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HYBRID MODELS AND SWITCHING CONTROL WITH CONSTRAINTS

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Abstract. We consider a class of hybrid control, relative to Markov-Feller processes, where the discrete and the continuous type variables exchange information when a signal arrives. These problems can also be studied as optimal stopping and impulse control problems for a Markov-Feller process where the controls are allowed only when a signal arrives. There are a few references of the authors in the last years, where the HJB equation was solved and an optimal control (for the optimal stopping problem and impulse control problem) was obtained, under suitable conditions, including a setting on a (locally) compact metric state space, a strictly positive cost-per-impulse, and without multiple simultaneous impulses. In this work, we use these results to discuss optimal switching problems for Markov-Feller processes on a locally compact state-space under weaker conditions, as a particular case of optimal hybrid control problems.

1. Introduction

The impulse control of Markov processes and its applications has been the subject of numerous studies (e.g., see Bensoussan and Lions [2], Davis [10], Menaldi [25], Robin [34], and the references therein, among other works). However, impulse control problems where constraints are imposed on the possible stopping times is not often considered. The constraint referred in the present paper could be expressed as 'control when a signal arrives'. For instance, in an impulse control setting, x_t is the Markov process to be controlled and the impulse times must be the jump times of another Markov process y_t, the 'signal process'. Perhaps the simplest model is the case when x_t is a Wiener process in $\mathbb{R}$ and y_t is a Poisson process as in Dupuis and Wang [11].

Let us mention that reference related to optimal stopping with constraint include Lempa [20] and Liang [22], who studied particular cases of the model considered here, and that other classes of constraint have been considered, e.g., in Egloff and Leippold [12]. Moreover, for impulse control with constraint, relevant references include Brémaud [6, 7], Liang and Wei [23], and Wang [37]. A different kind of constraint is considered in Costa et al. [9], where the constraints are written as infinite horizon expected discounted costs.

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Impulse control problems with constraint in this sense have been studied in [28, 29, 30], when x_t is a general Markov-Feller process and $y_t = t - \tau_n$, where τ_n is an increasing sequence of instants such that $T_n = \tau_n - \tau_{n-1}$, for $n \geq 1$ are, conditionally to x_t, IID random variables.

The aim of the present paper is to investigate the links of these problems with hybrid models and to apply previous results to some hybrid situations in particular when x_t is a switching process. Moreover, we consider extensions of previous works when the cost of impulses is not always strictly positive and when simultaneous impulses can occur. The paper is organized as follows: Section 2 introduces hybrid models. In section 3 the general framework and previous results are described. In section 4, we show the application of these results to switching control and, in Sections 5 and 6 we discuss weaker assumptions and the case of simultaneous impulses.

2. Hybrid Models

The state of a continuous-time hybrid model has a continuous-type variable x (with cad-lag paths) and a discrete-type variable n (with cad-lag piecewise constant paths). The 'signal' is represented by the 'jumps' of the n_t, and in general, this signal enable any possible change in setting of the model, not only the 'possibility of controlling' as studied in this paper (an others). The general idea is that the usual evolution of the system is described by the component x_t, and 'once in a while' (or under some specific conditions) a discrete transition (i.e., a jump of n_t occurs) and everything may change, and the evolution continues thereafter, and so, the paths $t \mapsto (x_t, n_t)$ are usually discontinuous in probability. With this in mind, the *signal* (to act, e.g., to control the system as in our model) is given by the 'hitting time' of a set of states S, i.e., $\tau = \inf\{t > 0 : (x_t, n_t) \in S\}$, which plays the role of a 'set-interface' for the continuous-type and discrete-type evolution. It is convenient to refer the expression defining the signal τ (hitting time of the set-interface) as the *signal functional*. This set-interface D may be given a priori or used as part of the parameters of control, however, in this paper only a very simple model is studied. Thus *set-interface* D may or may not be part of the control, if it is then there are a maximum set-interface $D^{\vee}$ and a minimum set-interface $D^{\wedge}$, which have the following meaning: (a) the state cannot remain in $D^{\wedge}$ so that an impulse is mandatory, (b) outside of $D^{\vee}$ no impulse controls are allowed, and (c) impulses are allowed (but not required) within the region $D^{\vee} \smallsetminus D^{\wedge}$. Always in a general context, the sequence of signals (or simply the signals) $\{\tau_k\}$ are given by a recurrence formula, beginning with a given state (x, n) at time $t = 0$ define

$$\tau_k = \inf\{t > \tau_{k-1} : (x_t, n_t) \in D\}, \quad \forall k = 1, 2, \dots,$$

and for convenience set $\tau_0 = 0$, which may or may not be a signal (depending on the specific model). Versions of this model can be found in Bensoussan and Menaldi [4, 5] and [27], and their references as well as many others. Also, a more complete abstract model is being studied in a book to appear, see Jasso-Fuentes et al. [19]).

It should be clear that this hybrid model includes almost all situations, but particular examples requires some specific details, e.g., suppose a very simple case where the signals $\{\tau_k\}$ is constructed from an independent identically distributed (IID) sequence $\{T_k\}$ of random variables (RVs), i.e., $\tau_k = \tau_{k-1} + T_k$, $k \geq 1$. If for instance, these RVs $\{T_k\}$ are exponentially distributed, as in Dupuis and Wang [11] or Liang [22] (and others), where the signals was given as the jumps of a Poisson process, then thanks to the memoryless property of the exponential distribution, there is no need to know the 'waiting time' (i.e., how much time the controller has been waiting for the signal) to have full information and be able to exert the control of the system. In the case of a stochastic differential equations (SDE) driven by a final dimensional Lévy process (i.e., a combination of a Wiener and a Poisson measure, both in an Euclidean space), it is not so simple to model jumps with a prescribed distribution (other than the exponential, e.g., see Çinlar and Jacod [8]) for the times between jumps, due to the fact that any Poisson measure (which acts as the source of jumps) is such that the times between two consecutive jumps has necessarily an exponential distribution. One way to overcome this difficulty is to allow jumps following a semi-Markov process instead of a Markov process.

