Communications on Stochastic Analysis
Vol. 15, No. 3-4 (2021)

Serials Publications
www.serialspublications.com

CAUCHY PROBLEM AND INTEGRAL REPRESENTATION ASSOCIATED TO THE POWER OF THE QWN-EULER OPERATOR

AYMEN ETTAIEB, HABIB OUERDIANE, AND HAFEDH RGUIGUI

Abstract. In this paper we give the integral representation of the power of the quantum white noise (QWN) Euler operator $(\Delta_E^Q)^\rho$, for $\rho \in \mathbb{N}$, in terms of the QWN-derivatives $\{D_t^-,\ D_t^+;\ t \in \mathbb{R}\}$ as a kind of functional integral acting on nuclear algebra of white noise operators. The solution of the Cauchy problem associated to $(\Delta_E^Q)^\rho$ is worked out in the basis of the QWN coordinate system.

1. Introduction

As an infinite dimensional analogue of the Euler operator defined on $\mathbb{R}^d$ by $\sum_{k=1}^d x_k \frac{\partial}{\partial x_k}$, the operator

$$\Delta_E := \Delta_G + N = \sum_{k=1}^{\infty} \langle \cdot, e_k\rangle \partial_{e_k}, \tag{1.1}$$

was investigated in [4, 5], where Δ_G and N are the infinite dimensional Laplacians initiated by Gross [8] and Piech [25], respectively, $\{e_n;\ n \geq 0\}$ is an arbitrary orthonormal basis for $L^2(\mathbb{R})$, ∂_{e_k} denotes the holomorphic derivative in the direction e_k acting on the test function space $\mathcal{F}_\theta(\mathcal{S}'_{\mathbb{C}}(\mathbb{R}^d))$. For details see [20].

In our previous paper [2], the existence of a solution of the Cauchy problem associated with the Euler operator Δ_E in the basis of nuclear algebra of entire functions is investigated. More precisely, for two linear continuous operators K_1 and K_2 from the complexification of some nuclear space into its topological dual space, the (infinite dimensional) Euler operator is defined as follows

$$\Delta_E(K_1, K_2) = \Delta_G(K_1) + N(K_2). \tag{1.2}$$

It is shown that under some appropriate conditions, $\Delta_E(K_1, K_2)$ is the generator of a one-parameter group transformation. Furthermore, by using the $\mathcal{G}_{K_1,K_2}$-transform studied in [10, 4], the solution of the Euler Cauchy Problem was worked out.

Received 2012-2-13; Communicated by the editors. Article is based on a lecture presented at the International Conference on Stochastic Analysis and Applications, Hammamet, Tunisia, October 10-15, 2011.

2000 *Mathematics Subject Classification.* Primary 60H40; Secondary 46A32, 46F25, 46G20.

Key words and phrases. Cauchy problem, QWN-conservation operator, QWN-Euler operator, QWN-derivatives.

Using the Hermite functions one can easily show that

$$(x\frac{d}{dx})^{\rho}=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\left(\sum_{j=0}^{m}\begin{pmatrix} m \\ j \end{pmatrix}(-1)^j j^{\rho}\right)x^m(\frac{d}{dx})^m \tag{1.3}$$

and the solution of the Cauchy problem $\frac{\partial}{\partial t}u_t=(x\frac{d}{dx})^{\rho}u_t$ with $u_0\in S(\mathbb{R})$ is given by

$$u_t(x)=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\left(\sum_{j=0}^{m}\begin{pmatrix} m \\ j \end{pmatrix}(-1)^j e^{tj^{\rho}}\right)x^m(\frac{d}{dx})^m u_0(x). \tag{1.4}$$

As a generalization of (1.3), the operator $(\alpha\Delta_G+\beta N)^{\rho}$, $\alpha,\beta\in\mathbb{C}$, $\rho\in\mathbb{N}$ is studied in [6] in the space of test and generalized white noise functionals. For each positive integer ρ, the explicitly one-parameter semigroup and cosine family of operators is given on an appropriate test space of which infinitesimal generator is $(\alpha\Delta_G+\beta N)^{\rho}$. As an application, the existence and uniqueness of solutions of the Cauchy problems for the first and second order differential equations associated with the operator $(\alpha\Delta_G+\beta N)^{\rho}$ are studied.

In [3], by using the new idea of QWN-derivatives pointed out by Ji-Obata in [15, 14], the quantum analogous Δ_E^Q of (1.2) is defined as the sum $\Delta_G^Q+N^Q$, where Δ_G^Q and N^Q stand for appropriate quantum counterparts of the Laplace operators. The functional integral representations of Δ_E^Q in terms of the QWN-derivatives $\{D_t^-,\ D_t^+;\ t\in\mathbb{R}\}$ on the class of white noise operators is given by

$$\begin{aligned}\Delta_E^Q(K_1,K_2) &= \sum_{j=1}^{\infty}M^{Q+}_{\langle\cdot,K_1e_j\rangle}D^+_{e_j}+\sum_{j=1}^{\infty}M^{Q-}_{\langle\cdot,K_2e_j\rangle}D^-_{e_j}\\ &= \int_{\mathbb{R}^2}\tau_{K_1}(s,t)M^{Q+}_{\langle\cdot,\delta_s\rangle}D_t^+dsdt+\int_{\mathbb{R}^2}\tau_{K_2}(s,t)M^{Q-}_{\langle\cdot,\delta_s\rangle}D_t^-dsdt\end{aligned}$$

where, for $z\in N'$,

$$M^{Q-}_{\langle\cdot,z\rangle}=\sigma^{-1}(M_{\langle\cdot,z\rangle}\otimes I)\sigma,\quad M^{Q+}_{\langle\cdot,z\rangle}=\sigma^{-1}(I\otimes M_{\langle\cdot,z\rangle})\sigma,$$

$M_{\langle\cdot,z\rangle}$ is the classical multiplication operator by the distribution $\langle\cdot,z\rangle$ and σ is the Wick symbol defined in (2.5).

In the present paper, by using the QWN-derivatives and their adjoints, the power of the QWN-Euler operator $(\Delta_E^Q)^{\rho}$, for $\rho\in\mathbb{N}$, is studied. The first main result is the functional integral representation of $(\Delta_E^Q)^{\rho}$ in terms of the QWN-derivatives $\{D_t^-,\ D_t^+;\ t\in\mathbb{R}\}$ on the class of white noise operators. The second remarkable feature is to solve the Cauchy problem associated to $(\Delta_E^Q)^{\rho}$.

The paper is organized as follows. In Section 2, we briefly recall well-known results on nuclear algebra of entire holomorphic functions, then we recall the creation derivative and annihilation derivative as well as their adjoints. In Section 3, we give an integral representation of the power of the QWN-Euler operator. In Section 4, we solve the Cauchy problem associated to the power of the QWN-Euler operator and we give an integral representation of the solution.

