International Review of Pure and Applied Mathematics
Volume 14 • Number 2 • July-December 2018

A STUDY OF *k-g-* FRAMES IN HILBERT SPACES

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Abstract: *K*-frames were recently introduced by Gãvruţa in Hilbert spaces to study atomic systems with respect to a bounded linear operator. K-g-frames are more general than of g-frames in Hilbert spaces. Some results on k-g-frames are studied.

$(K_1 \otimes K_2) - g - frame$ for the tensor product of Hilbert spaces $H_1 \otimes H_2$ is introduced and some results on it are established.

Mathematics Subject Classification (2010): 42C15

Key words: frame: g-frame, k-g-frame

1. INTRODUCTION

Frames are generalization of bases. D. Han and D.R. Larson [5] have developed a number of basic aspects of operator-theoretic approach to frame theory in Hilbert space. Peter G. Casazza [1] presented a tutorial on frame theory and he suggested the major directions of research in frame theory. A. Najati and A. Rahimi [6] have developed the generalized frame theory and introduced methods for generating g-frames of a Hilbert space. Sun[7] introduced the concepts of g-Riesz bzses and g-frames.Recently, K-frames in a Hilbert space is introduced by L.Gavruta [4]as a generalization of the notion of the frame in Hilbert space. Fahimeh Arabyani Neyshaburi and Ali Akber Arefijamaal[3] were characterize all duals of a given k-frame and given some approaches for constructing K-frames. In [8], the authors Y Zhou and Y.Zhu are put forward the concept of K-g- frames, which are more general than ordinary g-frames in Hilbert spaces. Dingli Hua and Yongdong Huang [2] are proposed for construction methods for K-g-frames. The g-frame operator for g-frame in Hilbert space is introduced and results of g-frame operator are presented by GU Reddy in [9] and in [10] the tensor product of g-frames in tensor product of Hilbert spaces were studied.

In this paper Some results on k-g-frames are studied. $(K_1 \otimes K_2) - g - frame$ for the tensor product of Hilbert spaces $H_1 \otimes H_2$ is introduced and some results on it are established.

2. PRELIMINARIES

Frames are generalizations of orthonormal basis in Hilbert spaces. We recall the basic definations of frames.

Definition 2.1: A sequence $\{f_j\}_{j\in J}$ of vectors in a Hilbert space H is called a frame if there exists two constants $0 < A \le B < \propto$, such that

$$A\|f\|^2 \le \sum_{j\in J}\left|\langle f, f_j\rangle\right|^2 \le B\|f\|^2 \ \forall f \in H$$

The above inequality is called a frame inequality. The numbers A and B are called the lower and upper frame bounds respectively. If $A = B$ then $\{f_j\}_{j\in J}$ is called tight frame, if A=B=1 then $\{f_j\}_{j\in J}$ is called normalized tight frame. A synthesis operator $T : l_2 \to H$ is defined as $Te_j = f_j$ where $\{e_j\}$ is an orthonormal basis for l_2. The analysis operator $T^* : H \to l_2$ is an adjoint of synthesis operator T and is defined as $T^* f = \sum_{j\in J} < f, f_j > e_j \ \ \forall f \in H$.

A frame operator $S = TT^* : H \to H$ is defined as $Sf = \sum_j < f, f_j > f_j \ \forall f \in H$

The following few theorems from [1, 5] are useful in our discussion.

Theorem 2.2: For an orthonormal system $\{e_i\}_{i=1}^{\infty}$, the following are equivalent

(i) $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis.

(ii) $f = \sum_{i=1}^{\infty} < f, e_i > e_i \ \ \forall \ f \in H$.

(iii) <f, g> $= \sum_{i=1}^{\infty} < f, e_i >< e_i, g >$, $\forall \ f, g \in H$.

(iv) $\sum_{i=1}^{\infty} |< f, e_i >|^2 = \|f\|^2, \ \forall \ f \in H$.

(v) Span $\{e_i\}_{i=1}^{\infty} = H$.

(vi) If <f, e_i> = 0 $\forall$ i then f = 0.

Theorem 2.3. Suppose $\{f_j\}_{j\in J}$ is a frame for H if and only if $AI_{op} \le S \le BI_{op}$ and $\{f_j\}_{j\in J}$ is normalized tight frame for H if and only if $S = I_{op}$, where I_{op} is an identity operator on H.

The following theorem gives the existence of inverse of frame operator.

