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THE FORCING TOTAL MONOPHONIC NUMBER OF A GRAPH

Abstract: *For a connected graph $G = (V,E)$ of order at least two. A total monophonic set of a graph G is a monophonic set M such that the subgraph induced by M has no isolated vertices. The minimum cardinality of a total monophonic set of Gis the total monophonic number of G and is denoted by $m_t(G)$. A subset T of a minimum total monophonic set M of G is a forcing total monophonic subset for M if M is the unique minimum total monophonic set containing T. A forcing total monophonic subset for M of minimum cardinality is a minimum forcing total monophonic subset of M. The forcing total monophonic number $fm_t(M)$ in G is the cardinality of a minimum forcing total monophonic subset of M.The forcing total monophonic number of G is $fm_t(G) = min\{fm_t(M)$, where the minimum is taken over all minimum total monophonic sets M in G. It is shown that forevery pair a, b of positive integers with $0 \leq a < b$ and $b > 2a + 1$, there exists a connected graph G such that $fm_t(G) = a$ and $m_t(G) = b$*

Keywords: *forcing monophonic number,forcing monophonic set,forcing total monophonic number,Monophonic set, monophonic number, total monophonic set, total monophonic number.*

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I. INTRODUTION

By a graph $G = (V,E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology we refer to Harary[2]. The distance $d(u, v)$ between two vertices $u$and$v$ in a connected graph G is the length of a shortest u-v path of length $d(u,v)$ is called a u-v geodesic.A vertex v is said to lie on an x-y geodesic P if v is a vertex of P including the vertices x and y. *A* set S of vertices is a geodetic set if$I[S]$ = V, and the minimum cardinality of a geodetic set is the geodetic number $g(G)$. A geodetic set of cardinality $g(G)$ is called *a* g-set. The geodetic number of a graph was introduced in [3].The neighborhood of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. The closed neighbourhood of a vertex v is the set $N(v) = N(v) \cup \{v\}$. A vertex v is an extreme vertex if the subgraph induced by its neighbours is complete.Avertex v is a semi-extreme vertex of G if $\Delta(\langle N(v)\rangle) = |N(v)| - 1$. In particular, every extreme vertex is a semi - extreme vertex and a semi - extreme vertex need not be an extreme vertex.

A chord of a path $u_1, u_2, \ldots, u_k$ in G is an edge $u_i u_j$ with $j \geq i + 2$. A path P is called a monophonic path if it is a chordless path. A set M of vertices is a monophonic set if every vertex of G lies on a monophonic path joining some pair of vertices in M, and the minimum cardinality of a monophonic set is the monophonic number $m(G)$. The monophonic number of a graph G was studied in [6]. A set S of vertices of a graph G is an edge monophonic set if every edge of G lies on an x-y monophonic for some elements x and y in S. The minimum cardinality of an edge monophonic set of G is the edge monophonic number of G, denoted by $em(G)$. The edge monophonic number of a graph G was studied in [5].

A total monophonic set of a graph G is a monophonic set M such that the subgraph induced by M has no isolated vertices. The minimum cardinality of a total monophonic set of G is the total monophonic number of G and is denoted by $m_t(G)$. A total edge monophonic set of a graph G is an edge monophonic set M such that the subgraph induced by M has no isolated vertices. The minimum cardinality of a total edge monophonic set of G is the total edge monophonic number of G and is denoted by $em_t(G)$. The total edge monophonic number of a graph was introduced and studied in [1].

Example 1.1. For the graph G given in Figure 1.1, $M_1=\{v_1, v_2, v_4, v_5\}$, $M_2 = \{v_1, v_2, v_5, v_6\}$, $M_3 = \{v_1, v_2, v_3, v_6\}$, $M_4 = \{v_1, v_3, v_4, v_5\}$ are the minimum total monophonic sets of G and so $m_t(G) = 4$.

