IJAMAA
January-June 2014, Volume 9, No. 1, pp. 21-30

TRANSIENT SOLUTION OF GENERALIZED FINITE BIRTH-DEATH PROCESS

MOHIT, D. V. GUPTA AND P. K. SHARMA

ABSTRACT: A transient single server queueing model with finite birth-death process is considered. The transient distribution of the number of customers in the system and the expected length of the system for a finite birth and death process are derived by solving the system of differential-difference equations using Laplace-Transforms and finding the inversion through the properties of tridiagonal symmetric matrices. Some numerical comparisons are made with the randomization method.

Keywords: Transient, Birth-Death process, Differential-difference equations, Laplace transform.

1. INTRODUCTION

Much of the vast literature on queueing models is confined to results describing steady-state operation only. But in many potential applications of queueing theory, the practitioner needs to know how the system will operate up to some instant '*t*'. Many systems begin operations and are stopped at some specified time *t*. Business or service operations such as rental agencies or medical clinics which open and close, never operate under steady-state conditions. Furthermore, if the system is empty initially, the fraction of time the server is busy and the initial rate of output etc will be below the steady-state values, and hence, the use of steady-state results to obtain these measures is not appropriate. Thus, the investigation of the transient behaviour of queueing processes is also important from the point of view of the theory and its applications.

The transient derivation of Markovian queueing models is quite a complicated procedure. Detailed analysis of transient solution of some of these models is discussed in Gross and Harris [5]. The solution of the $M/M/1/\infty$ model postdated that of the basic Erlang work by nearly half a century, with the first published solution due to Ledermann and Reuter [7], in which they used spectral analysis for the general birth-death process. In the same year, an additional paper appeared on the solution of this problem by Bailey [2], and later on by Champernowne [3]. Bailey's approach to the time-dependent problem was via generating functions for partial differential equation, and Champernowne's was via complex combinatorial methods. It is Bailey's approach that has been popular over the years.

In this paper, a finite transient state birth-death process is considered. The model has also been studied by Rosenlund [9]. The explicit solution for $M/M/1$ obtained by

Takacs [11] is cumbersome. The computational procedure described here for generalized birth-death process is simpler in application as well as in computation. Sharma and Dass [10] in 1988 simplified the earlier presentation of the method in considering the finite capacity $M/M/1$ model for finding the transient solution. Herein, method of Sharma and Dass [10] is followed with a little modification thus leading to a simpler computation of the model's solution. Krinik *et. al*. [6] have applied the randomization technique to solve transient single server queueing model. Al-Seedy [1] has obtained transient solution of $M/M/2$ queueing model with balking using generating function technique. But these techniques are not applicable to generalized birth-death transient queueing models.

2. The Mathematical Model

Consider the generalized birth-death process with $(N + 1)$ states viz. 0, 1, 2, …, N having birth-death rates λ_n and μ_n respectively, when it is in state n. Let it be assumed that the initial system size at time 0 is i. Let $P_n(t)$ denotes probability that the process is in state n at time t. Then differential-difference equations governing the system size are given by

$$\left.\begin{aligned}
&\frac{d}{dt}P_0(t) = \lambda_0 P_0(t) + \mu_1 P_1(t), \quad n = 0, \\
&\frac{d}{dt}P_n(t) = -(\lambda_n + \mu_n)P_n(t) + \lambda_{n-1}P_{n-1}(t) + \mu_{n+1}P_{n+1}(t), \quad 1 \le n \le (N-1), \\
&\frac{d}{dt}P_N(t) = \lambda_{N-1}P_{N-1}(t) - \mu_N P_N(t), \quad n = N
\end{aligned}\right\} \tag{1}$$

3. Solution of Mathematical Model

We solve these time-dependent equations by using combination of probability generating functions, partial differential equations and Laplace transforms. Let $\overline{P}_n(s)$ be the Laplace-transform of $P_n(t)$. Taking Laplace-transform of equation (1), we get

$$\left.\begin{aligned}
&(\lambda_0 + s)\overline{P}_0(s) - \mu_1\overline{P}_1(s) = \delta_{i0}, && n = 0 \\
&-\lambda_{n-1}\overline{P}_{n-1}(s) + (\lambda_n + \mu_n + s)\overline{P}_n(s) - \mu_{n+1}\overline{P}_{n+1}(s) = \delta_{in}, && 1 \le n \le (N-1) \\
&-\lambda_{N-1}\overline{P}_{N-1}(s) + (\mu_N + s)\overline{P}_N(s) = \delta_{iN}, && n = N
\end{aligned}\right\} \tag{2}$$

where δ_{in} is the Kronecker delta.