Theorem 2.4. [5] Let S be a frame operator for the frame $\{f_j\}_{j\in J}$ with frame bounds A and B in the Hilbert space H. Then S^{-1} exists, positive and $B^{-1}I_{op} \leq S^{-1} \leq A^{-1}I_{op}$.

Throughout this paper $\{H_j, j \in J\}$ will denote a sequence of Hilbert spaces. Let $L(H, H_j)$ be a collection all bounded linear operators from H to H_j and $\{\Lambda_j \in L(H, H_j) : j \in J\}$.

Definition 2.5. A sequence of operators $\{\Lambda_j\}_{j\in J}$ is said to be g-frame for Hilbert space H with respect to sequence of Hilbert spaces $\{H_j, j \in J\}$, if there exist two constants $0 < A \leq B < \infty$, such that

$$A\|f\|^2 \leq \sum_{j\in J}\|\Lambda_j f\|^2 \leq B\|f\|^2 \quad \forall f \in H.$$

The above inequality is called a g-frame inequality. The numbers A and B are called the lower frame bound and upper frame bound respectively. A g-frame $\{\Lambda_j\}_{j\in J}$ for H is said to be g-tight frame if $A = B$ and g-normalized tight frame for H if $A = B = 1$.

Definition 2.6. Let $\{\Lambda_j\}_{j\in J}$ be a g-frame for Hilbert space H. A g-frame operator

$S^g: H \to H$ is defined as $S^g f = \sum_{j\in J}\Lambda^*_j\Lambda_j f \;\; \forall f \in H$.

By using above definitions the following theorem on g-frame operator can be derived easily, so left to reader.

Theorem 2.7. If S^g is a g- frame operator , then we have

(i) $< S^g f, f> = \sum_{j\in J}\|\Lambda_j f\|^2$, for all f Î H.

(ii) S^g is a positive operator.

(iii) S^g is a self adjoint operator.

Theorem 2.8. Suppose $\{\Lambda_j\}_{j\in J}$ is a g-frame iff $A\, I_{op} \le S^g \le B\, I_{op}$ and $\{\Lambda_j\}_{j\in J}$ is g-normalized tight frame iff $S^g = I_{op}$ where I_{op} is an identity operator on *H*.

Proof. Since $\{\Lambda_j\}_{j\in J}$ is a g-frame so, we have,

$$A\,\|f\|^2 \le \sum_{j\in J}\|\Lambda_j f\|^2 \le B\,\|f\|^2, \text{ for all } f \in H.$$

Consider $< A\, I_{op}\, f, f > = A <f, f> = A\|f\|^2 \le \sum_{j\in J}\|\Lambda_j f\|^2 \le B\,\|f\|^2$

$$= B <f, f> = < B\, I_{op} f, f>$$

$\Rightarrow \quad A\, I_{op} \le S^g \le B\, I_{op}$

Conversely suppose $\quad A\, I_{op} \le S^g \le B\, I_{op}$

$$\Rightarrow <A\, I_{op} f, f> \le < S^g f, f> \le < I_{op} Bf, f>, \text{ for all } f \in H.$$

$$\Rightarrow A\|f\|^2 \le \sum_{j\in J}\|\Lambda_j f\|^2 \le B\,\|f\|^2$$

which implies $\{\Lambda_j\}_{j\in J}$ is a g-frame for *H*.

Suppose $\{\Lambda_j\}_{j\in J}$ is a g-normalized tight frame for *H*

$$\Leftrightarrow \sum_{j\in J}\|\Lambda_j f\|^2 = \|f\|^2 \text{ for all } f \in H.$$

$$\Leftrightarrow < S^g f, f> =< I_{op} f, f > \Leftrightarrow S^g = I_{op}.$$

We can easily seen that the frame operator S^g is invertible and $S^{g\,-1}$ is a positive operator.

The following theorem gives the existence of inverse of *g*-frame operator.

Theorem 2.9. Let S^g be a g-frame operator of the g-frame $\{\Lambda_j\}_{j\in J}$ with frame bounds A and B in the Hilbert space H. Then $B^{-1} I_{op} \le S^{g\,-1} \le A^{-1} I_{op}$.