Thus, in a simple situation, we may imagine that the hybrid state evolution (x_t, n_t) is a Markov process under a Markovian feedback control, even more, the continuous-type variable x_t could be a Markov process and the discrete-type variable n_t be a semi-Markov process. Within this assumption (as specified later), is included the existence of a component of x_t which is a non-negative process y_t representing the 'time elapsed since the last jumps' of n_t. Hence, the controlled Markov process governing the hybrid evolution may be better written as (x_t, y_t, n_t), where a specific component y_t (essentially needed to complete n_t from semi-Markov to Markov) has been identified. Again, making some simplifications (or decomposition), a (simple) typical hybrid model could have two components, (x_t, y_t) and (x_t, n_t), which are (each of them) a controlled Markov process conditioned to the other. This means that, in the case of a SDE, the coefficients defining the SDE for x_t may also depends on (y_t, n_t) and the coefficients defining the SDE for n_t may also depend on (x_t, y_t). However, it is not so clear what should be the SDE for y_t, since the 'waiting time' (called *signal process*) is usually defines as $y_t = t - \sup\{s \leq t : n_s \neq n_t\}$, see also Brémaud [6], Davis [10].

One of our main interests is on the signal process y_t, so that we try to minimize the possible actions of the control, i.e., only impulse controls as described below are allowed. Moreover, the discrete-type process (jump process) n_t is a semi-Markov (jump) process, which may be denoted by z_t. Thus, in short, we are going to discuss an impulse control problem with constraint as in [29], but impulse controls becomes switching controls, which are allowed only at the jump-times of the Markov process z_t. Precise assumptions are given in the next section.

Example 2.1. Before going further, let us make a more specific example of a hybrid (or switching) control model with state (x_t, n_t, y_t). Suppose x_t belonging to a Polish (i.e., complete, separable and metric) space E, represents, say, the difference between the production and the demand of a product. There are two modes of operation, $n = 1$ (normal) and $n = 2$ (degraded). When the mode is $n = 1$, the uncontrolled process x_t evolves as a Markov-Feller process with infinitesimal generator A_x^n. When $n = 1$, there is a transition to $n = 2$ after an exponentially distributed time (with parameter $a > 0$); and when $n = 2$, the mode remains at 2 (actually, the situation where this mode of operation is no more sustainable and a 'replacement' in necessary, could also be accommodated in this model, as well as many other variations). The control action is the transition from mode 2 to mode 1. There is a running cost $f(x,n)$, $n=1,2$, and a transition from 2 to 1 cost $c>0$. So the control can act only when $n=2$, and is an impulse (rather a jumps or switching) on the discrete component (the mode) n_t. In addition, there is a constraint on the impulse which have to be the jump times of the signal process y_t with infinitesimal generator $A_y\varphi(y) = \partial_y\varphi(y) + \lambda(y)[\varphi(0) - \varphi(y)]$,where ∂_y denotes the derivative with respect to y and λ is the intensity, i.e., a non-negative integrable function on any bounded interval, but with an infinite integral in the whole $[0,+\infty[$. *Thus* we have (x_t, n_t, y_t) as the state of the system at time $t \geq 0$, and we formulate the optimal control problem, for instance, as the minimization of the cost

$$J_{xny}(\nu) = \mathbb{E}^{\nu}_{xny}\Big\{\int_0^{\infty} \mathrm{e}^{-\alpha t} f(x_t, n_t)\mathrm{d}t + \sum_{i\geq 1} \mathrm{e}^{-\alpha\theta_i} c\Big\}$$

with a discount factor $\alpha > 0$ and an admissible impulse control $\nu = \{\theta_i\}$, which is an increasing sequence of stopping times, $\theta_i \to \infty$, satisfying $y_{\theta_i} = 0$ and $\theta_1 > 0$. The optimal cost $u_n(x,y) = \inf\{J_{xny}\}$, and the dynamic programming yields the following Hamilton-Jacobi-Bellman (HJB) equation

$$\begin{aligned}
&A^1_x u_1(x,y) - a[u_2(x,y) - u_1(x,y)] - \partial_y u_1 - \lambda(y)[u_1(x,0) - u_1(x,y)] + \\
&\qquad\qquad\qquad\qquad\qquad\qquad + \alpha u_1(x,y) = f_1(x), \\
&A^2_x u_2(x,y) - \partial_y u_2 - \lambda(y)[u_2(x,0) - u_2(x,y)] + \\
&\qquad\qquad + \lambda(y)[u_2(x,0) - Mu_1(x,0)]^+ + \alpha u_2(x,y) = f_2(x),
\end{aligned}$$

where $f_n(x) = f(x,n)$ and $Mu_1(x,0) = c + u_1(x,0)$.

The arguments in [29] and Jasso-Fuentes et al. [18] can be combined and extended (although not immediately) to the case E locally compact to treat the discounted cost 'switching case' just described above. For E compact, the arguments in [30] can be used to address the ergodic case, but for a locally compact E with an ergodic cost is more delicate. In other words, the switching models require a better discussion, they are not simple adaptation of previous arguments. Certainly, all this is very related to the switching diffusion models, e.g., the reader may check the books by Yin and Zhang [38, 39], Yin and Zhu [40], as well as several references therein.

3. Impulse Control Models

In short, a switching control problem for a Markov-Feller process with a discounted cost is considered, but all switching controls are allowed only when a signal arrives, however the details are many. First we need to describe the uncontrolled process and later to introduce the switching controls, all this in a general setting.

3.1. The uncontrolled process. For a purely impulse control model, the discrete-type component n_t used in the hybrid model is ignored (because only one possibility is involved) and the continuous-type component x_t is actually composed by two parts (x_t, y_t), as they were called, the reduced state x_t and the signal process y_t, which will serve to express the constraint. Thus, for the impulse control model, the uncontrolled state is (x_t, y_t), where x_t will later be impacted by the control, and certainly, x_t can include a component with discrete values (useful for the switching model as seen later).