2. Preliminaries

Let H be the real Hilbert space of square integrable functions on $\mathbb{R}$ with norm $|\cdot|_0$, $E \equiv \mathcal{S}(\mathbb{R})$ and $E' \equiv \mathcal{S}'(\mathbb{R})$ be the Schwartz space consisting of rapidly decreasing C^∞-functions and the space of the tempered distributions, respectively. Then the Gel'fand triple

$$E \subset H \subset E' \tag{2.1}$$

can be reconstructed in a standard way (see Ref. [20]) by the harmonic oscillator $A = 1 + t^2 - d^2/dt^2$ and H. The eigenvalues of A are $2n+2$, $n = 0, 1, 2, \cdots$, the corresponding eigenfunctions $\{e_n;\ n \geq 0\}$ form an orthonormal basis for $L^2(\mathbb{R})$ and each e_n is an element of E. In fact E is a nuclear space equipped with the Hilbertian norms

$$|\xi|_p = |A^p \xi|_0\,, \qquad \xi \in E, \quad p \in \mathbb{R}$$

and we have

$$E = \operatorname*{proj\,lim}_{p\to\infty} E_p\,, \qquad E' = \operatorname*{ind\,lim}_{p\to\infty} E_{-p},$$

where, for $p \geq 0$, E_p is the completion of E with respect to the norm $|\cdot|_p$ and E_{-p} is the topological dual space of E_p. We denote by $N = E + iE$ and $N_p = E_p + iE_p$, $p \in \mathbb{Z}$, the complexifications of E and E_p, respectively.

2.1. Spaces of holomorphic functions. Throughout the paper, we fix a Young function θ, i.e. a continuous, convex and increasing function defined on $\mathbb{R}_+$ and satisfies the two conditions: $\theta(0) = 0$ and $\lim_{x\to\infty} \theta(x)/x = +\infty$. The polar function θ^* of θ, defined by

$$\theta^*(x) = \sup_{t\geq 0}(tx - \theta(t)), \quad x \geq 0,$$

is also a Young function (see Refs. [7] and [21]). For a complex Banach space $(B, \|\cdot\|)$, let $\mathcal{H}(B)$ denotes the space of all entire functions on B, i.e. of all continuous $\mathbb{C}$-valued functions on B whose restrictions to all affine lines of B are entire on $\mathbb{C}$. For each $m > 0$ we denote by $\mathrm{Exp}(B, \theta, m)$ the space of all entire functions on B with $\theta-$exponential growth of finite type m, i.e.

$$\mathrm{Exp}(B, \theta, m) = \Big\{ f \in \mathcal{H}(B);\ \ \|f\|_{\theta,m} := \sup_{z\in B} |f(z)| e^{-\theta(m\|z\|)} < \infty \Big\}.$$

The projective system $\{\mathrm{Exp}(N_{-p}, \theta, m);\ p \in \mathbb{N},\ m > 0\}$ gives the space

$$\mathcal{F}_\theta(N') = \operatorname*{proj\,lim}_{p\to\infty; m\downarrow 0} \mathrm{Exp}(N_{-p}, \theta, m)\,. \tag{2.2}$$

It is noteworthy that, for each $\xi \in N$, the exponential function $e_\xi(z) := e^{\langle z, \xi\rangle}$, where $z \in N'$, belongs to $\mathcal{F}_\theta(N')$ and the set of such test functions spans a dense subspace of $\mathcal{F}_\theta(N')$. In the remainder of this paper we use the natation $\mathcal{F}_\theta$ to denote $\mathcal{F}_\theta(N')$. We are interested in continuous operators from $\mathcal{F}_\theta$ into its topological dual space $\mathcal{F}_\theta^*$. The space of such operators is denoted by $\mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$ and assumed to carry the bounded convergence topology. For $z \in N'$ and $\varphi(x)$ with Taylor expansions $\sum_{n=0}^\infty \langle x^{\otimes n}, f_n\rangle$ in $\mathcal{F}_\theta$, the holomorphic derivative of φ at $x \in N'$ in the direction z is defined by

$$(a(z)\varphi)(x) := \lim_{\lambda\to 0} \frac{\varphi(x + \lambda z) - \varphi(x)}{\lambda}. \tag{2.3}$$

We can check that the limit always exists, $a(z) \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta)$ and $a^*(z) \in \mathcal{L}(\mathcal{F}_\theta^*, \mathcal{F}_\theta^*)$, where $a^*(z)$ is the adjoint of $a(z)$, i.e., for $\Phi \in \mathcal{F}_\theta^*$ and $\phi \in \mathcal{F}_\theta$, $\langle\langle a^*(z)\Phi, \phi\rangle\rangle = \langle\langle \Phi, a(z)\phi\rangle\rangle$. If $z = \delta_t \in E'$ we simply write a_t instead of $a(\delta_t)$. By a straightforward computation we have

$$a_t e_\xi = \xi(t)\, e_\xi\,, \quad \xi \in N. \tag{2.4}$$

Similarly as above, for $\psi \in \mathcal{G}_{\theta^*}(N)$ with Taylor expansion $\psi(\xi) = \sum_n \langle \psi_n, \xi^{\otimes n}\rangle$ where $\psi_n \in N'^{\otimes n}$, we use the common notation $a(z)\psi$ for the derivative (2.3).

The Wick symbol of $\Xi \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$ is by definition [20] a $\mathbb{C}$-valued function on $N \times N$ defined by

$$\sigma(\Xi)(\xi, \eta) = \langle\langle \Xi e_\xi, e_\eta\rangle\rangle e^{-\langle \xi, \eta\rangle}, \quad \xi, \eta \in N. \tag{2.5}$$

By a density argument, every operator in $\mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$ is uniquely determined by its Wick symbol.

Let $\mathcal{H}_\theta(N \oplus N)$ denotes the restriction of the space $\mathcal{F}_\theta(N' \oplus N')$ over N, i.e.,

$$\mathcal{H}_\theta(N \oplus N) = \bigcap_{p \geq 0, \gamma_1, \gamma_2 > 0} \mathrm{Exp}(N_p \times N_p, \theta, \gamma_1, \gamma_2),$$

where $\mathrm{Exp}(N_p \oplus N_p, \theta, \gamma_1, \gamma_2)$ denotes the space of all entire functions on $N_p \times N_p$ such that

$$\sup_{(x_1, x_2) \in (N_p \times N_p)} |g(x_1, x_2)| e^{-\theta(\gamma_1 |x_1|_p) - \theta(\gamma_2 |x_2|_p)} < \infty.$$

In other words, all holomorphic functions g in $\mathcal{H}_\theta(N \oplus N)$ admit the Taylor expansions $g(x_1, x_2) = \sum_{l,m} \langle g_{l,m}, x_2^{\otimes l} \otimes x_1^{\otimes m}\rangle$ for $x_1, x_2 \in N$, where $g_{l,m} \in (N^{\otimes l} \otimes N^{\otimes m})_{sym(l,m)}$ such that for all $p \in \mathbb{N}$ and $\gamma_1,\ \gamma_2 > 0$

$$\|\overrightarrow{\sigma(\Xi)}\|^2_{\theta, p, (\gamma_1, \gamma_2)} := \sum_{l,m=0}^{\infty} (\theta_l \theta_m)^{-2} \gamma_1^{-l} \gamma_2^{-m} |g_{l,m}|_p^2 < \infty,$$

where $\theta_n = \inf_{r>0} \frac{e^{\theta(r)}}{r^n}$, for $n \in \mathbb{N}$. Then using the kernel theorem and the reflexivity of the space $\mathcal{F}_\theta$, we obtain the following characterization Theorem.