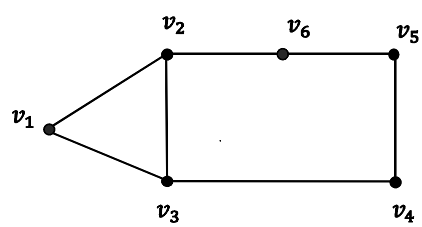

Figure 1.1

Let G be a connected graph and M be a minimum monophonic set of G. A subset $T \subseteq M$ is called a forcing subset for M if M is the unique minimum monophonic set containing T. A forcing subset for M of minimum cardinality is a minimum forcing subset of M. The forcing monophonic number of G, denoted by f_m (G), is (G) = min {where the minimum is taken over all minimum monophonic sets M in G. The forcing monophonic number of a graph was introduced and studied in [4].

A connected graph G may contain more than one minimum total monophonic sets. For example, the graph G given in Figure 1.1 contains four minimum total monophonic sets. Foreach minimum total monophonic set M in G there is always some subset T of M that uniquely determines M as the minimum total monophonic set containing T. Such sets are called forcing total monophonic subsets.

The following theorems used in the sequel.

Theorem 1.1 [1]. Each extreme vertex and each support vertex of a connected graph G belongs to every total edge monophonic set of G. If the set M of all extreme vertices and support vertices form a total edgemonophonic set, then M is the unique minimum total edge monophonic set of G.

Theorem 1.2[1]. For the complete graph $Kp(p$ 2), $em_t(Kp) = p$.

Theorem 1.3 [1]. For any non trivial tree T, the set of all end vertices and support vertices of T is the unique minimum total edge monophonic set of G.

Theorem 1.4 [1]. For any connected graph G, $em_t(G) = 2$ if and only if $G = K_2$.

Theorem 1.5 [1]. No cutvertex of a connected graph G belongs to any minimumtotal edge monophonic set of G.

Theorem 1.6[4]. For the complete graph $Kp(p$ 2),

Throughout this paper G denotes a connected graph with atleast two vertices.

2. FORCING TOTAL MONOPHONIC NUMBER OF A GRAPH

Definition 2.1. Let G be a connected graph and let M be a minimum total monophonic set of G. A subset T of a minimum total monophonic set M of G is a forcing total monophonic subset for M if M is the unique minimum total monophonic set containing T. A forcing total monophonic subset for M of minimum cardinality is a minimum forcing total monophonic subset of M. The forcing total monophonic number $fm_t(M)$

in G is the cardinality of a minimum forcing total monophonic subset of M. The forcing total monophonic number of G is $fm_t(G) = \min\{fm_t(M)\}$, where the minimum is taken over all minimum total monophonic sets in G.

Example 2.2. For the graph G given in Figure 2.1, $M_1 = \{u_1, u_3, u_6, u_7\}$, $M_2 = \{u_1, u_2, u_6, u_7\}$, $M_3= \{u_1, u_2, u_5, u_6\}$, $M_4 = \{u_1, u_3, u_5, u_6\}$ are the minimum total monophonic sets of G. It is clear that $fm_t(M_1) = 2$, $fm_t(M_2) = 2$, $fm_t(M_3) = 2$, $fm_t(M_4) = 2$, so that $fm_t(G) = 2$.

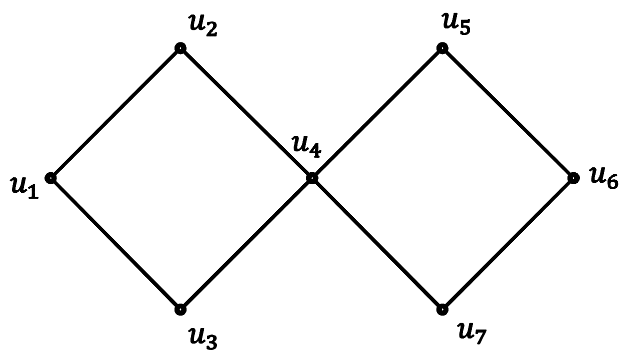

Figure 2.1

Theorem 2.3. For a connected graph G, $0 \leq fm_t(G) \leq m_t(G)\ p$.