Basic Notations: • $\mathbb{R}^+ = [0, \infty[$, E locally compact (actually, when convenient and to simplify or to reduce difficulties, we may impose 'compactness' and discuss later an extension to the locally compact case). separable and complete metric space (in short, a locally compact Polish space), and abusing notation $N = \{1, \ldots, N\}$, also $\mathbb{N}_0 = \{0, 1, \ldots\}$ (i.e., natural numbers $\mathbb{N} = \{1, 2, \ldots\}$ and 0), and the extended numbers $\overline{\mathbb{N}}_0 = \mathbb{N}_0 \cup \{\infty\}$, $\overline{\mathbb{R}}^+ = [0, \infty]$; • $\mathcal{B}(Z)$ the Borel σ-algebra of sets in Z, $B(Z)$ the space of real-valued Borel and bounded functions on Z, $C_b(Z)$ the space of real-valued continuous and bounded functions on Z, $C_0(Z)$ real-valued continuous functions vanishing at infinity on Z, i.e., a real-valued continuous function v belongs to $C_0(Z)$ if and only if for every $\varepsilon > 0$ there exists a compact set K of Z such that $|v(z)| < \varepsilon$ for every z in $Z \smallsetminus K$ (typically $E = \mathbb{R}^d$ and this means that $v(z) \to 0$ as $|z| \to \infty$), and also, if necessary, $B^+(Z)$, $C_b^+(Z)$, $C_0^+(Z)$ for non-negative functions; usually either $Z = E$ or $Z = E \times \mathbb{R}^+$; • the canonical space $D(\mathbb{R}^+, Z)$ of cad-lag functions, with its canonical process $z_t(\omega) = \omega(t)$ for any $\omega \in D(\mathbb{R}^+, Z)$, and its canonical filtration $\mathbb{F}^0 = \{\mathcal{F}_t^0 : t \geq 0\}$, $\mathcal{F}_t^0 = \sigma(z_s : 0 \leq s \leq t)$.

Assumption 3.1. Let $(\Omega, \mathbb{F}, x_t, y_t, P_{xy})$ be a (realization of a) strong and normal homogeneous Markov process , on $\Omega = D(\mathbb{R}^+, E \times \mathbb{R}^+)$ with its canonical filtration universally completed $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ with $\mathcal{F}_\infty = \mathcal{F}$, where (x_t, y_t) is the canonical process having values in $E \times \mathbb{R}^+$, and $\mathbb{E}_{xy}$ (or $\mathbb{E}_{x,y}$ when a confusion may arrive) denotes the expectation relative to P_{xy}.

(a) It is also assumed that x_t is a Markov process by itself (referred as the *reduced state*), with a C_0-semigroup $\Phi_x(t)$ (i.e., $\Phi_x(t) C_0(E) \subset C_0(E)$, $\forall t \geq 0$), and infinitesimal generator A_x with domain $\mathcal{D}(A_x) \subset C_0(E)$.

(b) The process y_t (referred to as the *signal* process) has jumps to zero at times $\tau_1, \ldots, \tau_n \to \infty$ and $y_t = t - \tau_n$ for $\tau_n \leq t < \tau_{n+1}$ (i.e., τ_1 is the time of the first jump –to zero– of y_t, each jump is 'the signal' and y_t is exactly the 'time elapsed since the last jump or signal'), and if $y_0 = 0$ and $\tau_0 = 0$ then it is assumed that conditionally to x_t, the intervals between jumps $T_n = \tau_n - \tau_{n-1}$ are independent, identically distributed random variables with a non-negative intensity function $\lambda(x, y)$, i.e.,

$$P_{x0}\{\tau_1 \geq t \mid x_s,\, s \leq t\} = \exp\Big(- \int_0^t \lambda(x_s, s) \mathrm{d}s \Big), \quad \forall t \geq 0, \, \forall x \in E,$$

which is continuous and bounded.

(c) Beside having $P_{x0}\{\tau_1 < \infty\} = 1$, and the intensity $\lambda(x, y)$ being a non-negative continuous and bounded function, the following equality and estimate

$$\mathbb{E}_{x0}\{\tau_1\} := \mathbb{E}_x\Big\{\int_0^\infty t\lambda(x_t, t)\exp\Big(-\int_0^t \lambda(x_s, s)\mathrm{d}s\Big)\mathrm{d}t\Big\} = \\ = \mathbb{E}_x\Big\{\int_0^\infty \exp\Big(-\int_0^t \lambda(x_s, s)\mathrm{d}s\Big)\mathrm{d}t\Big\} \le K, \quad \forall x,$$

hold true for some constant $K > 0$.

Remark 3.2. Actually, the condition (b) above does not forbid an intensity $\lambda(x, y)$ having a bounded support (or even being integrable on $]0, \infty[$), and therefore a signal such that $\tau_1 = \infty$ with a positive probability. Note also that under condition (c), a bounded intensity $\lambda(x, y)$ cannot have a compact support, since

$$\exp\Big(-\int_0^t \lambda(x_s, s)\mathrm{d}s\Big) \to 0 \quad \text{as} \quad t \to \infty.$$

Thus, adding condition (c) the current analysis is simplified, without assuming that the intensity $\lambda(x, y)$ is bounded below by a strictly positive constant. Some other generalizations will be the object of a further discussion elsewhere.