Theorem 2.1. [3] *The Wick symbol map realizes a topological isomorphism between the space $\mathcal{L}(\mathcal{F}_\theta^*, \mathcal{F}_\theta)$ and the space $\mathcal{H}_\theta(N \oplus N)$.*

2.2. `QWN`-Derivatives. It is a fundamental fact in `QWN` theory [20] (see, also Ref. [16]) that every white noise operator $\Xi \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$ admits a unique Fock expansion

$$\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}), \tag{2.6}$$

where, for each pairing $l, m \geq 0$, $\kappa_{l,m} \in (N^{\otimes(l+m)})'_{sym(l,m)}$ and $\Xi_{l,m}(\kappa_{l,m})$ is the integral kernel operator characterized via the Wick symbol transform by

$$\sigma(\Xi_{l,m}(\kappa_{l,m}))(\xi, \eta) = \langle \kappa_{l,m}, \eta^{\otimes l} \otimes \xi^{\otimes m}\rangle, \quad \xi, \eta \in N. \tag{2.7}$$

It is noteworthy that $\{\Xi^{a,b};\ a,\ b \in N\}$ spans a dense subspace of $\mathcal{L}(\mathcal{F}_\theta^*, \mathcal{F}_\theta)$, where

$$\Xi^{a,b} \equiv \sum_{l,m=0}^{\infty} \Xi_{l,m}\Big(\frac{1}{l!m!} a^{\otimes l} \otimes b^{\otimes m}\Big) \in \mathcal{L}(\mathcal{F}_\theta^*, \mathcal{F}_\theta).$$

From Refs. [13] and [14], (see also Refs. [15] and [1]), we summarize the novel formalism of QWN-derivatives. For $\zeta \in N$, then $a(\zeta)$ extends to a continuous linear operator from $\mathcal{F}_\theta^*$ into itself (denoted by the same symbol) and $a^*(\zeta)$ (restricted to $\mathcal{F}_\theta$) is a continuous linear operator from $\mathcal{F}_\theta$ into itself. Thus for any white noise operator $\Xi \in \mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$, the commutators

$$[a(\zeta), \Xi] = a(\zeta)\Xi - \Xi\, a(\zeta), \quad [a^*(\zeta), \Xi] = a^*(\zeta)\Xi - \Xi\, a^*(\zeta),$$

are well defined white noise operators in $\mathcal{L}(\mathcal{F}_\theta, \mathcal{F}_\theta^*)$. The *QWN-derivatives* are defined by

$$D_\zeta^+ \Xi = [a(\zeta), \Xi], \quad D_\zeta^- \Xi = -[a^*(\zeta), \Xi]. \tag{2.8}$$

These are called the *creation derivative* and *annihilation derivative* of Ξ, respectively.

The QWN-derivatives $D_z^\pm$ are natural QWN counterparts of the holomorphic partial derivatives $\partial_1 \equiv \frac{\partial}{\partial x_1}$ and $\partial_2 \equiv \frac{\partial}{\partial x_2}$ on the space of entire functions with two variables $\mathcal{H}_\theta(N \oplus N)$, for more details see [3].

Proposition 2.2. [3] *Let be given $z \in N$. The creation derivative and annihilation derivative of $\Xi \in \mathcal{L}(\mathcal{F}_\theta^*, \mathcal{F}_\theta)$ are given by*

$$D_z^- \Xi = \sigma^{-1} \partial_{1,z} \sigma(\Xi) \quad \text{and} \quad D_z^+ \Xi = \sigma^{-1} \partial_{2,z} \sigma(\Xi).$$

Moreover, their dual adjoints are given by

$$(D_z^-)^* \Xi = \sigma^{-1} \partial_{1,z}^* \sigma(\Xi) \quad \text{and} \quad (D_z^+)^* \Xi = \sigma^{-1} \partial_{2,z}^* \sigma(\Xi).$$

3. Power of the QWN-Euler Operator

Recall from [3] that the QWN-Euler operator $\Delta_E^Q(K_1, K_2) \in \mathcal{L}(\mathcal{L}(\mathcal{F}_\theta^*, \mathcal{F}_\theta))$ is defined by

$$\Delta_E^Q(K_1, K_2) = \Delta_G^Q(K_1, K_2) + N_{K_1,K_2}^Q, \quad K_1, K_2 \in \mathcal{L}(N', N'),$$

where $\Delta_G^Q(K_1, K_2)$ and N_{K_1,K_2}^Q stand for the QWN-$(K_1, K_2)-$Gross Laplacian and the QWN-conservation operator, respectively, given by

$$\Delta_G^Q(K_1, K_2) = \sum_{j=1}^{\infty} D_{e_j}^+ D_{K_1^* e_j}^+ + \sum_{j=1}^{\infty} D_{e_j}^- D_{K_2^* e_j}^-,$$

$$N_{K_1,K_2}^Q = \sum_{j=1}^{\infty} (D_{e_j}^+)^* (D_{K_1^* e_j})^+ + \sum_{j=1}^{\infty} (D_{e_j}^-)^* D_{K_2^* e_j}^-.$$

Throughout, for α_1 and α_2 non-zero complex numbers, we denote

$$\Delta_E^Q(\alpha_1, \alpha_2) = \Delta_E^Q(\alpha_1 I, \alpha_2 I), \quad N_{\alpha_1,\alpha_2}^Q = N_{\alpha_1 I, \alpha_2 I}^Q.$$

Lemma 3.1. *For any $\Xi \in \mathcal{L}(\mathcal{F}_\theta^*, \mathcal{F}_\theta)$ with $\Xi = \sum_{l,m=0}^\infty \Xi_{l,m}(\kappa_{l,m})$, we have*

$$N_{\alpha_1,\alpha_2}^Q \Xi = \sum_{l,m=0}^{\infty} (\alpha_1 l + \alpha_2 m) \Xi_{l,m}(\kappa_{l,m}). \tag{3.1}$$

Proof. For $z \in N$, by direct computation, the partial derivatives of the identity (2.7) in the direction z are given by

$$\partial_{1,z}\sigma(\Xi_{l,m}(\kappa_{l,m}))(\xi,\eta) \quad = \quad \sigma(m\Xi_{l,m-1}(\kappa_{l,m}\otimes_1 z))(\xi,\eta)$$

and

$$\partial_{2,z}\sigma(\Xi_{l,m}(\kappa_{l,m}))(\xi,\eta) \quad = \quad \sigma(l\Xi_{l-1,m}(z\otimes^1 \kappa_{l,m}))(\xi,\eta),$$

where, for $z_p \in (N^{\otimes p})'$, and $\xi_{l+m-p} \in N^{\otimes(l+m-p)}$, $p \leq l+m$, the contractions $z_p \otimes_p \kappa_{l,m}$ and $\kappa_{l,m} \otimes^p z_p$ are defined by

$$\langle z_p \otimes^p \kappa_{l,m}, \xi_{l-p+m}\rangle = \langle \kappa_{l,m}, z_p \otimes \xi_{l-p+m}\rangle$$

$$\langle \kappa_{l,m} \otimes_p z_p, \xi_{l+m-p}\rangle = \langle \kappa_{l,m}, \xi_{l+m-p} \otimes z_p\rangle.$$

Similarly, $\partial_{1,z}^*$ and $\partial_{2,z}^*$, the adjoint operators of $\partial_{1,z}$ and $\partial_{2,z}$ respectively, are given by

$$\partial_{1,z}^*\sigma(\Xi_{l,m}(\kappa_{l,m}))(\xi,\eta) = \sigma(\Xi_{l,m+1}(\kappa_{l,m}\otimes z))(\xi,\eta) \tag{3.2}$$

$$\partial_{2,z}^*\sigma(\Xi_{l,m}(\kappa_{l,m}))(\xi,\eta) = \sigma(\Xi_{l+1,m}(z\otimes \kappa_{l,m}))(\xi,\eta). \tag{3.3}$$