Proof. It is clear from the definition of $fm_t(G)$ that $fm_t(G) \geq 0$. Let M be a minimum total monophonic set of G. Since $fm_t(M) \leq m_t(G)$ and since $fm_t(G) = \min\{ fm_t(M)$: M is a minimum total monophonic set in $G\}$, it follows that $fm_t(G) \leq m_t(G)$. Thus $0 \leq fm_t(G) \leq m_t(G) \leq p$.

Remark 2.4. The bounds in Theorem 2.3 are sharp. By Theorems 1.2 and 1.6, for the complete graph $Kp(p \geq 2)$, $m_t(Kp) = p$, also $V(Kp)$ is the unique total monophonic set of Kp and so $fm_t(Kp) = 0$. The inequalities in Theorem 2.3 are strict.

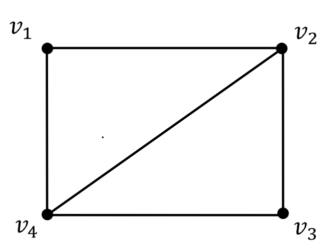

Figure 2.2

For the graph G given in Figure 2.2, $M_1 = \{v_1, v_2, v_3\}$, $M_2 = \{v_1, v_3, v_4\}$ are the minimum total monophonic sets of G so that $m_t(G) = 3$. It is clear that $fm_t(M_1) = 1$ and $fm_t(M_2) = 1$ so that $fm_t(G) = 1$. Thus $0\, fm_t(G) < m_t(G) < p$.

Theorem 2.5. Let G be a connected graph. Then

(i) $fm_t(G) = 0$ if and only if G has a unique minimum total monophonic set.

(ii) $fm_t(G) = 1$ if and only if G has at least two minimum total monophonic sets, one of which is a unique minimum total monophonic set containing one of its elements, and

(iii) $fm_t(G) = m_t(G)$ if and only if no minimum total monophonic set of G is the unique minimum total monophonic set containing any of its proper subsets.

Proof.(i) Let $fm_t(G) = 0$. Then by definition, $fm_t(M) = 0$ for some minimum total monophonic set M of G, so that the empty set Ø is the minimum forcing subset for M. Since the empty set Ø is a subset of every set, it follows that M is the unique minimum total monophonic set of G. The converse is clear.

(ii) Let $fm_t(G) = 1$. Then by Theorem 2.5 (i), G has atleast two minimum total monophonic sets. Since $fm_t(G) = 1$, there is a singleton subset T of a minimum total monophonic set M of G such that T is not a subset of any other minimum total monophonic set of G. Thus M is the unique minimum total monophonic set containing one of its elements. The converse is clear.

(iii) Let $fm_t(G) = m_t(G)$. Then $fm_t(M) = m_t(G)$ for every minimum total monophonic set in G. By Theorem 2.3, $m_t(G) \geq 2$, $fm_t(G) \geq 2$. Then by Theorem 2.5(i), G has at least two minimum total monophonic set of G. Since $fm_t(M) = m_t(G)$, no proper subset of M is a forcing subset of M. Thus no minimum total monophonic set of G is the unique minimum total monophonic set containing any of its proper subsets. Conversely, G contains more than one minimum total monophonic set and no subset of any minimum total monophonic set M other than M is a subset for M. Hence it follows that $fm_t(G) = m_t(G)$.

Definition 2.6. A vertex v of a connected graph G is said to be a total monophonic vertex of G if $v$belongs to every minimum total monophonic set of G.

Example 2.7. For the graph G given in Figure 2.3, $M_1 = \{v_1, v_4, v_5, v_8\}$, $M_2 = \{v_1, v_2, v_4, v_5\}$ are the only two total monophonic sets of G. It is clear that v_1, v_5 are total monophonic vertices of G.