Remark 3.3. Actually, we begin with a realization of the reduced state process x_t on the canonical space $D(\mathbb{R}^+, E)$ and the signal process y_t is constructed based on the given intensity $\lambda(x, y)$, and this procedure yields a $C_0(E \times \mathbb{R}^+)$-semigroup denoted by $\Phi_{xy}(t)$. Thus, in view of Palczewski and Stettner [32], all this implies that both semigroups $\Phi_x(t)$ and $\Phi_{xy}(t)$ have the Feller property, i.e., $\Phi_x(t)C_b(E) \subset C_b(E)$ and $\Phi_{xy}(t)C_b(E \times \mathbb{R}^+) \subset C_b(E \times \mathbb{R}^+)$, and since only a strong and normal Markov process is assumed, the semigroup $\Phi_{xy}(t)$ is (initially) acting on $B(E\times\mathbb{R}^+)$ and so, weak (or stochastic) continuity is deduced from the assumption of a cad-lag realization, which means that

$$(x, y, t) \mapsto \mathbb{E}_{xy}\{h(x_t, y_t)\} \quad \text{is a continuous function,} \tag{3.1}$$

for any h in $C_b(E \times \mathbb{R}^+)$. In [28, 29, 30] a probabilistic construction of the signal process y_t was described, but there are other ways to constructing $\Phi_{xy}(t)$. For instances, begin with the process $(x_t, \tilde{y}_t)$ with $\tilde{y}_t = y + t$ having infinitesimal generator $A^0 = A_x + \partial_y$ and a $C_0(E\times\mathbb{R}^+)$-semigroup. Then, add the perturbation $Bh(x, y) = \lambda(x, y)[h(x, 0) - h(x, y)]$, which is a bounded operator generating a $C_0(E \times \mathbb{R}^+)$-semigroup, with domain $\mathcal{D}(B) = C_0(E \times \mathbb{R}^+)$. Hence $A_{xy} = A^0 + B$ generates a $C_0(E \times \mathbb{R}^+)$-semigroup, with $\mathcal{D}(A_{xy}) = \mathcal{D}(A^0)$, e.g., see Ethier and Kurtz [13, Section 1.7, pp. 37–40, Thm 7.1]. Therefore A_{xy} will also denote the weak infinitesimal generator in $C_b(E \times \mathbb{R}^+)$, in several places of the following sections.

Remark 3.4. Note that Assumption 3.1 (b) on the signal process y_t means, in particular, that

$$P_{x0}\big\{T_n \in (t, t+\ dt) \mid x_s,\ 0 \le s \le t\big\} = \lambda(x_t, t)\exp\Big(-\int_0^t \lambda(x_s, s)\mathrm{d}s\Big)\mathrm{d}t, \tag{3.2}$$

and then it is deduced that $\Phi_{xy}(t)$ has an infinitesimal generator $A_{xy} = A_x + A_y$ with

$$A_y\varphi(x,y) = \partial_y\varphi(x,y) + \lambda(x,y)[\varphi(x,0) - \varphi(x,y)], \tag{3.3}$$

and recall that ∂_y denotes the derivative with respect to y, and that $\lambda \geq 0$ and $\lambda \in C_b(E \times \mathbb{R}^+)$. Moreover, using the law of T_1 as in (3.2) and the Feller property of (x_t, y_t), it is also deduced that

$$(x,y) \mapsto \mathbb{E}_{xy}\big\{\mathrm{e}^{-\alpha\tau_1} g(x_{\tau_1})\big\} \text{ belongs to } C_b(E \times \mathbb{R}^+), \tag{3.4}$$

for any g in $C_b(E)$ and any $\alpha \geq 0$. Note that if $y_0 = y$ then τ_1 is random variable independent of $T_1, T_2, \ldots$ with distribution $P_{x0}\{T_1 \in \cdot \mid y_0 = y\}$. Furthermore, in turn, by applying Dynkin's formula to $A_{xy}\varphi(x,y) + \alpha\varphi(x,y) = f(x,y)$, it follows that

$$(x,y) \mapsto \mathbb{E}_{xy}\Big\{\int_0^{\tau_1} \mathrm{e}^{-\alpha t} f(x_t, y_t)\mathrm{d}t\Big\} \text{ is in } C_b(E \times \mathbb{R}^+), \tag{3.5}$$

for any f in $C_b(E \times \mathbb{R}^+)$ and any $\alpha > 0$.

Remark 3.5. Note that because $\lambda(x,y)$ is bounded (it suffices for y near 0), there exists a constant a such that $P_{x0}\{\tau_1 \geq a > 0\} \geq a > 0$, for any x in E. Moreover, from Assumption 3.1 (b) and (c) on the signal process y_t we have: if $\lambda(x,y) \leq k_1 < \infty$, for every $y \geq 0$, and $x \in E$, then $\mathbb{E}_{x0}\{\tau_1\} \geq a_1 = 1/k_1$. Also, the condition $\mathbb{E}_{x0}\{\tau_1\} \leq a_2$ is satisfied if, for instance $\lambda(x,y) \geq k_0 > 0$ for $y \geq y_0$, $x \in E$, then $a_2 = y_0 + 1/k_0$. Moreover, since $\lambda(x,y)$ is a continuous function in $E \times \mathbb{R}^+$, the continuity of $E_{xy}\{\tau_1\}$ follows.

Definition 3.6 (with comments)**.** If, for some $\alpha > 0$, the evolution $\dot{e} = -\alpha t$ in $[0,1]$ is added to the homogeneous Markov process $\{(x_t, y_t) : t \geq 0\}$ then the expression

$$\{(X_n, e_n) = (x_{\tau_n}, \mathrm{e}^{-\alpha\tau_n}),\ n = 0, 1, \ldots\}, \tag{3.6}$$

with $e_0 = 1$, $\tau_0 = 0$ and $X_0 = x$, becomes a *homogeneous Markov chain* in $E\times]0,1]$ with respect to the filtration $\mathbb{G} = \{\mathcal{G}_n : n = 0, 1, \ldots\}$ obtained from $\mathbb{F}$, namely, $\mathcal{G}_n = \mathcal{F}_{\tau_n}$. Note that $\{x_{\tau_n} : n \geq 0\}$ and $\{(x_{\tau_n}, \tau_n) : n \geq 0\}$ are also a Markov chain with respect to $\mathcal{G}_n$. In this context, if

$$\tau = \inf\{t > 0 : y_t = 0\}, \tag{3.7}$$

is considered as a functional on Ω, then the *sequence of signals* (i.e., the instants of jumps for y_t) is defined by recurrence

$$\tau_{k+1} = \inf\{t > \tau_k : y_t = 0\}, \quad \forall k = 1, 2, \ldots, \tag{3.8}$$

with $\tau_1 = \tau$, and by convenience, set $\tau_0 = 0$. An $\mathbb{F}$-stopping time $\theta > 0$ satisfying $y_\theta = 0$ when $\theta < \infty$ is called an *admissible stopping time*, in other words, if and only if there exists a discrete (i.e., $\overline{\mathbb{N}}_0$-valued) $\mathbb{G}$-stopping time η such that $\theta = \tau_\eta$ with the convention that $\tau_\infty = \infty$. Moreover, if the condition $\theta > 0$ (or equivalently $\eta \geq 1$) is dropped then θ is called a *zero-admissible stopping time.*