Then using Proposition 2.2, we get

$$\sigma(N^Q_{\alpha_1,\alpha_2}\Xi^{a,b})(\xi,\eta) = (\alpha_1\langle a,a\rangle + \alpha_2\langle b,b\rangle)\sigma(\Xi^{a,b})(\xi,\eta).$$

On the other hand, we denote the right hand side of (3.1) by A^Q, then we get

$$\begin{aligned}\sigma(A^Q\Xi^{a,b})(\xi,\eta) \quad &= \quad \sum_{l,m}\alpha_1 l\frac{\langle a,\eta\rangle^l}{l!}\frac{\langle a,\xi\rangle^m}{m!} + \sum_{l,m}\alpha_2 m\frac{\langle a,\eta\rangle^l}{l!}\frac{\langle b,\xi\rangle^m}{m!}\\ &= \quad (\alpha_1\langle a,\eta\rangle + \alpha_2\langle b,\xi\rangle)\sigma(\Xi^{a,b})(\xi,\eta).\end{aligned}$$

Hence by a density argument we complete the proof. □

Motivated by Lemma 3.1, we get

$$(N^Q_{\alpha_1,\alpha_2})^\rho\Xi = \sum_{l,m=0}^{\infty}(\alpha_1 l + \alpha_2 m)^\rho \Xi_{l,m}(\kappa_{l,m}) \tag{3.4}$$

for all $\Xi = \sum_{l,m=0}^{\infty}\Xi_{l,m}(\kappa_{l,m})$ and $\rho \in \mathbb{N}$. We observe that $(N^Q_{\alpha_1,\alpha_2})^\rho$ is a linear continuous operator from $\mathcal{L}(\mathcal{F}_\theta^*, \mathcal{F}_\theta)$ into itself.

Recall that, (see [11] and [3]), the QWN-Fourier-Gauss transform $G^Q_{K_1,K_2;B_1,B_2}$ is defined by

$$G^Q_{K_1,K_2;B_1,B_2}\Xi = \sum_{l,m}^{\infty}\Xi_{l,m}(g_{l,m}) \tag{3.5}$$

where $g_{l,m}$ is given by

$$g_{l,m} = \sum_{j,k=0}^{\infty}\frac{(l+2k)!(m+2j)!}{l!m!k!j!}\left(B_1^{\otimes l}\otimes B_2^{\otimes m}\right)\left(\tau_{K_1}^{\otimes k}\otimes^{2k}\kappa_{l+2k,m+2j}\otimes_{2j}\tau_{K_2}^{\otimes j}\right).$$

Note that $G^Q_{K_1,K_2;B_1,B_2}$ is a continuous linear operator from $\mathcal{L}(\mathcal{F}_\theta^*, \mathcal{F}_\theta)$ into itself. In the following we use the notation $\mathcal{G}^Q$ for

$$\mathcal{G}^Q := G^Q_{-\frac{1}{2}I,-\frac{1}{2}I;-iI,-iI}.$$

Motivated by the classical case (see [19]), we can show that $\mathcal{G}^Q$ is a topological isomorphism from $\mathcal{L}(\mathcal{F}_\theta^*, \mathcal{F}_\theta)$ into itself. Moreover,

$$(\mathcal{G}^Q)^{-1}\Xi = G^Q_{-\frac{1}{2}I,-\frac{1}{2}I;iI,iI}\Xi, \qquad \forall \Xi \in \mathcal{L}(\mathcal{F}_\theta^*, \mathcal{F}_\theta). \tag{3.6}$$

Theorem 3.2. *For any* $\Xi \in \mathcal{L}(\mathcal{F}_\theta^*, \mathcal{F}_\theta)$*, we have*

$$(\Delta_E^Q(\alpha_1,\alpha_2))^\rho \Xi = (\mathcal{G}^Q)^{-1} \circ (N^Q_{\alpha_1,\alpha_2})^\rho \circ \mathcal{G}^Q \Xi.$$

Proof. It suffices to prove the fact:

$$\Delta_E^Q(\alpha_1,\alpha_2) = (\mathcal{G}^Q)^{-1} \circ (N^Q_{\alpha_1,\alpha_2}) \circ \mathcal{G}^Q. \tag{3.7}$$

To prove (3.7) we need to prove the following identities

$$\mathcal{G}^Q(\Delta_G^Q(\alpha_1,\alpha_2)\Xi) = -\Delta_G^Q(\alpha_1,\alpha_2)(\mathcal{G}^Q\Xi), \tag{3.8}$$

$$\mathcal{G}^Q(N^Q_{\alpha_1,\alpha_2}\Xi) = (\Delta_G^Q(\alpha_1,\alpha_2) + N^Q_{\alpha_1,\alpha_2})\mathcal{G}^Q\Xi. \tag{3.9}$$

Let us start by the proof of (3.8). Let a, $b \in N$, then we have

$$\begin{aligned}
&\sigma(\mathcal{G}^Q(\Delta_G^Q(\alpha_1,\alpha_2)\Xi^{a,b}))(\xi,\eta)\\
&= (\alpha_1\langle a,a\rangle + \alpha_2\langle b,b\rangle)\exp\{-\frac{1}{2}\langle a,a\rangle - \frac{1}{2}\langle b,b\rangle - i\langle a,\eta\rangle - i\langle b,\xi\rangle\}\\
&= (\alpha_1\langle a,a\rangle + \alpha_2\langle b,b\rangle)e^{-\frac{1}{2}\langle a,a\rangle - \frac{1}{2}\langle b,b\rangle}\sigma(\Xi^{-ia,-ib})(\xi,\eta).
\end{aligned}$$

So that, we get

$$\mathcal{G}^Q(\Delta_G^Q(\alpha_1,\alpha_2)\Xi^{a,b}) = (\alpha_1\langle a,a\rangle + \alpha_2\langle b,b\rangle)e^{-\frac{1}{2}\langle a,a\rangle - \frac{1}{2}\langle b,b\rangle}\Xi^{-ia,-ib}.$$

On the other hand,

$$\begin{aligned}
&\sigma(-\Delta_G^Q(\alpha_1,\alpha_2)(\mathcal{G}^Q\Xi^{a,b}))(\xi,\eta)\\
&= -\sigma(e^{-\frac{1}{2}\langle a,a\rangle - \frac{1}{2}\langle b,b\rangle}(\Delta_G^Q(\alpha_1,\alpha_2)\Xi^{-ia,-ib}))(\xi,\eta)\\
&= -(\alpha_1\langle -ia,-ia\rangle + \alpha_2\langle -ib,-ib\rangle)e^{-\frac{1}{2}\langle a,a\rangle - \frac{1}{2}\langle b,b\rangle}\sigma(\Xi^{-ia,-ib})(\xi,\eta)
\end{aligned}$$

which is equivalent to

$$-\Delta_G^Q(\alpha_1,\alpha_2)(\mathcal{G}^Q\Xi^{a,b}) = (\alpha_1\langle a,a\rangle + \alpha_2\langle b,b\rangle)e^{-\frac{1}{2}\langle a,a\rangle - \frac{1}{2}\langle b,b\rangle}\Xi^{-ia,-ib}. \tag{3.10}$$