Proof. Since $\{\Lambda_j\}_{j\in J}$ is a g-frame for Hilbert space H, so by theorem 2.8, we have

$$A\, I_{op} \leq S^{g} \leq B\, I_{op}$$

Since $A\, I_{op} \leq S^{g} \Rightarrow 0 \leq (S^{g} - A\, I_{op})\, S^{g\,-1} \Rightarrow 0 \leq I_{op} - A\, S^{g\,-1}$

(2.10) $\Rightarrow S^{g\,-1} \leq A^{-1}\, I_{op}$

Similarly, we can prove that

(2.11) $B^{-1}\, I_{op} \leq S^{g-1}$

From the equations (2.10) and (2.11), we get $B^{-1}\, I_{op} \leq S^{g\,-1} \leq A^{-1}\, I_{op}$.

Theorem 2.12. Let $\{\Lambda_j\}_{j\in J}$ be a g- frame for Hilbert space H with respect to $\{H_j\}_{j\in J}$ and $V \in B(H)$ be an invertible operator. Then $\{\Lambda_j V\}_{j\in J}$ is a g-frame for H with respect to $\{H_j\}_{j\in J}$ and its g-frame operator is $V^* S^g V$.

Proof. Since $V \in B(H)$, $\forall\ f \in H$, we have $Vf \in H$.

Given that $\{\Lambda_j\}_{j\in J}$ is a g- frame for H, by 2.5 for all $Vf \in H$, we have,

$$A\, \|Vf\|^2 \leq \sum_{j\in J} \|\Lambda_j Vf\|^2 \leq B\|Vf\|^2$$

Since $V \in B(H)$, therefore we have,

$$\|Vf\|^2 \leq \|V\|^2\, \|f\|^2 \text{ and } \|V^{-1}\|^{-2}\, \|f\|^2 \leq \|Vf\|^2$$

by using above inequalities, the equation (2.14) becomes

$$A\|V^{-1}\|^{-2}\, \|f\|^2 \leq \sum_{j\in J} \|\Lambda_j Vf\|^2 \leq B\|V\|^2\, \|f\|^2,\ \forall\ f \in H$$

$\Rightarrow \{\Lambda_j V\}_{j\in J}$ is g-frame for H.

For each $f \in H$, we have

$$S^g\, Vf = \sum_{j\in J} \Lambda^*_j \Lambda_j\, Vf$$

$$\Rightarrow V^*\, S^g\, Vf = \sum_{j\in J} V^* \Lambda^*_j \Lambda_j\, Vf$$

$\Rightarrow V^*\, S^g\, V$ is a g-frame operator for the frame $\{\Lambda_j V\}_{j\in J}$.

3. K-G-FRAMES

In this paper *L*(*H*) is the family of all linear bounded operators on *H* and $K \in L(H)$

Definition 3.1. Let $K \in L(H)$. A sequence $\{f_j\}_{j\in J}$ in Hilbert space *H* is said to be a *K*-frame for H if there exists two constants $0 < A \le B < \propto$, such that $A\|K^* f\|^2 \le \sum_{j\in J} |< f, f_j >|^2 \le B\|f\|^2, \ \forall f \in H$.

Where A and B are called lower and upper frame bounds for k-frame respectively. If K=I, then k-frames are just ordinary frames. If A=B then $\{f_j\}_{j\in J}$ is called k-tight frame, if A=B=1 then $\{f_j\}_{j\in J}$ is called normalized k-tight frame. The frame operator is given by S^k : H ® H is defined as $S^k f = \sum_{j\in J} \langle f, f_j \rangle f_j$, for all $f \in H$

Definition 3.2. Let $K \in L(H)$ and $\Lambda_j \in L(H, H_j)$ $_{j\in J}$. A sequence of operators $\{\Lambda_j\}_{j\in J}$ is said to be K-g-frame for Hilbert space H with respect to sequence of Hilbert spaces $\{H_j\}_{j\in J}$ if there exist two constants $0 < A \le B <$µ, such that

$$A\|K^* \ f\|^2 \le \sum_{j\in J} \|\Lambda_j f\|^2 \le B\|f\|^2 , \ \forall \ f \in H. \ .$$

The above inequality is called a K-g-frame inequality. The numbers A and B are called the lower and upper frame bounds of K-g-frame respectively. When K=I, K-g-frame is a g-frame.

A k-g- frame is said to be tight if there exist a constant a positive constant A such that

$$\sum_{j\in J} \|\Lambda_j f\|^2 = A\|K^* f\|^2 , \ \forall \ f \in H.$$

If A=1 then $\{\Lambda_j\}_{j\in J}$ is said to be parseval tight k-g-frame.