Let us also mention that as in Remark 3.5 we deduce:

Remark 3.7. Because $\lambda(x,y)$ is bounded (by k_1), for any given $\alpha > 0$ we also have

$$\mathbb{E}_{x0}\big\{\mathrm{e}^{-\alpha\tau}\big\} \le a = \frac{k_1}{\alpha + k_1} < 1, \quad \forall x \in E,$$

and, this implies that the operator $Pw(x) = \mathbb{E}_{x0}\{\mathrm{e}^{-\alpha\tau} w(x_\tau)\}$ is a contraction mapping on $C_b(E)$ with the sup-norm. Moreover, after iterating and using Markov property, this proves that

$$\mathbb{E}_{x0}\big\{\mathrm{e}^{-\alpha\tau_k}\big\} \le a^k < 1, \quad \forall x \in E, \quad \forall k = 1, 2, \ldots,$$

where $\{\tau_k\}$ is the sequence of signals (3.8).

3.2. Common assumptions. It is assumed that there are a running cost $f(x,y)$ and a cost-per-impulse $c(x,\xi)$ satisfying

$$\begin{aligned} &f : E \times \mathbb{R}^+ \to \mathbb{R}^+ \ \text{ bounded and continuous,} \quad \alpha > 0, \\ &c : E \times E \to [c_0, +\infty[, \ c_0 > 0, \ \text{ bounded and continuous.} \end{aligned} \tag{3.9}$$

Moreover, for any $x \in E$, the possible impulses must be in $\Gamma(x) = \{\xi \in E : (x,\xi) \in \Gamma\}$, where Γ is a given analytic set in $E \times E$ such that for every x in E the following properties hold true

$$\begin{aligned} &\emptyset \ne \Gamma(x) \text{ is compact}^1, \quad \forall \xi \in \Gamma(x), \ \Gamma(\xi) \subset \Gamma(x), \quad \text{and} \\ &c(x,\xi) + c(\xi,\xi') \ge c(x,\xi'), \quad \forall \xi \in \Gamma(x), \ \forall \xi' \in \Gamma(\xi) \subset \Gamma(x). \end{aligned} \tag{3.10}$$

Finally, defining the operator M

$$Mv(x) = \inf_{\xi \in \Gamma(x)} \big\{c(x,\xi) + v(\xi)\big\}, \tag{3.11}$$

the condition

$$\begin{aligned} &M \text{ maps } C_b(E) \text{ into } C_b(E), \text{ and there exists a measurable} \\ &\text{selector } \hat{\xi}(x) = \hat{\xi}(x,v) \text{ realizing the infimum in } Mv(x), \ \forall x, v. \end{aligned} \tag{3.12}$$

is assumed.

Remark 3.8. (a) The last condition in (3.10) is to ensure that simultaneous impulses is never optimal. (b) Some regularity property of $\Gamma(x)$ are implicitly required when (3.12) is assumed , e.g., see Davis [10]. (c) It is possible (but not necessary) that x belongs to $\Gamma(x)$, actually, even $\Gamma(x) = E$ whenever E is compact. However, an impulse occurs when the system moves from a state x to another state $\xi \ne x$, so that, it suffices to avoid (or not to allow) impulses that moves x to itself, since they have a higher cost.

3.3. The controlled process. For a detailed construction we refer to Bensoussan and Lions [3] (see also Davis [10], Lepeltier and Marchal [21], Robin [34], Stettner [36]). To describe this construction, let us consider $\Omega^\infty = [D(\mathbb{R}^+; E \times \mathbb{R}^+)]^\infty$, and define $\mathcal{F}_t^0 = \mathcal{F}_t$ and $\mathcal{F}_t^{n+1} = \mathcal{F}_t^n \otimes \mathcal{F}_t$, for $n \ge 0$, where $\mathcal{F}_t$ is the universal completion of the canonical filtration as previously. Hence, an *arbitrary impulse control* ν (not necessarily admissible at this stage) is a sequence $(\theta_n, \xi_n)_{n \ge 1}$, where θ_n is a stopping time of $\mathcal{F}_t^{n-1}$, $\theta_n \ge \theta_{n-1}$, and the impulse ξ_n is $\mathcal{F}_{\theta_n}^{n-1}$ measurable random (referred to as 'adapted') variable with values in E.

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[1]compactness is not really necessary, but it is convenient

The coordinate in Ω^∞ has the form $(x_t^0, y_t^0, x_t^1, y_t^1, \ldots, x_t^n, y_t^n, \ldots)$, and for any impulse control ν there exists a probability P_{xy}^ν on Ω^∞ such that the evolution of the controlled process (x_t^ν, y_t^ν) is given by the coordinates (x_t^n, y_t^n) of Ω^∞ when $\theta_n \leq t < \theta_{n+1}$, $n \geq 0$ (setting $\theta_0 = 0$), i.e., $(x_t^\nu, y_t^\nu) = (x_t^n, y_t^n)$ for $\theta_n \leq t < \theta_{n+1}$. Note that clearly (x_t^ν, y_t^ν) is defined for any $t \geq 0$, but (x_t^i, y_t^i) is only used for any $t \geq \theta_i$, and $(x_{\theta_i}^{i-1}, y_{\theta_i}^{i-1})$ is the state at time θ_i just before the impulse (or jump) to $(\xi_i, y_{\theta_i}^{i-1}) = (x_{\theta_i}^i, y_{\theta_i}^i)$, as long as $\theta_i < \infty$. Remark that the impulse control $\nu = \{(\theta_i, \xi_i) : i \geq 1\}$ and the probability P_{xy}^ν are constructed by means of a sequential (or inductive) procedure, and it may be convenient to add $\theta_0 = 0$ and $\xi_0 = x$, which is not considered as an impulse. Hence, $\{(x_t^0, y_t^0) : t \geq 0\}$ is the uncontrolled Markov evolution (of the state) and $\{(x_t^i, y_t^i) : t \geq \theta_i\}$ denotes the Markov evolution after the i-impulse, i.e., only the first i impulses are applied and the Markov process restart anew at time $\theta_i < \infty$ with initial condition $(x_{\theta_i}^i, y_{\theta_i}^i) = (\xi_i, 0)$, since $y_{\theta_i}^{i-1} = 0$. Also the sequence $\{\tau_k^i : k \geq 1\}$ of signals after θ_i is given by the functional $\tau_{k+1}^i = \inf\{t > \tau_k^i : y_t^i = 0\}$, beginning with $\tau_0^i = \theta_i < \infty$, and using the convention $\inf\{\emptyset\} = \infty$. For the sake of simplicity, we will not always indicate, in the sequel, the dependency of (x_t^ν, y_t^ν) with respect to ν.