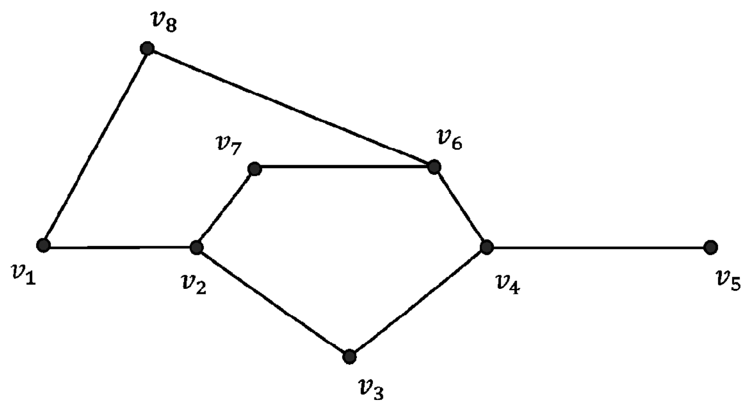

Figure 2.3

Corollary 2.8. Let G be a connected graph and let M be a minimum total monophonic set of G. Then no total monophonic vertex of G belongs to any minimum forcing total monophonic subset of M.

Theorem 2.9. Let G be a connected graph and let S be the set of all total monophonic vertices of G. Then $fm_t(G) \leq m_t(G) - |S|$.

Proof. Let M be any minimum total monophonic set of G. Then $m_t(G) = |M|$, $S \subseteq M$ and M is the unique minimum total monophonic set containing M–S.

Thus $fm_t(G) \leq |M–S| = |M| - |S| = m_t(G) - |S|$.

Corollary 2.10. If G is a connected graph with l extreme vertices and k support vertices, then $fm_t(G) \leq m_t(G) - (l+k)$. **Proof**. This follows from Theorem 1.1 and Theorem 2.9.

Remark 2.11. The bound in Theorem 2.9 is sharp. For the graph G given in Figure 2.2, $m_t(G) = 3$ and $fm_t(G) = 1$.

Also $S = \{v_1, v_3\}$ is the set of all total monophonic vertices of G and $fm_t(G) = m_t(G) - |S|$. Also the inequality in Theorem 2.9 can be strict.

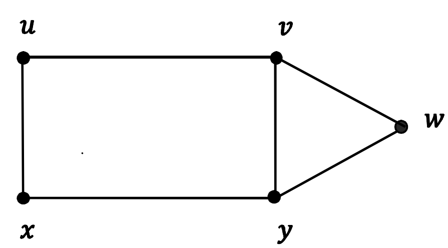

Figure 2.4

For the graph G given in Figure 2.4, $M_1=\{u, v, w\}$, $M_2 = \{x, y, w\}$ are the minimum total monophonic sets of G and so that $m_t(G) = 3$. Since M_1 is the unique minimum total monophonic set contains the subset $\{u\}$ so that $fm_t(M_1) = 1$ and M_2 is the unique minimum total monophonic set contains the subset$\{x\}$ so that $fm_t(M_2) =.1$ Hence, we have $fm_t(G) = 1$. Also, the vertex w is the unique total monophonic vertex of G, we have $fm_t(G) <m_t(G)\ |S|$.

Theorem 2.12. Let G be a connected graph and let M be a minimum total monophonic set of G. Then no cut vertex of G belongs to any minimum forcing total monophonic subset of M.

Proof: Let v be a cut vertex of G which is not a support vertex. By Theorems 1.1 and 1.5, v does not belong to any minimum total monophonic set of G. Since any minimum forcing total monophonic subset of $M$is a subset of minimum total monophonic set. It follows that, no cut vertex of G belongs to any minimum forcing total monophonic subset of M.

Theorem 2.13: For any complete graph $G = Kp(p \geq 2)$ or any non trivial tree $G = T, fm_t(G) = 0$.