Hence by density argument we complete the proof of (3.8). To prove (3.9), let a, $b \in N$. Then by Lemma 3.1 we get

$$\begin{aligned}
\sigma(\mathcal{G}^Q(N^Q_{\alpha_1,\alpha_2}\Xi^{a,b}))(\xi,\eta) &= \sum_{l,m,j,k=0}^{\infty} \frac{(\alpha_1(l+2k)+\alpha_2(m+2j))}{j!k!}(-i)^l(-i)^m\\
&\quad \times(-\frac{1}{2}\langle a,a\rangle)^k(-\frac{1}{2}\langle b,b\rangle)^j\langle \frac{a^{\otimes l}}{l!}\otimes\frac{b^{\otimes m}}{m!}, \eta^{\otimes l}\otimes\xi^{\otimes m}\rangle\\
&= \{-i\alpha_1\langle a,\eta\rangle - i\alpha_2\langle b,\xi\rangle - \alpha_1\langle a,a\rangle - \alpha_2\langle b,b\rangle\}\\
&\quad \times e^{-\frac{1}{2}\langle a,a\rangle - \frac{1}{2}\langle b,b\rangle}\sigma(\Xi^{-ia,-ib})(\xi,\eta).
\end{aligned} \tag{3.11}$$

On the other hand,

$$\begin{aligned}
&\sigma(N^Q_{\alpha_1,\alpha_2}(\mathcal{G}^Q\Xi^{a,b}))(\xi,\eta)\\
&= \{-i\alpha_1\langle a,\eta\rangle - i\alpha_2\langle b,\xi\rangle\}e^{-\frac{1}{2}\langle a,a\rangle - \frac{1}{2}\langle b,b\rangle}\sigma(\Xi^{-ia,-ib})(\xi,\eta).
\end{aligned}$$

Therefore, using (3.10), we obtain

$$\begin{aligned}&\sigma((N^Q_{\alpha_1,\alpha_2}+\Delta^Q_G(K_1,K_2))\mathcal{G}^Q\Xi^{a,b})(\xi,\eta)\\&\quad=\{-i\alpha_1\langle a,\eta\rangle-i\alpha_2\langle b,\xi\rangle-\alpha_1\langle a,a\rangle-\alpha_2\langle b,b\rangle\}\\&\qquad\times e^{-\frac{1}{2}\langle a,a\rangle-\frac{1}{2}\langle b,b\rangle}\sigma(\Xi^{-ia,-ib})(\xi,\eta).\end{aligned}$$

Hence by a density argument we complete the proof. □

Theorem 3.3. *The power of the* QWN*-Euler operator admits on* $\mathcal{L}(\mathcal{F}^*_\theta,\mathcal{F}_\theta)$ *the following representation*

$$\begin{aligned}&(\Delta^Q_E(\alpha_1,\alpha_2))^\rho\\&=\sum_{j,k,l,m=0}^{\infty}\frac{(-1)^{l+m+j+k}}{j!k!l!m!2^{l+m}}\Big(\sum_{0\le r\le l,0\le s\le m}\ \sum_{0\le i\le j,0\le n\le k}\binom{l}{r}\binom{m}{s} \qquad (3.12)\\&\binom{j}{i}\binom{k}{n}(-1)^{r+s+i+n}(2\alpha_1 r+2\alpha_2 s+\alpha_1 i+\alpha_2 n)^\rho\Big)\\&\int_{\mathbb{R}^{2(l+j+k+m)}}\tau(u_1,u_2)\cdots\tau(u_{2l-1},u_{2l})\tau(v_1,v_2)\cdots\tau(v_{2m-1},v_{2m})\\&\tau(s_1,u_{2l+1})\cdots\tau(s_j,u_{2l+j})\tau(t_1,v_{2m+1})\cdots\tau(t_k,v_{2m+k})\\&(D^+_{s_1})^*\cdots(D^+_{s_j})^*(D^-_{t_1})^*\cdots(D^-_{t_k})^*D^+_{u_1}\cdots D^+_{u_{2l+j}}D^-_{v_1}\cdots D^-_{v_{2m+k}}\\&ds_1\cdots ds_j dt_1\cdots dt_k du_1\cdots du_{2l+j}dv_1\cdots dv_{2m+k}. \qquad (3.13)\end{aligned}$$

Type your text

Proof. Using (3.4) and (3.5), we can show that

$$\begin{aligned}(N^Q_{\alpha_1,\alpha_2})^\rho\mathcal{G}\Xi^{a,b}&=\sum_{j,k,l,m=0}^{\infty}\frac{(l+2k)!(m+2j)!}{l!m!j!k!2^{k+j}}(-1)^{j+k}(-i)^{l+m}\\&\quad\times(\alpha_1 l+\alpha_2 m)^\rho\Xi_{l,m}(\tau^{\otimes k}\otimes^{2k}\kappa_{l+2k,m+2j}\otimes_{2j}\tau^{\otimes j}).\end{aligned}$$

Then by (3.6) and (3.5), we get

$$\begin{aligned}&(\mathcal{G}^Q)^{-1}(N^Q_{\alpha_1,\alpha_2})^\rho\mathcal{G}^Q\Xi\\&=\sum_{i,j,k,l,m,n=0}^{\infty}\frac{(l+2k+2i)!(m+2j)!}{j!k!l!m!i!n!}\\&\quad\times\frac{(-1)^{j+k}}{2^{j+k+l+m}}\{\alpha_1(l+2i)+\alpha_2(m+2n)\}^\rho\\&\quad\times\Xi_{l,m}\Big(\tau^{\otimes(i+k)}\otimes^{2(i+k)}\kappa_{l+2i+2k,m+2j+2n}\otimes_{2(j+n)}\tau^{\otimes 2(j+n)}\Big).\end{aligned}$$

By a change of variables $i+k=r$ and $j+n=s$, we obtain

$$\begin{aligned}(\Delta^Q_E(\alpha_1,\alpha_2))^\rho\Xi&=\sum_{r,s,l,m=0}^{\infty}\frac{(l+2r)!(m+2s)!(-1)^{r+s}}{l!m!r!s!2^{r+s}}\\&\quad\times\Big(\sum_{0\le i\le r,0\le n\le s}\binom{r}{i}\binom{s}{n}\{\alpha_1(l+2i)+\alpha_2(m+2n)\}^\rho(-1)^{i+n}\Big)\\&\quad\times\Xi_{l,m}(\tau^{\otimes r}\otimes^{2r}\kappa_{l+2r,m+2s}\otimes_{2s}\tau^{\otimes s}).\end{aligned}$$

Then for a,b $\in N$, the Wick symbol of $(\Delta_E^Q(\alpha_1,\alpha_2))^\rho \Xi^{a,b}$ is given by

$$\begin{aligned}&\sigma((\Delta_E^Q(\alpha_1,\alpha_2))^\rho \Xi^{a,b})(\xi,\eta)\\ &= \sum_{r,s,l,m=0}^{\infty} \frac{(-1)^{r+s}\langle a,a\rangle^r\langle b,b\rangle^s\langle a,\eta\rangle^l\langle b,\xi\rangle^m}{l!m!r!s!2^{r+s}}\\ &\times\Big(\sum_{0\le i\le r, 0\le n\le s}\tbinom{r}{i}\tbinom{s}{n}\{\alpha_1 l+2\alpha_1 i+\alpha_2 m+2\alpha_2 n\}^\rho(-1)^{i+n}\Big).\end{aligned}$$