Let $C_N(s)$ be the determinant of the coefficient matrix of (2). Let $T_r(s)$ and $B_r(s)$ be the determinants of the $r \times r$ matrices formed at the top left corner and at the bottom right corner of the coefficient matrix. Also let $T_0(s) = B_0(s) = 1$.

The solution of equation (1) is given as

$$\bar{P}_n(s) = \left(\prod_{j=n+1}^{i} u_j\right)\frac{T_n(s)\,B_{N-i}(s)}{A_N(s)}, \quad 0 \le n \le i$$

$$= \left(\prod_{j=i}^{n-1} \lambda_j\right)\frac{T_i(s)\,B_{N-n}(s)}{A_N(s)}, \quad (i+1) \le n \le N \tag{3}$$

where the first product may be interpreted as $\prod_{j=k}^{n} u_j = 1$ whenever $n < k$.

Our aim is to express (3) as a partial fraction and we know that $C_N(s)$ is given by

$$C_N(s) = \begin{vmatrix} \lambda_0 + s & -\mu_1 & & & & & \\ -\lambda_0 & \lambda_1 + \mu_1 + s & -\mu_2 & & & & \\ & -\lambda_1 & \lambda_2 + \mu_2 + s & -\mu_3 & & & \\ & & -\lambda_2 & \lambda_3 + \mu_3 + s & -\mu_4 & & \\ & & & & \vdots & & \\ & & & & & \lambda_{N-1} + \mu_{N-1} + s & -\mu_N \\ & & & & & -\lambda_{N-1} & \mu_N + s \end{vmatrix}$$

Notice that s is zero of $C_N(s)$ if and only if $-s$ is an eigenvalue of the matrix E_N, where

$$E_N = \begin{vmatrix} \lambda_0 & -\mu_1 & & & & & \\ -\lambda_0 & \lambda_1 + \mu_1 & -\mu_2 & & & & \\ & -\lambda_1 & \lambda_2 + \mu_2 & -\mu_3 & & & \\ & & -\lambda_2 & \lambda_3 + \mu_3 & -\mu_4 & & \\ & & & & \vdots & & \\ & & & & & \lambda_{N-1} + \mu_{N-1} & -\mu_N \\ & & & & & -\lambda_{N-1} & \mu_N \end{vmatrix}$$

Observe that $s = 0$ is an eigenvalue of E_N. However, since each off-diagonal element is non-zero, all eigenvalues are distinct. Thus, s is the only zero of $C_N(s)$. Also, since all minors of elements of E_N are positive, by the strum sequence property, all other eigenvalues are positive. Therefore, all the eigenvalues of the positive semi-definite matrix E_N are real, distinct zeros one of which is zero and the rest are the negatives of the eigenvalues of E_N. Let the zeros of $C_N(s)$ be z_k, $k = 0, 1, \ldots, N$, with $z_0 = 0$. $\bar{P}_n(s)$ in

(3) can also be expressed as:

$$\overline{P}_n(s) = \frac{G_n(s)}{A_n(s)}, \quad n = 0, 1, ..., N, \tag{4}$$

The partial fraction form is given as

$$\overline{P}_n(s) = \sum_{k=0}^{N} \frac{\beta_{n,k}}{s - z_k}, \quad n = 0, 1, ..., N, \tag{5}$$

where

$$\beta_{n,k} = \frac{G_n(z_k)}{\prod\limits_{\substack{j=0 \\ j \neq k}}^{N} (z_k - z_j)}, \quad k = 0, 1, ..., N \tag{6}$$

The distribution of the state can be obtained by inverting (5) which is

$$P_n(t) = \beta_{n,0} + \sum_{k=1}^{N} \beta_{n,k} e^{z_k t}, \quad n = 0, 1, ..., N, \tag{7}$$

where $\beta_{n,k}$'s are given by (6).

4. RESULTS AND DISCUSSIONS

Extensive computations are made to test the method and obtain the performance measures of the transient queueing models. Value of N is taken as 4. Here C_4 is of the form

$$C_4(s) = \begin{vmatrix} \lambda_0 + s & -\mu_1 & & & \\ -\lambda_0 & \lambda_1 + \mu_1 + s & -\mu_2 & & \\ & -\lambda_1 & \lambda_2 + \mu_2 + s & -\mu_3 & \\ & & -\lambda_2 & \lambda_3 + \mu_3 + s & -\mu_4 \\ & & & \lambda_3 & \mu_4 + s \end{vmatrix}$$

It is known that s is a zero of $C_4(s)$ if and only if $-s$ is an eigen-value of the matrix E_4 where

$$E_4 = \begin{vmatrix} \lambda_0 & -\mu_1 & & & \\ -\lambda_0 & \lambda_1 + \mu_1 & -\mu_2 & & \\ & -\lambda_1 & \lambda_2 + \mu_2 & -\mu_3 & \\ & & -\lambda_2 & \lambda_3 + \mu_3 & -\mu_4 \\ & & & \lambda_3 & \mu_4 \end{vmatrix}$$

Here z_k, $k = 0, 1, 2, 3, 4$ are the roots of $C_4(s)$.