Proof: For $G = Kp$, it follows from Theorem 1.6 that the set of all vertices of G is the unique minimum total monophonic set of G. Now, it follows from Theorem 2.5 (i) that $fm_t(G) = 0$. If G is a non trivial tree, then by Theorem 1.3, the set of all end vertices and support vertices of G is the unique minimum total monophonic set of G and by Thorem 2.5(i) $fm_t(G) = 0$.

Theorem 2.14: If G is a connected graph with $m_t(G) = 2$, then $fm_t(G) = 0$.

Proof: This follows from Theorem 1.4 and Theorem 2.13.

Theorem 2.15: For the complete bipartite graph $G = (m,n \geq 2)$, $fm_t(G) =$

$$\left\{\begin{matrix} 1\, if\ 2=m=n \\ 4\, if\ 3\leq m\leq n \\ 1\, if\ 2=m<n \\ 0\, if\ m=1\, and\, n=2 \end{matrix}\right\}$$

Proof: We prove this theorem by considering four cases.

Case 1: If $m = 1$ and $n = 2$. Let $U = \{u_1, u_2, \ldots, u_m\}$ and $W = \{w_1, w_2, \ldots, w_n\}$ be the bipartition of G. If $m = 1$, then G is a tree and its forcing total monophonic number is 0.

Case 2: If $2 = m = n$, then $U = \{u_1, u_2\}$ and $W = \{w_1, w_2\}$ and any minimum total monophonic set of G is of the following forms: (i) $U \cup \{w_i\}$ or (ii) $W \cup \{u_i\}$. Hence in both cases $fm_t(G) = 1$.

Case 3: If $2 = m < n$, then for any $j (1 \leq j \leq n)$, $S_j = U \cup \{w_j\}$ is a minimum total monophonic set of G. Since $n \geq 3$, then by Theorem 2.5 (ii), we have $fm_t(G) = 1$.

Case 4: If $3 = m = n$, then any minimum total monophonic set of G is of the following forms: (i) $U \cup \{w_j\}$ for some j $(1 \leq j \leq n)$. (ii) $W \cup \{u_i\}$ for some i $(1 \leq i \leq m)$, or (iii) any set got by chossing any two elements from each of U and W. If $3 = m < n$, then any minimum total monophonic set of G is either $U \cup \{w_j\}$ for some j $(1 \leq j \leq n)$ or any set got by chossing any two elements from each of U and W. Hence in both cases , we have $m_t(G) = 4$. Clearly, no minimum total monophonic set of G is the unique minimum total monophonic set containing any of its proper subsets. Then by Theorem 2.5 (iii), we have $fm_t(G) = m_t(G) = 4$.

3. REALIZATION RESULTS

Theorem 3.1: For every pair *a, b* of positive integers with and , there exists a connected graph G such that $fm_t(G) = a$ and $m_t(G) = b$.

Proof: If $a = 0$, let $G = K_b$. Then by Theorem 2.13, $fm_t(G) = 0$ and by Theorem 1.2, $m_t(G) = b$. Thus we assume that $0 < a < b$.

For each i with $1 \leq i \leq a$, let $C_i : u_{i,1}, u_{i,2}, u_{i,3}, u_{i,4}, u_{i,1}$ be a cycle of order 4. Let $K_{1,b-2a-1}$ be a star with the cutvertex x and $V(K_{1,b-2a-1}) = \{x, v_1, v_2 \ldots v_{b-2a-1}\}$. The graph G is obtained from C_i $(1 \leq i \leq a)$ and $K_{1,b-2a-1}$ by joining the vertices x and $u_{i,1}$ and by joining the vertices $u_{i,2}$ and $u_{i,4}$ $(1 \leq i \leq a)$. The graph G is given in Figure 3.1.