A *Markov impulse control* ν is identified by a closed subset S of $E \times \mathbb{R}^+$ and a Borel measurable function $(x, y) \mapsto \xi(x, y)$ from S into $C = E \times \mathbb{R}^+ \smallsetminus S$, with the following meaning: intervene only when the the process (x_t, y_t) is leaving the continuation region C and then apply an impulse $\xi(x, y)$, while in the stopping region S, moving back the process to the continuation region C, i.e., $\theta_{i+1} = \inf\{t > \theta_i : (x_t^i, y_t^i) \in S\}$, with the convention that $\inf\{\emptyset\} = \infty$, and $\xi_{i+1} = \xi(x_{\theta_{i+1}}^i, y_{\theta_{i+1}}^i)$, for any $i \geq 0$, as long as $\theta_i < \infty$.

Now, recalling that τ_n are the arrival times of the signal given by (3.8), the admissible controls are defined as follows:

Definition 3.9. **(i)** As mentioned earlier, a stopping time θ is called 'admissible' if almost surely there exists $n = \eta(\omega) \geq 1$ such that $\theta(\omega) = \tau_{\eta(\omega)}(\omega)$, or equivalently if θ satisfies $\theta > 0$ and $y_\theta = 0$ a.s.
(ii) An impulse control $\nu = \{(\theta_i, \xi_i), i \geq 1\}$ as above is called 'admissible', if each θ_i is admissible (i.e., $\theta_i > 0$ and $y_{\theta_i} = 0$), and $\xi_i \in \Gamma(x_{\theta_i}^{i-1})$. The set of admissible impulse controls is denoted by $\mathcal{V}$.
(iii) If $\theta_1 = 0$ is allowed, then ν is called 'zero-admissible'. The set of zero-admissible impulse controls is denoted by $\mathcal{V}_0$.
(iv) An 'admissible Markov' impulse control corresponds to a stopping region $S = S_0 \times \{0\}$ with $S_0 \subset E$, and an impulse function satisfying $\xi(x, 0) = \xi_0(x) \in \Gamma(x)$, for any $x \in S_0$, and therefore, if $\{(x_t^0, y_t^0) : t \geq 0\}$ is the uncontrolled Markov evolution (of the state) and $\{(x_t^i, y_t^i) : t \geq \theta_i\}$ denotes the Markov evolution after the i-impulse then $\eta_0 = 0$, $\tau_0^0 = 0$, $\theta_0 = \tau_0^0$, $\xi_0 = x$, $\tau_k^0 = \inf\{t > \tau_{k-1}^0 : y_t^0 = 0\}$ $(\forall k \geq 1)$, $\eta_1 = \inf\{k > \eta_0 : x_{\tau_k^0}^0 \in S_0\}$, $\theta_1 = \tau_{\eta_1}^0$, $\tau_{\eta_1}^1 = \theta_1$, $\xi_1 = \xi(x_{\theta_1}^0, 0)$, and next, $\tau_k^1 = \inf\{t > \tau_{k-1}^1 : y_t^1 = 0\}$ $(\forall k > \eta_1)$, $\eta_2 = \inf\{k > \eta_1 : x_{\tau_k^1}^1 \in S_0\}$, $\theta_2 = \tau_{\eta_2}^1$, $\tau_{\eta_2}^2 = \theta_2$, $\xi_2 = \xi(x_{\theta_2}^1, 0)$, and so forth. For a 'zero-admissible Markov' impulse control, it suffices to use $\eta_1 = \inf\{k \geq \eta_0 : x_{\tau_k^0}^1 \in S_0\}$, i.e., to replace $k > \eta_0$ with $k \geq \eta_0$, within the construction of η_1 in the previous iteration.

As seen later, it will be useful to consider an auxiliary problem in discrete time, for the Markov chain $X_n = x_{\tau_n}$, with the filtration $\mathbb{G} = \{\mathcal{G}_n, n \geq 0\}$, $\mathcal{G}_n = \mathcal{F}^{n-1}_{\tau_n}$. The impulses occurs at the stopping times η_i with values in the set $\mathbb{N} = \{0, 1, 2, \dots\}$ and are related to θ_i by $\eta_i = \inf\{k \geq \eta_{i-1} : \theta_i = \tau^i_k\}$ for admissible controls $\{\theta_i\}$ and similarly for zero-admissible controls with $\eta_i = \inf\{k \geq \eta_{i-1} : \theta_i = \tau^i_k\}$. The discrete time impulse control problem has been consider in Bensoussan [1], Stettner [35]. Thus,

Definition 3.10. If $\nu = \{(\eta_i, \xi_i), i \geq 1\}$ is a sequence of $\mathbb{G}$-stopping times and $\mathcal{G}_{\eta_i}$-measurable random variables ξ_i, with $\xi_i \in \Gamma(x_{\tau_{\eta_i}})$, η_i increasing and $\eta_i \to +\infty$ a.s., then ν is referred to as an 'admissible discrete time' impulse control if $\eta_1 \geq 1$. If $\eta_i \geq 0$ is allowed, it is referred as an 'zero-admissible discrete time' impulse control.