Definition 3.3: Let $\{\Lambda_j\}_{j\in J}$ be a K-g-frame for H. A synthesis operator $T : l^2\left(\{H_j\}_{j\in J}\right)\to H$

is defined as $T\left(\{g_j\}_{j\in J}\right)=\sum_{j\in J}\Lambda^*_j g_j \quad \forall \{g_j\}_{j\in J} \in l^2\left(\{H_j\}_{j\in J}\right)$.

Definition 3.4: Let $\{\Lambda_j\}_{j\in J}$ be a K-g-frame for *H*. The analysis operator $T^* \ H \to: l^2\left(\{H_j\}_{j\in J}\right)$ is the adjoint of synthesis operator *T* and is defined as $T^* f=\{\Lambda_j f\}_{j\in J} \ \forall f \in H$

Definition 3.5: Let $\{\Lambda_j\}_{j\in J}$ be a K-g-frame for Hilbert space H. A K- g-frame operator

$S^{kg} : H \to H$ is defined as $S^{kg} f = \sum_{j\in J}\Lambda^*_j\Lambda_j f, \qquad \forall f \in H$.

Note that $\langle S^{kg} f, f\rangle = \sum_{j\in J}\|\Lambda_j f\|^2$.

Theorem 3.6: If $K \in L(H)$ and $\{\Lambda_j\}_{j\in J}$ is a K-g-frame for Hilbert space *H* with respect to $\{H_j\}_{j\in J}$ then $S^{kg} \geq AKK^*$.

Proof. Suppose $\{\Lambda_j\}_{j\in J}$ is a K-g-frame for H

$$\Rightarrow \quad A\|K^* f\|^2 \leq \sum_{j\in J}\|\Lambda_j f\|^2 \leq B\|f\|^2, \ \forall \ f \in H.$$

$$\Rightarrow \quad A\langle K^* f, K^* f\rangle \leq \langle S^{kg} f, f\rangle \ \forall \ f \in H$$

$$\Rightarrow \quad \langle AKK^* f, f\rangle \leq \langle S^{kg} f, f\rangle \ \forall \ f \in H$$

$$\Rightarrow \quad S^{kg} \geq AKK^*.$$

Theorem 3.7. Let $\{\Lambda_j\}_{j\in J}$ be a K-g-frame for Hilbert space H with respect to $\{H_j\}_{j\in J}$. If $V : H \to H$ is any bounded linear invertible operator such that V^{-1} is commutes with K^*, then $\{\Lambda_j V\}_{j\in J}$ is k-g-frame for *H*.

Proof. Suppose $\{\Lambda_j\}_{j\in J}$ is a K-g-frame for *H*, by definition we have $A\|K^* f\|^2 \le \sum_{j\in J}\|\Lambda_j f\|^2 \le B\|f\|^2$, $\forall\ f \in H$.

If $V: H \to H$ is a bounded linear invertible operator $\forall\ f \in H$ *implies* $Vf \in H$

Therefore $$A\|K^* Vf\|^2 \le \sum_{j\in J}\|\Lambda_j Vf\|^2 \le B\|Vf\|^2$$

$$\le B\|V\|^2\|f\|^2 \ \forall\ f \in H \tag{3.8}$$

Now $$A\|K^* f\|^2 = A\|K^* V^{-1} Vf\|^2 = A\|V^{-1}K^* Vf\|^2 \le \|V^{-1}\|^2 A\|K^* Vf\|^2$$

$$\le \|V^{-1}\|^2 \sum_{j\in J}\|\Lambda_j Vf\|^2 \ \forall\ f \in H$$

$$\Rightarrow A\|V^{-1}\|^{-2}\|K^* f\|^2 \le \sum_{j\in J}\|\Lambda_j Vf\|^2 \ \forall\ f \in H \tag{3.9}$$

By using (3.8) and (3.9) we have

$$\Rightarrow A\|V^{-1}\|^{-2}\|K^* f\|^2 \le \sum_{j\in J}\|\Lambda_j Vf\|^2 \le B\|V\|^2\|f\|^2 \ \forall\ f \in H$$

$\Rightarrow \{\Lambda_j V\}_{j\in J}$ is a k-g-frame for *H*.

Theorem 3.10: If $K \in L(H)$ and $\{\Lambda_j\}_{j\in J}$ is a K-g-frame for Hilbert space H with respect to $\{H_j\}_{j\in J}$ If $V: H \to H$ is any bounded linear operator then $\{\Lambda_j V^*\}_{j\in J}$ is a Vk-g-frame for H with respect to $\{H_j\}_{j\in J}$.