Let $S = \{v_1, v_2, \ldots, v_{b-2a-1}, u_{1,3}, u_{2,3}, \ldots, u_{a,3,} x\}$ be the set of all extreme vertices and support vertex of G. By Theorem 1.1, every total monophonic set of G contains S. It is easily verified that S is not a total monophonic set of G. Hence $M' = S \cup \{u_{1,2}, u_{2,2}, \ldots, u_{a,2}\}$ is a total monophonic set of G. Thus $m_t(G) = b$.

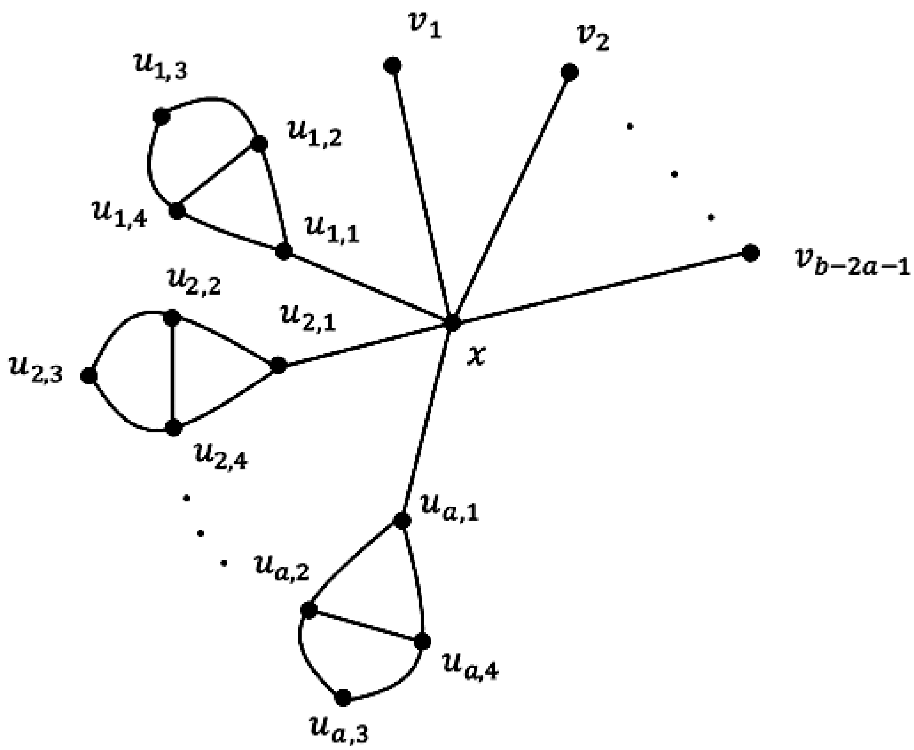

Figure 3.1

Next we show that $fm_t(G) = a$. It is observed that S is the set of all total monophonic vertices of G. Then by Theorem 2.9, $fm_t(G) \leq m_t(G) - = b - (b - a) = a$.

Now, since $m_t(G) = b$ and every total monophonic set of G contains S, it is easily seen that every minimum total monophonic set M_1 of G is of the form $S \cup \{x_1, x_2, \ldots, x_a\}$, where $x_i \in \{u_{i,2}, u_{i,4}\}$ for every $i(1 \leq i \leq a)$ Let T be any proper subset of M_1 with $|T| < a$. Then there is a vertex $x \in M_1 - S$ such that $x \notin T$. If $x = u_{i,2}$ $(1 \leq i \leq a)$, then $M_2 = (M_1 - \{u_{i,2}\} \cup \{u_{i,4}\}$ is a minimum total monophonic set of G containing T.

Similarly, if $x = u_{i,4}$ $(1 \leq i \leq a)$, then $M_3 = (M_1 - \{u_{i,4}\} \cup \{u_{i,2}\})$ is a minimum total monophonic set of G containing T. Thus M_1 is not the unique minimum total monophonic set of G containing T and so T is not a forcing total monophonic subset of M_1. Since this is true for all minimum total monophonic set of G, it follows that $fm_t(G) \geq a$ and so $fm_t(G) = a$.