Here $T_r(s)$ and $B_r(s)$ are determinants of the $r \times r$ matrices at the top left corner and bottom right corner of the coefficient matrix respectively and $T_0(s) = B_0(s) = 1$.

$$G_n(s) = \left(\prod_{j=i}^{n-1} \lambda_j \right) B_{4-n}(s), \quad 0 \le n \le 4$$

and

$$\beta_{n,k} = \frac{G_n(z_k)}{\prod_{\substack{j=0 \\ j \ne k}}^{N} (z_k - z_j)}$$

Transient probabilities are given by

$$P_n(t) = \beta_{n,0} + \sum_{k=1}^{N} \beta_{n,k} e^{z_k t}, \quad n = 0, 1, ..., N.$$

Expected number of customers in the system are given by

$$L_s = \sum_{n=0}^{N} n P_n(t)$$

First of all, following parameters are chosen

$$\lambda_i = 1, \qquad 0 \le i \le (N-1)$$
$$\mu_i = 1, \qquad 1 \le i \le N$$

Probabilities at different times $P_n(t)$ and expected length of the system L_s are computed until the following criterion is satisfied

$$|P_n(t+h) - P_n(t)| < \varepsilon \quad \forall n$$

i.e. difference of probabilities at the consecutive time-steps is less than a specified tolerance parameter ε. In the present computations, ε is taken as 10^{-3}. The results are reported in Table 1. It is clear from the table that after a certain time, probabilities $P_n(t)$ reach the steady state. Next μ_i are chosen as 2, 3 and 4 $\forall\, i$ for the same value of λ_i and the entire computational procedure is repeated. The results are reported in Tables 2, 3 and 4 respectively. From the tables, it is clear that as μ_i increases, expected length of the system L_S decreases. To investigate the effect of change of arrival rate, following values of λ_i and μ_i are taken

$$\lambda_i = 2, \qquad \mu_i = 2 \;\forall\; i$$
$$\lambda_i = 3, \qquad \mu_i = 2 \;\forall\; i$$
$$\lambda_i = 4, \qquad \mu_i = 2 \;\forall\; i,$$

Effect of t on P_0, P_1, P_2, P_3, P_4 & L_S.

Table 1

$\lambda = 1$ and $\mu = 1$

t	P_0	P_1	P_2	P_3	P_4	L_S
0.0	0.999	0.000	0.000	0.000	0.000	0.000
1.0	0.523	0.308	0.122	0.035	0.009	0.699
2.0	0.385	0.296	0.179	0.090	0.047	1.117
3.0	0.319	0.269	0.194	0.127	0.089	1.397
4.0	0.279	0.248	0.198	0.151	0.122	1.588
5.0	0.253	0.232	0.199	0.166	0.146	1.719
6.0	0.236	0.222	0.199	0.177	0.163	1.808
7.0	0.224	0.215	0.199	0.184	0.175	1.868
8.0	0.217	0.210	0.199	0.189	0.182	1.910
9.0	0.211	0.207	0.199	0.192	0.188	1.938
10.0	0.207	0.204	0.199	0.195	0.192	1.957
11.0	0.205	0.203	0.199	0.196	0.194	1.970
12.0	0.203	0.202	0.199	0.197	0.196	1.980
13.0	0.202	0.201	0.199	0.198	0.197	1.986
14.0	0.201	0.201	0.199	0.198	0.198	1.990
15.0	0.201	0.201	0.199	0.199	0.198	1.993
16.0	0.201	0.201	0.199	0.199	0.199	1.995
17.0	0.201	0.201	0.199	0.199	0.199	1.997
18.0	0.201	0.201	0.199	0.199	0.199	1.998

Table 2

$\lambda = 1$ and $\mu = 2$

t	P_0	P_1	P_2	P_3	P_4	L_S
0.0	0.999	0.000	0.000	0.000	0.000	0.000
1.0	0.633	0.257	0.082	0.021	0.008	0.517
2.0	0.565	0.263	0.111	0.043	0.017	0.687
3.0	0.539	0.261	0.121	0.054	0.024	0.764
4.0	0.527	0.259	0.125	0.059	0.028	0.801
5.0	0.521	0.258	0.127	0.061	0.030	0.820
6.0	0.518	0.258	0.128	0.060	0.031	0.820
7.0	0.517	0.258	0.128	0.063	0.031	0.827
8.0	0.516	0.258	0.128	0.064	0.031	0.830
9.0	0.516	0.258	0.128	0.064	0.032	0.834
10.0	0.516	0.258	0.128	0.064	0.032	0.834