Therefore, using the fact that

$$z^\rho = \frac{d^\rho}{dt^\rho}|_{t=0} e^{tz} \quad for \quad z\in\mathbb{C}, \tag{3.14}$$

we get the following equality

$$\sigma((\Delta_E^Q(\alpha_1,\alpha_2))^\rho \Xi^{a,b})(\xi,\eta) =$$

$$\frac{d^\rho}{dt^\rho}|_{t=0}\exp\{\frac{1}{2}(e^{2\alpha_1 t}-1)\langle a,a\rangle+\frac{1}{2}(e^{2\alpha_2 t}-1)\langle b,b\rangle+e^{\alpha_1 t}\langle a,\eta\rangle+e^{\alpha_2 t}\langle b,\xi\rangle\}.$$

On the other hand, denoting the right hand side of (3.13) by I^Q, then we have

$$\begin{aligned}\sigma(I^Q\Xi^{a,b})(\xi,\eta) &= \sum_{j,k,l,m=0}^{\infty}\frac{(-1)^{j+k+l+m}}{j!k!l!m!2^{l+m}}\\ &\quad\times\sum_{0\le r\le l,0\le s\le n}\ \sum_{0\le i\le j,0\le n\le k}\begin{pmatrix} l\\ r\end{pmatrix}\begin{pmatrix} m\\ s\end{pmatrix}\begin{pmatrix} j\\ i\end{pmatrix}\begin{pmatrix} k\\ n\end{pmatrix}\\ &\quad(-1)^{r+s+i+n}(2\alpha_1 r+2\alpha_2 s+\alpha_1 i+\alpha_2 n)^\rho\\ &\quad\times\langle a,a\rangle^l\langle b,b\rangle^m\langle a,\eta\rangle^j\langle b,\xi\rangle^k\sigma(\Xi^{a,b})(\xi,\eta).\end{aligned}$$

Therefore, using (3.14), we get

$$\begin{aligned}\sigma(I^Q\Xi^{a,b})(\xi,\eta) &= \frac{d^\rho}{dt^\rho}|_{t=0}\exp\{\frac{1}{2}(e^{2\alpha_1 t}-1)\langle a,a\rangle+\frac{1}{2}(e^{2\alpha_2 t}-1)\langle b,b\rangle\\ &+ (e^{\alpha_1 t}-1)\langle a,\eta\rangle+(e^{\alpha_2 t}-1)\langle b,\xi\rangle\}\sigma(\Xi^{a,b})(\xi,\eta).\end{aligned}$$

Hence from the fact that

$$\sigma(\Xi^{a,b})(\xi,\eta)=\exp\{\langle a,\eta\rangle+\langle b,\xi\rangle\}$$

we get the desired statement by a density argument. □

Remark 3.4. For $\rho = 1$ we find the integral representation of Δ_E^Q appeared in [3]. The representation (3.13) is the QWN analogue of the following classical integral representation on the nuclear space $\mathcal{F}_\theta$

$$\begin{aligned}(\Delta_E)^\rho &= \sum_{l,m=0}^{\infty}\frac{(-1)^{l+m}}{l!m!2^l}\Big(\sum_{0\le j\le l,0\le i\le m}\begin{pmatrix} l\\ j\end{pmatrix}\begin{pmatrix} m\\ i\end{pmatrix}(-1)^{j+i}(2j+i)^\rho\Big)\\ &\quad\int_{\mathbb{R}^{2(l+m)}}\tau(u_1,u_2)\cdots\tau(u_{2l-1},u_{2l})\tau(s_1,u_{2l+1})\cdots\tau(s_m,u_{2l+m})\\ &\quad a_{s_1}^*\cdots a_{s_m}^* a_{u_1}\cdots a_{u_{2l+m}}ds_1\cdots ds_m du_1\cdots du_{2l+m}.\end{aligned}$$

Which gives for $\rho = 1$

$$\Delta_E = \int_{\mathbb{R}^2} \tau(u_1, u_2) a_{u_1} a_{u_2} du_1 du_2 + \int_{\mathbb{R}^2} \tau(s_1, u_1) a^*_{s_1} a_{u_1} ds_1 du_1 = \Delta_G + N.$$

4. Cauchy Problem Associated to the Power of the QWN-Euler Operator

Motivated by the classical Cauchy problem associated to the power of the Euler operator studied in [6], we fix two non-zero complex numbers α_1 and α_2 such that $\Re(\alpha_1^i \alpha_2^{\rho-i}) \leq 0$ for all $1 \leq i \leq \rho$. For $\Xi \in \mathcal{L}(\mathcal{F}^*_\theta, \mathcal{F}_\theta)$ with $\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m})$, let

$$X_{\alpha_1,\alpha_2,\rho;t}\Xi = G^Q_{0,0;e^{\alpha_1 t}I, e^{\alpha_2 t}I}\Xi = \sum_{l,m=0}^{\infty} e^{t(\alpha_1 l + \alpha_2 m)^\rho} \Xi_{l,m}(\kappa_{l,m}), \quad t \geq 0. \tag{4.1}$$

The Wick symbol of $X_{\alpha_1,\alpha_2,\rho;t}\Xi$ is given by

$$\sigma(X_{\alpha_1,\alpha_2,\rho;t}\Xi)(\xi, \eta) = \sum_{l,m=0}^{\infty} e^{t(\alpha_1 l + \alpha_2 m)^\rho} \langle \kappa_{l,m}, \eta^{\otimes l} \otimes \xi^{\otimes m} \rangle.$$

Lemma 4.1. *The following properties hold true:*

(1) $X_{\alpha_1,\alpha_2,\rho;t} \in \mathcal{L}(\mathcal{F}^*_\theta, \mathcal{F}_\theta)$ *for any* $t \geq 0$.
(2) $X_{\alpha_1,\alpha_2,\rho;t} = e^{t(N^Q_{\alpha_1,\alpha_2})^\rho}$ *for any* $t \geq 0$.
(3) $\{X_{\alpha_1,\alpha_2,\rho;t}\}_{t\geq 0}$ *is a differentiable one-parameter semigroup of operators on* $\mathcal{L}(\mathcal{F}^*_\theta, \mathcal{F}_\theta)$ *with infinitesimal generator* $(N^Q_{\alpha_1,\alpha_2})^\rho$.

Proof. 1. For $\Xi \in \mathcal{L}(\mathcal{F}^*_\theta, \mathcal{F}_\theta)$ with $\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m})$, any $\gamma_1, \gamma_2 > 0$ and any $q \geq 0$, we have

$$\begin{aligned}
\|\overrightarrow{\sigma(X_{\alpha_1,\alpha_2,\rho;t}\Xi)}\|^2_{\theta;q;(\gamma_1,\gamma_2)} &:= \sum_{l,m=0}^{\infty} (\theta_l \theta_m)^{-2} \gamma_1^{-l} \gamma_2^{-m} |e^{t(\alpha_1 l + \alpha_2 m)^\rho}|^2 |\kappa_{l,m}|^2_q \\
&= \sum_{l,m=0}^{\infty} (\theta_l \theta_m)^{-2} \gamma_1^{-l} \gamma_2^{-m} e^{2Re(t(\alpha_1 l + \alpha_2 m)^\rho)} |\kappa_{l,m}|^2_q \\
&\leq \|\overrightarrow{\sigma(\Xi)}\|^2_{\theta;q;(\gamma_1,\gamma_2)}, \quad t \geq 0.
\end{aligned}$$