Proof. Given $\{\Lambda_j\}_{j\in J}$ is a K-g-frame for Hilbert space H with respect to $\{H_j\}_{j\in J}$, we have

$$A\left\|K^{*}\ f\right\|^{2}\leq\sum_{j\in J}\left\|\Lambda_{j}f\right\|^{2}\leq B\left\|f\right\|^{2},\ \forall\ f\in H.$$

Since $V\in L(H)$ implies $V^{*}f\in H$ for any $f\in H$, then we have

$$A\left\|K^{*}V^{*}f\right\|^{2}\leq\sum_{j\in J}\left\|\Lambda_{j}V^{*}f\right\|^{2}\leq B\left\|V^{*}f\right\|^{2},\ \forall\ f\in H.$$

$$\Rightarrow A\left\|(VK)^{*}f\right\|^{2}\leq\sum_{j\in J}\left\|(\Lambda_{j}V^{*})f\right\|^{2}\leq B\left\|V\right\|^{2}\left\|f\right\|^{2},\ \forall\ f\in H.$$

$\Rightarrow\{\Lambda_{j}V^{*}\}_{j\in J}$ is a Vk-g-frame for H with respect to $\{H_{j}\}_{j\in J}$.

Theorem 3.11: Let $K_{1},K_{2}\in L(H)$ and $\{\Lambda_{j}\}_{j\in J}$ is a $K_{1}-g-frame\ and\ K_{2}-g-frame\ for\ H$ and α,β are scalars then $\{\Lambda_{j}\}_{j\in J}$ is a $(\alpha K_{1}+\beta K_{2})-g-frame\ for\ H$ and $K_{1}K_{2}-g-frame\ for\ H$.

Proof. Suppose $\{\Lambda_{j}\}_{j\in J}$ is a $K_{1}-g-frame\ for\ H$

$$\Rightarrow A_{1}\left\|K_{1}^{*}f\right\|^{2}\leq\sum_{j\in J}\left\|\Lambda_{j}f\right\|^{2}\leq B_{1}\left\|f\right\|^{2},\ \forall\ f\in H.$$

Given $\{\Lambda_{j}\}_{j\in J}$ is a $K_{1}-g-frame\ for\ H$

$$\Rightarrow A_{2}\left\|K_{2}^{*}f\right\|^{2}\leq\sum_{j\in J}\left\|\Lambda_{j}f\right\|^{2}\leq B_{2}\left\|f\right\|^{2},\ \forall\ f\in H.$$

Consider

$$\left\|(\alpha K_{1}+\beta K_{2})^{*}f\right\|^{2}=\left\|\overline{\alpha}K_{1}^{*}f+\overline{\beta}K_{2}^{*}f\right\|^{2}=2\left\|\overline{\alpha}K_{1}^{*}f\right\|^{2}+2\left\|\overline{\beta}K_{2}^{*}f\right\|^{2}-\left\|\overline{\alpha}K_{1}^{*}f-\overline{\beta}K_{2}^{*}f\right\|^{2}$$

$$\leq 2\left\|\overline{\alpha}K_{1}^{*}f\right\|^{2}+2\left\|\overline{\beta}K_{2}^{*}f\right\|^{2}=2\left|\alpha\right|^{2}\left\|K_{1}^{*}f\right\|^{2}+2\left|\beta\right|^{2}\left\|K_{2}^{*}f\right\|^{2}$$

$$=\frac{2\left|\alpha\right|^{2}}{A_{1}}A_{1}\left\|K_{1}^{*}f\right\|^{2}+\frac{2\left|\beta\right|^{2}}{A_{2}}A_{2}\left\|K_{2}^{*}f\right\|^{2}\leq\left(\frac{2\left|\alpha\right|^{2}}{A_{1}}+\frac{2\left|\beta\right|^{2}}{A_{2}}\right)\sum_{j\in J}\left\|\Lambda_{j}f\right\|^{2}$$

$$= 2\left(\frac{A_2|\alpha|^2 + A_1|\beta|^2}{A_1 A_2}\right)\sum_{j\in J}\|\Lambda_j f\|^2$$

$$\Rightarrow \frac{A_1 A_2}{2(A_2|\alpha|^2 + A_1|\beta|^2)}\|(\alpha K_1 + \beta K_2)^* f\|^2 \le \sum_{j\in J}\|\Lambda_j f\|^2$$

