\omega) \right|^2 d\omega < \eta,$$

Now by equation (16), we have

$$\sum_{l} \sum_{j \in \mathbb{Z}} 2^{j} \int_{0}^{T} \left| \sum_{h \in \mathbb{Z}} \hat{f}(2^{j}(\omega + hT)) \overline{\hat{\psi}_{l}(\omega + hT)} \right|^{2} d\omega$$

$$\leq \sum_{l} \sum_{j=-M+1}^{M} 2^{j} \int_{0}^{T} \left| \sum_{h \in \mathbb{Z}} \hat{f}(2^{j}(\omega + hT)) \overline{\hat{\psi}_{l}(\omega + hT)} \right|^{2} d\omega$$

$$+ C_{\eta} + \frac{C}{2\epsilon} \int_{\omega_0 - \epsilon}^{\omega_0 + \epsilon} \sum_{l} \sum_{j=-\infty}^{-M} \left| \hat{\psi}_{l}(2^{-j}\omega) \right|^{2} d\omega. \tag{17}$$

Therefore, by the definition of wavelet packet frame and equation (17), we have

$$R = \sum_{l} \sum_{j=-M+1}^{\infty} 2^{j} \int_{0}^{T} \left| \sum_{h \in \mathbb{Z}} \hat{f}(2^{j}(\omega + hT)) \overline{\hat{\psi}_{l}(\omega + hT)} \right|^{2} d\omega$$

$$\geq A - C_{\eta} + \frac{C}{2\epsilon} \int_{(\omega_0 - \epsilon)}^{(\omega_0 + \epsilon)} \sum_{l} \sum_{j=-\infty}^{-M} \left| \hat{\psi}_{l}(2^{-j}\omega) \right|^{2} d\omega. \tag{18}$$

On the other hand, for all sufficient small $\epsilon > 0$, it is clear that

$$R = \sum_{l} \sum_{j=-M+1}^{\infty} 2^{j} \int_{2^{-j}(\omega_0 - \epsilon)}^{2^{-j}(\omega_0 + \epsilon)} \left| \hat{f}(2^{j}(\omega) \overline{\hat{\psi}_{l}(\omega)} \right|^{2} d\omega$$

$$= \frac{1}{2\epsilon} \int_{\omega_0 - \epsilon}^{\omega_0 + \epsilon} \sum_{l} \sum_{j=-M+1}^{\infty} \left| \hat{\psi}_{l}(2^{-j}\omega) \right|^{2} d\omega,$$

where, $\hat{f} = \left(\frac{1}{\sqrt{2\epsilon}}\right) \chi_{[\omega_0 - \epsilon, \omega_0 + \epsilon]}$. Hence, in view of the boundedness property in equation (14), we may take $\epsilon \to 0$ in equation (18) to arrive at

$$\sum_{l} \sum_{j=-M+1}^{\infty} \left| \hat{\psi}_{l}(2^{-j}\omega_0) \right|^{2} \geq A - C_{\eta} - C \sum_{l} \sum_{j=-\infty}^{-M} \left| \hat{\psi}_{l}(2^{-j}\omega_0) \right|^{2} \tag{19}$$

for almost all $\omega_0 > 0$. Since $\eta > 0$ is arbitrary, so from equations (14) and (19), we get

$$\sum_{l} \sum_{j \in \mathbb{Z}} \left| \hat{\psi}_{l}(2^{-j}\omega) \right|^{2} \geq A \tag{20}$$

for almost all $\omega_0 > 0$. Hence, by equations (14) and (20), we get the desired result.

REFERENCES

[1] O. Christensen, (2001), Frames, Riesz Bases and Discrete Gabor/Wavelet Expansions, *Bull. Amer. Math. Soc*., **38(3)**: 273–291.

[2] C. K. Chui, and X. L. Shi, (1993), Inequalities of Littlewood-Paley Type for Frames and Wavelets, *SIAM J. Math. Anal*., **24(1)**: 263–277.

[3] A. Cohen, I. Daubechies, and J. C. Feauveau, (1992), Biorthogonal Bases of Compactly Supported Wavelets, Comm. Pure Appl. Math., **45**: 485–560.

[4] R. R. Coifman, and R. Rochberg, (1980), Representation for Holomorphic and Harmonic Functions in Lp, *Asterisque,* **77**: 11–66.

[5] I. Daubechies, (1988), Orthonormal Bases of Compactly Supported Wavelets, Comm. Pure Appl. Math., **41**: 909–996.

[6] I. Daubechies, (1990), The Wavelet Transform, Time-Frequency Localization and Signal Analysis, *IEEE Trans. Inform. Theory*, **36(5)**: 961–1005.

[7] I. Daubechies, (1992), Ten Lectures on Wavelets, *SIAM*, Philadelphia.

[8] L. Debnath, (2002), *Wavelet Transforms and Their Applications*, Birkhauser.

[9] R. J. Duffin, and A. C. Schaeffer, A Class of Non Harmonic Fourier Series, *Trans. Amer. Math. Soc*., **72**: 341–366.

[10] C. Heil, and D. Walnut, (1989), Continuous and Discrete Wavelet Transforms, SIAM Review, **31**: 628–666.

[11] A. Karoui, (2003), Wavelets Properties and Approximate Solution of a Second Kind Integral Equation, *Computers and Mathematics with Applications*, **46**: 263–277.

[12] Z. Shang, and X., Zhou (2007), Dual Generators for Weighted Irregular Wavelet Frames and Reconstruction Error, *Appl. Comp. Harm. Anal*., **22(3)**: 356–367.

[13] W. Sun, and X. Zhou, (2002), Irregular wavelet/Gabor frames, *Appl. Comp. Harm. Anal.*, **13:** 63–76.

[14] M. V. Wickerhauser, (1991), Acoustic Signal Compression with Wavelet Packets, In C. K. Chui, (Editor), 679–700.

M. K. Ahmad & Javid Iqbal
Department of Mathematics,
Aligarh Muslim University, Aligarh, India.
E-mails: ahmad_kalimuddin@yahoo.co.in, & javid2iqbal@yahoo.co.in