Hence $X_{\alpha_1,\alpha_2,\rho;t} \in \mathcal{L}(\mathcal{L}(\mathcal{F}^*_\theta, \mathcal{F}_\theta))$.
2. For $\Xi \in \mathcal{L}(\mathcal{F}^*_\theta, \mathcal{F}_\theta)$ with $\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m})$, we have

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} (N^Q_{\alpha_1,\alpha_2})^{\rho n} \Xi = \sum_{l,m=0}^{\infty} e^{t(\alpha_1 l + \alpha_2 m)^\rho} \Xi_{l,m}(\kappa_{l,m}).$$

3. Obviously $\{e^{t(N^Q_{\alpha_1,\alpha_2})^\rho}\}_{t\geq 0}$ is an one-parameter semigroup of operators on $\mathcal{L}(\mathcal{F}^*_\theta, \mathcal{F}_\theta)$. By direct computation, we can easily show that

$$\lim_{t\to 0^+} \sup_{\|\overrightarrow{\sigma(\Xi)}\|, t\leq 1} \left\| \overrightarrow{\frac{\sigma(e^{t(N^Q_{\alpha_1,\alpha_2})^\rho}\Xi) - \sigma(\Xi)}{t} - \sigma((N^Q_{\alpha_1,\alpha_2})^\rho \Xi)} \right\|^2_{\theta;q;(\gamma_1,\gamma_2)} = 0.$$

Thus $\{X_{\alpha_1,\alpha_2,\rho;t}\}_{t\geq 0}$ is a differentiable semigroup with infinitesimal generator $(N^Q_{\alpha_1,\alpha_2})^\rho$. $\square$

Let α_1 and α_2 be two non-zero complex numbers with $\Re(\alpha_1^i\alpha_2^{\rho-i}) \leq 0$ for all $1 \leq i \leq \rho$. For $t \geq 0$, let $Y_{\alpha_1,\alpha_2,\rho;t}$ be given by

$$Y_{\alpha_1,\alpha_2,\rho;t} = (\mathcal{G}^Q)^{-1} \circ X_{\alpha_1,\alpha_2,\rho;t} \circ \mathcal{G}^Q.$$

It is obvious that $Y_{\alpha_1,\alpha_2,\rho;t} \in \mathcal{L}(\mathcal{L}(\mathcal{F}_\theta^*, \mathcal{F}_\theta))$.

Theorem 4.2. *Let α_1 and α_2 be two non-zero complex numbers with $\Re(\alpha_1^i\alpha_2^{\rho-i}) \leq 0$ for all $1 \leq i \leq \rho$. For $\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}) \in \mathcal{L}(\mathcal{F}_\theta^*, \mathcal{F}_\theta)$, the following Cauchy problem*

$$\frac{\partial}{\partial t}U_t = (\Delta_E^Q(\alpha_1,\alpha_2))^\rho U_t, \quad U_0 = \Xi \tag{4.2}$$

has a unique solution in $\mathcal{L}(\mathcal{F}_\theta^*, \mathcal{F}_\theta)$ given by

$$\begin{aligned} U_t &= Y_{\alpha_1,\alpha_2,\rho;t}\Xi \\ &= \sum_{j,k,l,m=0}^{\infty} \frac{(l+2k)!(m+2j)!}{j!k!l!m!2^{j+k}}(-1)^{j+k+l+m} \\ &\quad \times \left(\sum_{0\leq r\leq k, 0\leq s\leq j} \binom{k}{r}\binom{j}{s} e^{t\{\alpha_1(l+2r)+\alpha_2(m+2s)\}^\rho}(-1)^{r+s} \right) \\ &\quad \times \Xi_{l,m}(\tau^{\otimes k} \otimes^{2k} \kappa_{l+2k,m+2j} \otimes_{2j} \tau^{\otimes j}). \end{aligned} \tag{4.3}$$

Remark 4.3. Note that the solution (4.3) is the QWN analogue of the solution of the classical Cauchy problem studied in [6].

Proof. From the continuity of $\mathcal{G}^Q$ and Lemma 4.1, we deduce that $\{Y_{\alpha_1,\alpha_2,\rho;t}\}_{t\geq 0}$ is a differentiable one-parameter semigroup of operators on $\mathcal{L}(\mathcal{F}_\theta^*, \mathcal{F}_\theta)$ with infinitesimal generator $(\mathcal{G}^Q)^{-1} \circ (N^Q_{\alpha_1,\alpha_2})^\rho \circ \mathcal{G}^Q$ which is equal to $(\Delta_E^Q(\alpha_1,\alpha_2))^\rho$ by Theorem 4.2. Then we deduce that $Y_{\alpha_1,\alpha_2,\rho;t}\Xi$ is the unique solution of (4.2).

To prove (4.3), we use (3.5) to get

$$\begin{aligned} X_{\alpha_1,\alpha_2,\rho;t}(\mathcal{G}^Q\Xi) &= \sum_{j,k,l,m=0}^{\infty} \frac{(l+2k)!(m+2j)!}{j!k!l!m!2^{j+k}}(-1)^{j+k}(-i)^{l+m} \\ &\quad \times e^{t(\alpha_1 l+\alpha_2 m)^\rho}\Xi_{l,m}(\tau^{\otimes^k} \otimes^{2k} \kappa_{l+2k,m+2j} \otimes_{2j} \tau^{\otimes j}). \end{aligned}$$

Then applying $(\mathcal{G}^Q)^{-1}$ we obtain

$$\begin{aligned} Y_{\alpha_1,\alpha_2,\rho;t}\Xi &= \sum_{i,j,k,l,m,n=0}^{\infty} \frac{(l+2i+2k)!(m+2j+2n)!}{i!j!k!l!m!n!2^{i+j+k+n}} \\ &\times (-1)^{j+k}e^{t\{\alpha_1(l+2k)+\alpha_2(m+2j)\}^\rho} \\ &\times \Xi_{l,m}(\tau^{\otimes^{(k+i)}} \otimes^{2(k+i)} \kappa_{l+2k+2i,m+2j+2n} \otimes_{2(j+n)} \tau^{\otimes(j+n)}). \end{aligned}$$

By a simple change of variables $i+k=r$ and $j+n=s$, we get

$$
\begin{aligned}
Y_{\alpha_1,\alpha_2,\rho;t}\Xi &= \sum_{j,k,l,m=0}^{\infty} \sum_{r\geq k, s\geq j} \frac{(l+2r)!(m+2s)!}{j!k!l!m!(r-k)!(s-j)!2^{r+s}}(-1)^{j+k} \\
&\quad \times e^{t\{\alpha_1(l+2k)+\alpha_2(m+2j)\}^\rho} \Xi_{l,m}(\tau^{\otimes^r} \otimes^{2r} \kappa_{l+2r,m+2s} \otimes_{2s} \tau^{\otimes s}) \\
&= \sum_{r,s,l,m} \frac{(l+2r)!(m+2s)!}{r!s!l!m!2^{r+s}}(-1)^{r+s+l+m} \\
&\quad \times \left(\sum_{0\leq k\leq r, 0\leq j\leq s} \begin{pmatrix} r \\ k \end{pmatrix} \begin{pmatrix} s \\ j \end{pmatrix} e^{t\{\alpha_1(l+2k)+\alpha_2(m+2j)\}^\rho} (-1)^{j+k} \right) \\
&\quad \times \Xi_{l,m}(\tau^{\otimes r} \otimes^{2r} \kappa_{l+2k+2i,m+2j+2n} \otimes_{2s} \tau^{\otimes s}).
\end{aligned}
$$