$$= \frac{1}{2}\sum_{j\in J}\|\Lambda_j f\|^2 + \frac{1}{2}\sum_{j\in J}\|\Lambda_j f\|^2$$

$$\le \frac{1}{2}B_1\|f\|^2 + \frac{1}{2}B_2\|f\|^2 = \left(\frac{B_1 + B_2}{2}\right)\|f\|^2$$

$$\Rightarrow \frac{A_1 A_2}{2(A_2|\alpha|^2 + A_1|\beta|^2)}\|(\alpha K_1 + \beta K_2)^* f\|^2 \le \sum_{j\in J}\|\Lambda_j f\|^2 \le \left(\frac{B_1 + B_2}{2}\right)\|f\|^2 \quad \forall\ f \in H$$

Which shows that $\{\Lambda_j\}_{j\in J}$ is a $(\alpha K_1 + \beta K_2) - g - frame\ for\ H$.

Now $\|(K_1 K_2)^* f\|^2 \le \|K_2^*\|^2\ \|K_1^* f\|^2$

$$\Rightarrow \frac{A_1}{\|K_2^*\|^2}\|(K_1 K_2)^* f\|^2 \le A_1\ \|K_1^* f\|^2 \le \sum_{j\in J}\|\Lambda_j f\|^2 \le B_1\|f\|^2$$

$$\Rightarrow \frac{A_1}{\|K_2^*\|^2}\|(K_1 K_2)^* f\|^2 \le \sum_{j\in J}\|\Lambda_j f\|^2 \le B_1\|f\|^2\ \forall\ f \in H$$

Which shows that $\{\Lambda_j\}_{j\in J}$ is a $K_1 K_2 - g - frame\ for\ H$.

4. TENSOR PRODUCT OF K- G-FRAMES

In this section the tensor product of K-g-frames in tensor product of Hilbert spaces is introduced. It was shown that the tensor product of two K- g-frames is a K-g-frame for the tensor product of Hilbert spaces.

Let H_1 and H_2 be two Hilbert spaces with inner products $<.,.>_1$, $<.,.>_2$ and norms $\|.\|_1, \|.\|_2$ respectively. The tensor product of H_1 and H_2 is denoted by $H_1 \otimes H_2$ and is an inner product space with respect to the inner product

$$\langle f_1 \otimes g_1 , f_2 \otimes g_2 \rangle = \langle f_1 , f_2 \rangle_1 \langle g_1, g_2 \rangle_2$$

for all $f_1, f_2 \in H_1$ and $g_1, g_2 \in H_2$. The norm on $H_1 \otimes H_2$ is defined by

$$\| f \otimes g \| = \| f \|_1 \|g\|_2 \quad \forall f \in H_1, g \in H_2.$$

The space $H_1 \otimes H_2$ is clearly completion with the above inner product. Therefore the space $H_1 \otimes H_2$ is a Hilbert space. We denote $L(H_1, H_2)$ be the space of all bounded linear operators from $H_1 \to H_2$. Let $M \in L(H_1)$ and $N \in L(H_2)$ be two operators, then the tensor product of operator $M \otimes N$ acts on $H_1 \otimes H_2$ as

$$(M \otimes N)(f \otimes g) = Mf \otimes Ng$$

for every $\forall f \in H_1, g \in H_2$ *and* $f \otimes g \in H_1 \otimes H_2$.

We note that if $M_1, M_2 \in L(H_1)$, $N_1, N_2 \in L(H_2)$ and $M_1 \otimes N_1, M_2 \otimes N_2 \in L(H_1 \otimes H_2)$ then $(M_1 \otimes N_1)(M_2 \otimes N_2) = M_1 M_2 \otimes N_1 N_2$.

In this paper we denote I_{H_1} is an identity operator on H_1 and I_{H_2} is an identity operator on H_2 then $I_{H_1} \otimes I_{H_2} = I_{H_1 \otimes H_2}$ is an identity operator on $H_1 \otimes H_2$.

The following is the extension of (3.2) to the sequence of operators $\{ \Lambda_i \otimes ß_j \}$.