Table 3

$\lambda = 1$ and $\mu = 3$

t	P_0	P_1	P_2	P_3	P_4	L_S
0.0	0.999	0.000	0.000	0.000	0.000	0.000
1.0	0.710	0.215	0.056	0.012	0.002	0.379
2.0	0.679	0.222	0.069	0.021	0.006	0.450
3.0	0.672	0.222	0.073	0.023	0.007	0.470
4.0	0.669	0.223	0.073	0.024	0.008	0.476
5.0	0.669	0.223	0.074	0.024	0.008	0.478
6.0	0.669	0.223	0.074	0.024	0.008	0.478

Table 4

$\lambda = 1$ and $\mu = 4$

t	P_0	P_1	P_2	P_3	P_4	L_S
0.0	0.999	0.000	0.000	0.000	0.000	0.000
1.0	0.766	0.183	0.040	0.008	0.001	0.294
2.0	0.753	0.187	0.045	0.011	0.002	0.322
3.0	0.751	0.187	0.046	0.011	0.002	0.327
4.0	0.750	0.187	0.046	0.011	0.002	0.327
5.0	0.750	0.187	0.046	0.011	0.002	0.327
6.0	0.750	0.187	0.046	0.011	0.002	0.327

Only the steady-state probabilities and expected length of customers in the system are reported in Table 5. Table 2 and Table 5 clearly demonstrate that as arrival rate λ_i increases, L_S also increases. Table 6 contains the computational results for different values of N corresponding to the same values of λ_i and μ_i when steady state is achieved. Again it is clear from the table that as N increases, expected length of customers in system L_S also increases. Also the above results are reported when values of λ_i and μ_i are same $\forall$ i. The method works equally well for varying values of λ_i and μ_i. It is assumed that customers arrive from a single infinite source in accordance with parameter λ. Customers are served by one of the c-servers. The capacity of the system is limited to N (including those being served) i.e. there is a waiting room with capacity $(N - c)$.

Table 5

λ_i	μ_i	P_0	P_1	P_2	P_3	P_4	L_S
2.0	2	0.200	0.200	0.199	0.199	0.199	1.999
3.0	2	0.075	0.115	0.170	0.255	0.383	2.759
4.0	2	0.032	0.064	0.129	0.257	0.515	3.159

Table 6

$\lambda_i = 1$ and $\mu_i = 2 \; \forall \; i$

N	P_0	P_1	P_2	P_3	P_4	P_5	L_S
3.0	0.533	0.266	0.132	0.066	–	–	0.731
4.0	0.516	0.258	0.128	0.064	0.032	–	0.834
5.0	0.508	0.254	0.127	0.063	0.031	0.015	0.896

An arriving customer who finds all the c-servers busy on arrival, but the waiting room not full, may balk with probability q or may join the system with probability p where $p + q = 1$. Thus λq is the instantaneous balking rate. After joining the queue, a customer may renege i.e. he will wait a certain length of time for the service to begin, otherwise he will depart from the system. The length of time he will wait is a random variable having exponential distribution with parameter α. In this case,

$$\lambda_n = \begin{cases} \lambda, & 0 \le n \le (c-1) \\ p\lambda, & c \le n \le N \end{cases}$$

$$\mu_n = \begin{cases} n\mu, & 0 \le n \le (c-1) \\ c\mu + (n-c)\alpha, & c \le n \le N \end{cases}$$

For the purpose of computational results, following parameters are chosen

$$N = 3,\ c = 2,\ p = 0.5,\ \alpha = 5,\ \lambda = 1,\ \mu = 2$$

Values of λ_i and μ_i reduce to

$$\lambda_0 = \lambda,\ \lambda_1 = \lambda,\ \lambda_2 = p\lambda,\ \lambda_3 = p\lambda$$
$$\mu_1 = \mu,\ \mu_2 = 2\,\mu,\ \mu_3 = 2\,\mu + \alpha,\ \mu_4 = 2\,\mu + 2\alpha$$

The above values of λ_i and μ_i correspond to the model with reneging and balking, computational procedure is carried out and the results are reported in Table 7. Further,