Hence we get the desired statement. □

Theorem 4.4. *The solution of the Cauchy problem (4.2) admits the following representation*

$$
\begin{aligned}
U_t &= \sum_{j,k,l,m=0}^{\infty} \frac{(-1)^{l+m+j+k}}{j!k!l!m!2^{l+m}} \\
&\quad (\sum_{0\leq r\leq l, 0\leq s\leq m} \sum_{0\leq i\leq j, 0\leq n\leq k} \begin{pmatrix} l \\ r \end{pmatrix} \begin{pmatrix} m \\ s \end{pmatrix} \begin{pmatrix} j \\ i \end{pmatrix} \begin{pmatrix} k \\ n \end{pmatrix} \\
&\quad (-1)^{r+s+i+n} e^{t(2\alpha_1 r+2\alpha_2 s+\alpha_1 i+\alpha_2 n)^\rho}) \\
&\quad \int_{\mathbb{R}^{2(l+j+k+m)}} \tau(u_1,u_2)\cdots\tau(u_{2l-1},u_{2l})\tau(v_1,v_2)\cdots\tau(v_{2m-1},v_{2m}) \\
&\quad \tau(s_1,u_{2l+1})\cdots\tau(s_j,u_{2l+j})\tau(t_1,v_{2m+1})\cdots\tau(t_k,v_{2m+k}) \\
&\quad (D^+_{s_1})^*\cdots(D^+_{s_j})^*(D^-_{t_1})^*\cdots(D^-_{t_k})^* D^+_{u_1}\cdots D^+_{u_{2l+j}} D^-_{v_1}\cdots D^-_{v_{2m+k}}\Xi \\
&\quad ds_1\cdots ds_j dt_1\cdots dt_k du_1\cdots du_{2l+j} dv_1\cdots dv_{2m+k}. \qquad (4.4)
\end{aligned}
$$

on $\mathcal{L}(\mathcal{F}^*_\theta, \mathcal{F}_\theta)$.

Remark 4.5. Note that for $\rho=1$, the solution in (4.4) coincides with the solution of the Cauchy problem associated to the QWN-Euler operator studied in [3].

Proof. From Theorem 4.2 we get the desired statement using the same technic of calculus used in Theorem 3.3. □

References

1. Accardi, L., Barhoumi, A. and Ji, U. C.: Quantum Laplacians on Generalized Operators on Boson Fock space, *Probability and Mathematical Statistics* **31** (2011) 1–24.
2. Barhoumi, A., Ouerdiane, H. and Rguigui, H.: Generalized Euler heat equation, *Quantum Probability and White Noise Analysis* **25** (2010) 99–116.
3. Barhoumi, A., Ouerdiane, H. and Rguigui, H.: QWN-Euler Operator And Associated Cauchy Problem, *Infinite Dimensional Analysis Quantum Probability and Related Topics* **15**, No. 1 (2012) 1250004 (20 pages).

4. Chung, D. M. and Ji, U. C.: Transform on white noise functionals with their application to Cauchy problems, *Nagoya Math. J.* **147** (1997) 1–23.
5. Chung, D. M. and Ji, U. C.: Transformation groups on white noise functionals and their application, *Appl. Math. Optim.* **37**, No. 2 (1998) 205–223.
6. Chung, D. M. and Ji, U. C.: Some cauchy problems in white noise analysis and associated semigroups of operators, *Stochastic Analysis and Applications* **17**, No. 1 (1999) 1–22.
7. Gannoun, R., Hachaichi, R., Ouerdiane, H. and Rezgi, A.: un théorème de dualité entre espace de fonction holomorphes à croissance exponentielle, *J. Funct. Anal.* **171** (2000) 1–14.
8. Gross, L.: Abstract Wiener spaces, *Proc. 5-th Berkeley Symp. Math. Stat. Probab.* **2** (1967) 31–42.
9. Hida, T.: *Introduction to White Noise*, Lecture Notes, Nagoya University, 1990.
10. Ji, U. C.: Integral kernel operators on regular generalized white noise functions, *Bull. Korean Math. Soc.* **37**, No. 3 (2000) 601–618.
11. Ji, U. C.: Quantum Extensions of Fourier-Gauss and Fourier-Mehler Transforms, *J. Korean Math. Soc.* **45**, No. 6 (2008) 1785–1801.
12. Ji, U. C., and Obata, N.: Unitary of Kuo's Fourier Mehler Transform, *Infinite Dimensional Analysis Quantum Probability and Related Topics* **7**, No. 1 (2004) 147–154.
13. Ji, U. C., and Obata, N.: Generalized white noise operator fields and quantum white noise derivatives, *Séminaires & Congrès* **16** (2007) 17–33.
14. Ji, U. C., and Obata, N.: Annihilation-derivative, creation-derivative and representation of quantum martingales, *Commun. Math. Phys.* **286** (2009) 751–775.
15. Ji, U. C., and Obata, N.: Quantum stochastic integral representations of Fock space operators, *Stochastics: An International Journal of Probability and Stochastics Processes* **81**, Nos. 3-4 (2009) 367–384.
16. Ji, U. C., Obata, N. and Ouerdiane, H.: Analytic characterization of generalized Fock space operators as two-variable entire function with growth condition, *Infinite Dimensional Analysis Quantum Probability and Related Topics* **5**, No. 3 (2002) 395–407.
17. Kuo, H.-H.: Potential theory associated with Uhlenbeck Ornstein process, *J. Funct. Anal* **21** (1976) 63–75.
18. Kuo, H.-H.: On Laplacian operator of generalized Brownian functionals, *Lect. Notes in Math* 1203 (1986) 119–128.
19. Kuo, H.-H.: *White noise distribution theory*, CRC press, Boca Raton, 1996.
20. Obata, N.: *White noise calculus and Fock spaces*, Lecture notes in Mathematics 1577, Spriger-Verlag, 1994.
21. Obata, N.: Quantum white noise calculus based on nuclear algebras of entire function, *Trends in Infinite Dimensional Analysis and Quantum Probability, RIMS*, No. 1278 (2001) 130–157.
22. Ouerdiane, H.: Fonctionnelles analytiques avec condition de croissance, et application l'analyse gaussienne, *Japan. J. Math. (N.S.)* **20**, No. 1 (1994) 187–198.
23. Ouerdiane, H.: Noyaux et symboles d'opérateurs sur des fonctionelles analytiques gaussiennes, *Japon. J. Math.* **21** (1995) 223–234.
24. Ouerdiane, H. and Rguigui, H.: QWN-Conservation Operator And Associated Differential Equation, *Communication on stochastic analysis* **6**, No. 3 (2012) 437–450.
25. Piech, M. A.: Parabolic equations associated with the number operator, *Trans. Amer. Math. Soc.* **194** (1974) 213–222.

Aymen Ettaieb: Department of Mathematics, Faculty of Sciences of Tunis University of Tunis El-Manar, 1060 Tunis, Tunisia
E-mail address: ettaieb97134@gmail.com

Habib Ouerdiane: Department of Mathematics, Faculty of Sciences of Tunis University of Tunis El-Manar, 1060 Tunis, Tunisia
E-mail address: habib.ouerdiane@fst.rnu.tn

Hafedh Rguigui: Department of Mathematics, Faculty of Sciences of Tunis University of Tunis El-Manar, 1060 Tunis, Tunisia
E-mail address: hafedh.rguigui@yahoo.fr