3.4. HJB equation. The *discounted* cost of an impulse control (or policy) $\nu = \{(\theta_i, \xi_i) : i \geq 1)\}$ is given by

$$J_{x,y}(\nu) = \mathbb{E}^\nu_{x,y}\Big\{ \int_0^\infty \mathrm{e}^{-\alpha t} f(x_t, y_t)\mathrm{d}t + \sum_{i=0}^\infty \mathrm{e}^{-\alpha\theta_i} c(x^{i-1}_{\theta_i}, \xi_i)\Big\}, \tag{3.13}$$

where $\mathbb{E}^\nu_{xy}$ is the P^ν_{xy}-expectation of the process under the impulse control ν with initial conditions $(x_0, y_0) = (x, y)$, and $x^{i-1}_{\theta_i}$ is the value of the process just before the impulse. Note that the process $\{y_t : t \geq 0\}$ is not subject to any impulse, and the condition $y_\theta = 0$ determines admissibility of the impulse time θ.

Thus, the optimal cost is defined by

$$u(x, y) = \inf\big\{J_{x,y}(\nu) : \nu \in \mathcal{V}\big\}, \quad \forall (x, y) \in E \times [0, \infty[, \tag{3.14}$$

and its associated auxiliary impulse control problem (referred to as the 'time-homogeneous' impulse control) has an optimal cost given by

$$u_0(x, y) = \inf\big\{J_{x,y}(\nu) : \nu \in \mathcal{V}_0\big\}, \quad \forall (x, y) \in E \times [0, \infty[. \tag{3.15}$$

As with the optimal stopping time problems, since $u(x, y) = u_0(x, y)$ for any $x \in E$ and $y > 0$, it may be convenient to write $u_0(x) = u_0(x, 0)$ as long as no confusion arrives.

The Dynamic Programming Principle shows (heuristically) (see [29, Section 3]) that

$$u(x, y) = \mathbb{E}_{xy}\Big\{ \int_0^\tau \mathrm{e}^{-\alpha t} f(x_t, y_t)\mathrm{d}t + \mathrm{e}^{-\alpha\tau} \min\{Mu, u\}(x_\tau, y_\tau)\Big\}, \tag{3.16}$$

and

$$\begin{aligned} u_0(x, y) &= \mathbb{E}_{xy}\Big\{ \int_0^\tau \mathrm{e}^{-\alpha t} f(x_t, y_t)\mathrm{d}t + \mathrm{e}^{-\alpha\tau} u_0(x_\tau, y_\tau)\Big\}, \quad y > 0, \\ u_0(x) &= \min\Big\{\mathbb{E}_{x0}\Big\{ \int_0^\tau \mathrm{e}^{-\alpha t} f(x_t, y_t)\mathrm{d}t + \mathrm{e}^{-\alpha\tau} u_0(x_\tau)\Big\}, Mu_0(x)\Big\}, \end{aligned} \tag{3.17}$$

are the corresponding Hamilton-Jacobi-Bellman (HJB) equations, which are referred to as quasi-variational inequalities (QVI) in a weak form. Note that M is an operator in the variable x alone, so that $Mu(x, y) = [Mu(\cdot, y)](x)$. In any case,

$\min\{Mu, u\}(x_\tau, y_\tau) = \min\{Mu, u\}(x_\tau, 0)$, because $y_\tau = 0$. Also, both problems are related (logically) by the condition

$$u(x, y) = \mathbb{E}_{xy}\Big\{ \int_0^\tau \mathrm{e}^{-\alpha t} f(x_t, y_t)\mathrm{d}t + \mathrm{e}^{-\alpha\tau} u_0(x_\tau)\Big\}, \tag{3.18}$$

and so, if $u_0(x)$ is known then the last equality yields $u(x, y)$ and $u_0(x, y)$.

The optimal cost $u_0(x) = u_0(x, 0)$ can be expressed as a discrete time optimal impulse control similar to Bensoussan [1, Chapter 8, 89–132] (ignoring the constraint), as discussed in [28], will be used in Section 4 to solve the HJB equation. As a key point, let us mention that $u_0(x) = u_0(x, 0)$ is given by either (3.15), and that in view of the relation (3.18), the other costs $u(x, y)$ and $u_0(x, y)$ can be obtained directly.

General discrete time hybrid models in Borel spaces with non-constant discount factor have been discussed in Jasso-Fuentes et al. [16, 17], but this time, having a non-empty intersection with the discrete time models considered in this work, and again, with another set of assumptions. Also, the interested reader may check (among many others) the books by Hernández-Lerma and Lasserre [15] for a classic study on discrete time Markov control processes, and by Peskir and Shiryaev [33] for a deep analysis of optimal stopping and free-boundary problems (which are connected with impulse control problems). As mentioned in the abstract, an extensive analysis on these type of impulse control problems with constraints was developed in a series of works [28, 29, 30, 31] under several assumptions. The results include solving the HJB equations in a suitable way so that it agrees with the optimal costs, as well as a description and construction of an optimal (admissible) impulse control (as the first exit times of the continuation region with an optimal impulse).

4. Switching Control Models

As mentioned earlier, in a switching model there is a 'state' x_t and a (operation) 'mode' n_t, and in most situation, some components of process n_t are control and state variables at the same time. This means that the actions of a switching control alter in some sense the typical continuous order: (a) read state of the system, (b) take a decision, (c) run the dynamic evolution, and iterate (a), (b), (c). This alteration is better recognized in a continuous time model, and it corresponds to another step (b'), in between (b) and (c), which produces a 'instantaneous' (relative to the iteration) modification of the state. Essentially, because a certain continuity in time is technically necessary (i.e., uncontrolled evolution and regarded as Markov processes with trajectories continuous in probability), the switching control produces discontinuity, that needs to be discussed. Indeed, this is a key point of hybrid control models, this discontinuity is followed via the set-interface, which acts as the region where discontinuity occurs.