Definition 4.1. Let $K_1 \otimes K_2 \in L(H_1 \otimes H_2)$ and $\{ \Lambda_i \}$ and $\{ß_j\}$ be the sequences of operators in Hilbert spaces H_1 and H_2 respectively. Then the sequence of operators $\{ \Lambda_i \otimes ß_j \}$ is said to be a $(K_1 \otimes K_2) - g -$ *frame* for the tensor product of Hilbert spaces $H_1 \otimes H_2$, if there exist two constants 0 < Ad” B <µ, such that

$$\mathrm{A}\left\|(K_1 \otimes K_2)^* (f \otimes g\right\|^2 \le \sum_{i,j} \left\|(\Lambda_i \otimes \beta_j)(f \otimes g)\right\|^2 \le B\left\|f \otimes g\right\|^2, \forall\, f \otimes g \in H_1 \otimes H_2$$

The numbers A and B are called lower and upper frame bounds of $(K_1 \otimes K_2) - g - frame$.

Theorem 4.2: Let $K_1 \in L(H_1)$, $K_2 \in L(H_2)$ and $\{\Lambda_i\}$, $\{ß_j\}$ be $K_1 - g - frame$ and $K_2 - g - frame$ for Hilbert spaces H_1, H_2 with respect to $\{H_{1i}\}$ and $\{H_{2j}\}$, respectively. Then $\{\Lambda_i \otimes \beta_j\}$is a $(K_1 \otimes K_2) - g - frame$ for $H_1 \otimes H_2$ with respect to $\{H_{1i} \otimes H_{2j}\}$.

Proof: Let $\{\Lambda_i\}$ be a $K_1 - g - frame$ for H_1 with frame bounds A_1 and B_1 with respect to $\{H_{1i}\}$ then, for all $f \in H_1$ and $K_1 \in L(H_1)$

(4.3) $$A_1\left\|K_1^* f\right\|^2 \le \sum_{j \in J} \left\|\Lambda_i f\right\|^2 \le B_1\left\|f\right\|^2, \ \forall\ f \in H_1.$$

Let $\{ß_j\}$ be a $K_2 - g - frame$ for H_2 with frame bounds A_2 and B_2 with respect to $\{H_{2j}\}$, then, for all $g \in H_2$ and $K_2 \in L(H_2)$

(4.4) $$A_2\left\|K_2^* g\right\|^2 \le \sum_{j \in J} \left\|\beta_j g\right\|^2 \le B_2\left\|g\right\|^2, \ \forall\ g \in H_2.$$

multiplying the equations (4.3) and (4.4), we get

$$A_1 A_2 \left\|K_1^* f\right\|^2 \left\|K_2^* g\right\|^2 \le (\sum_{i \in I} \left\|\Lambda_i f\right\|^2)(\sum_{j \in J} \left\|\beta_j g\right\|^2) \le B_1 B_2 \left\|f\right\|^2 \left\|g\right\|^2 \ \ \forall\ f \in H_1,\ g \in H_2$$

$$\Rightarrow\ A_1 A_2 \left\|K_1^* f \otimes K_2^* g\right\|^2 \le \sum_{i,j \in J} \left\|\Lambda_i f\right\|^2 \left\|\beta_j g\right\|^2 \le B_1 B_2 \left\|f \otimes g\right\|^2 \ \ \forall\ f \otimes g \in H_1 \otimes H_2$$

$$\Rightarrow\ A_1 A_2 \left\|(K_1 \otimes K_2)^* (f \otimes g)\right\|^2 \le \sum_{i,j \in J} \left\|\Lambda_i f \otimes \beta_j g\right\|^2 \le B_1 B_2 \left\|f \otimes g\right\|^2 \ \ \forall\ f \otimes g \in H_1 \otimes H_2$$

$$\Rightarrow\ A_1 A_2 \left\|(K_1 \otimes K_2)^* (f \otimes g)\right\|^2 \le \sum_{i,j \in J} \left\|(\Lambda_i \otimes \beta_j)(f \otimes g)\right\|^2 \le B_1 B_2 \left\|f \otimes g\right\|^2 \ \ \forall\ f \otimes g \in H_1 \otimes H_2$$

$\Rightarrow$ $\{\Lambda_i \otimes \beta_j\}$ is a $(K_1 \otimes K_2) - g - frame$ for $H_1 \otimes H_2$.

Theorem 4.3: If $\{\Lambda_i \otimes \beta_j\}$ is a $(K_1 \otimes K_2)-g-frame$ for $H_1 \otimes H_2$ with respect to $\{H_{1i} \otimes H_{2j}\}$. Then $\{\Lambda_i\}$ and $\{\beta_j\}$ are K-g-frames for Hilbert spaces H_1 and H_2 with respect to $\{H_{1i}\}$ and $\{H_{2j}\}$ respectively.