Theorem 3.2: For every integers a and b with $a < b$, and $b - 2a - 2 > 0$, there exists a connected graph G such that, $fm_t(G) = a$ and $m_t(G) = b$.

Proof: Case 1. $a = 0$, $b \geq 2$. Let $G = K_{1,b-1}$. Then by Theorem 2.13, $fm_t(G) = 0$ and $m_t(G) = b$.

Case 2. $0 < a < b$. Let F_i: s_i, r_i, u_i, t_i, s_i be a copy of C_4. Let H be a graph obtained from F_i's by identifying t_{i-1} of F_{i-1} and s_i of F_i $(2 \leq i \leq a)$. Let G be a graph obtained from H by adding $b - 2a - 1$ new vertices $x, v_1, v_2, \ldots, v_{b-2a-2}$ and joining the edges $xs_1, t_a v_1, \ldots, t_a v_{b-2a-2}$ as shown in Figure 3.2.

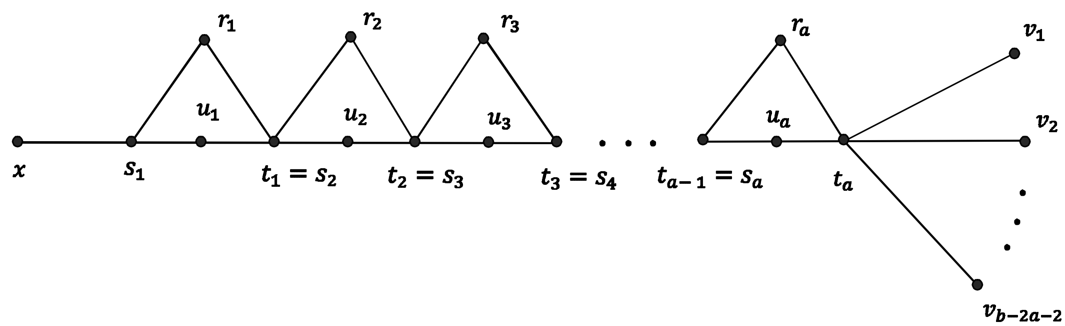

Figure 3.2

Let $Z = \{x, v_1, v_2, \ldots, v_{b-2a-2}\}$ be the set of end vertices of G. It is clear that Z is not a total monophonic set of G. By Theorem 1.5, $Z' = Z \cup \{s_1, s_2 \ldots, s_a, t_a\}$ is a subset of every total monophonic set of G. Let $H_i = \{u_i, r_i\}$ $(1 \leq i \leq a)$. We observe that every $m_t(G)$ must contain at least one vertex from each H_i, so that $m_t(G) \geq b - 2a - 1 + a + 1 + a = $ b. Now, $M = Z' \cup \{r_1, r_2, \ldots, r_a\}$ is a total monophonic set of G, so that $m_t(G) \leq b - 2a - 1 + a + 1 + a = b$. Thus $m_t(G) = b$. Next, we show that $fm_t(G) = a$. Since every m_t – set contains Z', it follows from Theorem 2.9, $fm_t(G) \leq m_t(G) - (b - 2a - 1 + a + 1) = a$.

It is easily seen that every m_t- set of G is of the form $Z' \cup \{r_1, r_2, \ldots, r_a\}$, where $r_i \in H_i (1 \leq i \leq a)$. Let T be any proper subset of M with $|T| < a$. Then there exists $r_i (1 \leq i \leq a)$such that $r_i \notin T$. Let e_i be the vertex of H_i distinct from r_i. Then $W = (M - \{r_i\}) \cup \{e_i\}$ is am_t-set properly containing T. Thus M is not the unique m_t-set containing T, so that T is not a forcing subset of M. This is true for all m_t-sets, so that $fm_t(G) = a$.

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