Table 7

$N = 3, \mathrm{c} = 2, p = 0.5, \alpha = 5, \lambda = 1, \mu = 2$

t	P_0	P_1	P_2	P_3	L_S
0.0	0.999	0.000	0.000	0.000	0.000
1.0	0.650	0.283	0.062	0.006	0.428
2.0	0.616	0.303	0.075	0.008	0.478
3.0	0.613	0.306	0.076	0.008	0.484
4.0	0.612	0.306	0.076	0.008	0.484
5.0	0.612	0.306	0.076	0.008	0.484
6.0	0.612	0.306	0.076	0.008	0.484

for the same values of N, c, λ and μ; values of p and α are chosen as 1 and 0 respectively and the results are reported in Table 8. These results correspond to the case when there is no balking and reneging. Expected length of customers in the system in this case is greater than that with balking and reneging. The results depict the same behaviour as reported by Mohanty *et al.*, [8] and are in agreement with the observed pattern.

Table 8

$N = 3, c = 2, p = 1, \alpha = 0, \lambda = 1, \mu = 2$

t	P_0	P_1	P_2	P_3	L_S
0.0	1.000	0.000	0.000	0.000	0.000
1.0	0.648	0.297	0.059	0.012	0.452
2.0	0.610	0.298	0.073	0.017	0.497
3.0	0.604	0.301	0.075	0.018	0.507
4.0	0.603	0.301	0.075	0.018	0.508
5.0	0.603	0.301	0.075	0.018	0.508
6.0	0.603	0.301	0.075	0.018	0.508

5. CONCLUSION

The classical treatment of the transient behaviour is usually more complex than the steady state behaviour. A generalised finite-state birth-death process is considered and transient distribution of the number of customers is derived using Laplace transform of differential-difference equations and inverting again using the properties of tri-diagonal matrices. The numerical computations are carried out until the desired convergence criterion is satisfied, thus arriving at the steady–state distribution of the queueing model. The probability distribution of the number of customers and expected length of the system are given for various values of parameters viz. λ, μ and N. It is observed that as service rate μ increases, expected length of the system L_S decreases. Further, as arrival rate λ increases, L_S increases. Similarly as N increases, L_S again increases. The queueing model with balking and reneging can also be tackled using generalized birth-death model. Expected length of customers in the system is greater in the queueing model with balking and reneging than the model without these. The computational results presented here exhibit the similar pattern as randomization method (Grassmann [4]). Moreover, the computational method presented here is efficient and easier to implement.

REFERENCES

[1] Al-Seedy R., (2004), "A Transient Solution of a Non-Truncated Queue *M*/*M*/2 with balking and an additional server for longer queues (Krishnamoorthi discipline)", *Applied Mathematics and Computation*, **156**, (2004), 763-769.

[2] Bailey N. T. J., (1954), "A Continuous Time Treatment of a Single Queue Using Generating Functions], *J. Roy Statist. Soc. Ser.*, **B-16**, 288-291.

[3] Champernowne D. G., (1956), "An Elementary Method of Solution of the Queueing Problem with a Single Server and a Constant Parameter', *J. Roy Statist. Soc. Ser.*, **B-18**, 125-128.

[4] Grassmann W. K., (1990), "Computational Methods in Probability Theory", Chapter-5, *Handbooks in OR & MS*, Elsevier, North-Holland, **2**, 200.

[5] Gross D., and Harris C., (1985), '*Fundamentals of Queueing Theory*", John Wiley & Sons, Inc., New York .

[6] Krinik A., *et al*., (2002), 'Transient Probabilities of a Single Server Priority Queueing System", *Journal of Statistical Planning and Inference*, **101**, 185-190.

[7] Ledermann W., and Reuter G., (1954), 'Spectral Theory for the Differential Equations of Simple Birth and Death Process", *Phil. Trans. R. Soc. Lond. Ser.*, **A-246**, 321-369.

[8] Mohanty S. G., *et al*., (1993), 'On the Transient Behaviour of a Finite Birth-Death Process with an Application.' *Computers and Operations Research*, **20**, 239-248.

[9] Rosenlund S., (1978), 'Transition Probabilities for a Truncated Birth-Death Process", *Scand. J. Statist*., **5**, 119-122.

[10] Sharma O. P., and Dass J., (1988), 'Multi-Server Markovian Queue with Finite Waiting Space", Sankhya-Series, **B-50**, 428-431.

[11] Takacs L., (1962), "Introduction to the Theory of Queues", Oxford University Press, Oxford.

Mohit* & D. V. Gupta
Department of Mathematics
College of Engineering Roorkee, Roorkee, India
*E-mail: *mohit2692003@yahoo.co.in*

P. K. Sharma
Department of Mathematics
D.A.V. (P.G.) College, Dehradun, India