Proof: Suppose that $\{\Lambda_i \otimes \beta_j\}$ is a $(K_1 \otimes K_2)-g-frame$ for $H_1 \otimes H_2$ with frame bounds A and B. Then for each $f \otimes g \in H_1 \otimes H_2 - \{0 \otimes 0\}$

$$A\|(K_1 \otimes K_2)^*(f \otimes g)\|^2 \leq \sum_{i,j\in J}\|(\Lambda_i \otimes \beta_j)(f \otimes g)\|^2 \leq B\|f \otimes g\|^2 \quad \forall f \otimes g \in H_1 \otimes H_2$$

$$\Rightarrow A\|(K_1^* f \otimes K_2^* g)\|^2 \leq \sum_{i,j\in J}\|\Lambda_i f \otimes \beta_j g\|^2 \leq B\|f \otimes g\|^2 \quad \forall f \otimes g \in H_1 \otimes H_2$$

$$\Rightarrow A\|K_1^* f\|^2 \|K_2^* g\|^2 \leq (\sum_{i,\in J}\|\Lambda_i f\|^2)(\sum_{j\in J}\|\beta_j\|^2) \leq B\|f\|^2\|g\|^2 \quad \forall f \in H_1, g \in H_2$$

Consider $f \otimes g$ is a non zero vector i.e. f and g are non zero vectors, therefore the above inequality becomes

$$\Rightarrow \frac{A\|K_2^* g\|^2}{\sum_{j\in J}\|\beta_j g\|^2}\|K_1^* f\|^2 \leq \sum_{i\in J}\|\Lambda_i f\|^2 \leq \frac{B\|g\|^2}{\sum_{j\in J}\|\beta_j g\|^2}\|f\|^2 \quad \forall f \in H_1, g \in H_2$$

$$\Rightarrow A_1\|K_1^* f\|^2 \leq \sum_{i\in J}\|\Lambda_i f\|^2 \leq B_1\|f\|^2, \quad \forall f \in H_1.$$

$$\text{where } A_1 = \frac{A\|K_2^* g\|^2}{\sum_{j\in J}\|\beta_j g\|^2} \text{ and } B_1 = \frac{A\|g\|^2}{\sum_{j\in J}\|\beta_j g\|^2}$$

which shows that $\{\Lambda_i\}$ is a $K_1-g-frame$ for H_1. Similarly we can prove that $\{ß_j\}$ is a $K_2-g-frame$ for H_2.

ACKNOWLEDGEMENT

The research of the first author is partially supported by the UGC (India) [Letter No. F.20-4(1)/ 2012(BSR)].

REFERENCES

[1] P. G. Casazza, "The Art of Frame theory", *Taiwanese Journal of Mathematics*, Vol. 4, No. 2(2000), pp 129-201.

[2] Dingli Hua and Yongdong Huang, k-G-Frames and Stability of K-G-Frames in Hilbert Spaces, *J. Korean Math, Soc,* 53(2016), No.6, pp. 1331-1345.

[3] Fahimeh Arabyani Neyshaburi and Ali Akber Arefijamaal, Construction for K-frames and their Duals in Hilbert Spaces, The extended abstracts of The 4th Seminar on Functional Analysis and its Applications, 2-3rd March 2016, Ferdowsi University of Mashhad, Iran.

[4] L.Gavruta, Frames for Operators, Appl. Comput. Harmon. Anal. 32(2012),139-144.

[5] D. Han, and D.R. Larson, "Frames, Bases and Group Representations", Memories, Ams Nov 7(2000), Providence RI.

[6] A. Najati, and A. Rahimi, "Generalized frames in Hilbert spaces", *Bulletin of the Iranian Mathematical Society,* Vol. 35, No. 1(2009), pp. 97-109.

[7] W. Sun, G-Frames and G-Riesz bases, *J. Math. Annal. Appl,* 322(2006), No.1, 437-452.

[8] Y. Zhou and Y. Zhu, K-g-frames and dual g-frames for closed subspaces, Acta Math, Sinica, 56(2013), No.5, 799-806.

[9] G.U. Reddy and N.G.Reddy, Some Results on g-frame Operator in Hilbert Space, *Int. J. Contemp. Math. Sciences*, Vol.5, 2010, No.40, 1977-1984.

[10] G.U. Reddy and N.G.Reddy, "A Note on Tensor product of G-Frames", *International Journal of Computational Science and Mathematics* (ISSN 0974-3189), Volume 4, Number 1(2012), pp. 57-62.

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