Due to the constraint 'interventions are allowed only at the jump-times of a (semi-)Markov process', the full state should contain information about the jumping times, i.e., the reduced state x_t needs some complement to describe the impulse model. Thus, the signal process y_t is used to translated the constraint as 'interventions are allowed only when $y_t = 0$', and since it is assumed that there are not jump at the initial time $t = 0$, it is also added (the constraint conditions) the condition $t > 0$. As mentioned earlier, the Markov process (x_t, n_t) represents the uncontrolled evolution in the switching control model without constraints; and hence, a Markov process (x_t, n_t, y_t) represents the uncontrolled evolution for the constrained model.

This was essentially the notation in the Example 2.1, but in order to harmonize this notation with that of the impulse control model used in Section 3.3, it seems better to use a Markov process (x'_t, n_t, y_t) as the evolution of state in a switching control problem, i.e., $x_t = (x'_t, n_t)$. However, if (conditioned to x'_t) instead of a Markov process n_t, a semi-Markov process n_t is needed then, by adding $\mathfrak{s}_t = t - \sup\{s \le t : n_s \ne n_t\}$, the notation $x_t = (x'_t, z_t)$, with $z_t = (\mathfrak{s}_t, n_t)$, yields a Markov process representing the reduced state of a switching control model.

As mentioned earlier, the discrete-type component n_t used in the hybrid model is not necessary for (or used in) the impulse control model, where an instantaneous change is interpreted as a jump in the state, rather than as a change of mode (of operation) as in a switching control model. Thus, the 'controlled' process n_t plays a more visible role in a switching control model, and even it may be a totally controlled variable (i.e., a state and a control variable at the same time).

Nevertheless, it could be important to point-out that the meaning of the process n_t in the previous hybrid model is not exactly the same as in the switching model, actually, n_t is going to be part of the reduced state x_t as either $x_t = (x'_t, n_t)$ in a Markov case, or $x_t = (x'_t, z_t)$ with $z_t = (\mathfrak{s}_t, n_t)$ in semi-Markov case. Remark that in our hybrid model, the continuous-type (alternately, discrete-type) component refers to processes having piecewise continuous (alternately, constant) trajectories, i.e., a continuous-type component like x_t is supposed continuous in probability and may include a discrete-valued component (a jump component), but the actions of an impulse/switching control produces discontinuities in probability like the so-called discrete-type component n_t in hybrid models.

In practical situations and for computational purposes, the switching model and the impulse model are treated as different formulations. However, both are essentially the same, but as expected, the difference is mainly on the assumptions imposed on the data of the problems. In the remainder of this section, a discussion on how to convert a switching model into an impulse model, both with constraints, is given in some details.

4.1. Process without intervention. In this model, there is only one type of controls, namely switching from one operation mode to another one. So, without intervention means the uncontrolled process (x_t, y_t) as given in Section 3.1.

Let us describe a switching model that includes Example 2.1, but with a more general discrete component and based on the notation of the impulse model. This is, within Assumption 3.1 (a) on the Markov process x_t, assume that, (1) the state the reduced state $x_t = (x'_t, \mathfrak{s}_t, n_t)$ belongs to $E = E' \times \mathbb{R}^+ \times \mathbb{N}$, for a Polish space E'); (2) x'_t is like the x'_t in the example, n_t indicates the operation mode of x'_t, i.e., x'_t is a Markov process with a C_0-semigroup $\Phi^i(t)$ when $n = i$; (3) n_t is a semi-Markov process with values in $\mathbb{N}$, and $\mathfrak{s}_t = t - \sup\{s \le t : n_s \ne n_t\}$ is its *waiting time* (time elapsed since the last switching, similar to a signal process, but not necessarily equal to y_t), so that $z_t = (\mathfrak{s}_t, n_t)$ is Markov processes, with a C_0-semigroup $\Phi_z(t)$, i.e., $\Phi_z(t)C_0(\mathbb{R}^+ \times \mathbb{N}) \subset C_0(\mathbb{R}^+ \times \mathbb{N})$, $\forall t \geq 0$, and infinitesimal generator A_z with domain $\mathcal{D}(A_z) \subset C_0(\mathbb{R}^+ \times \mathbb{N})$,

$$A_z\varphi(\mathfrak{s}, n) = \partial_{\mathfrak{s}}\varphi(\mathfrak{s}, n) + r(\mathfrak{s}, n)\sum_{i\neq n} p_i(\mathfrak{s}, n)[\varphi(0, i) - \varphi(\mathfrak{s}, n)],$$

where $\sum_{i\neq n} p_i(\mathfrak{s}, n) = 1$, $p_n(\mathfrak{s}, n) = 0$, and $p_i(\mathfrak{s}, n)$ are non-negative continuous functions, and the intensity $r(\mathfrak{s}, n)$ is a continuous function such that $0 < c_0 \leq r(\mathfrak{s}, n) \leq c_1$, and for simplicity, we may assume $p_i(x, n) = 0$ for any $i \geq N$, i.e., switching withing the modes $n = 1, \ldots, N$. The semi-Markov process n_t may be called the *automatic switching process*, and if it is a Markov process then the waiting time process $\mathfrak{s}_t$ is not a necessary information for the reduced state of the system. Actually, if n_t is a purely jump Markov process then $r(\mathfrak{s}, n) = r$ constant, i.e., n_t is a compound Poisson process.

The constraint on the control is enforced via the signal process y_t, and with these data, Assumptions 3.1 is satisfied. If only a finite of modes is necessary then mode-state space $\mathbb{N}$ is replaced by a finite set $\mathcal{N}$, where the modes can be labeled $\{1, 2, \ldots, N\}$.

There is an interesting situation called *automaton*, which is produced by a fixed set-interface. For instance, when the state reaches a certain value (including the constraint) then the mode n_t changes to another fixed value. This would be the case in Example 2.1 if we allow a mode 3 meaning 'no operational mode' (or idle) that required an immediate switching. Actually, depending on the model, even a switching-rate from mode 2 to 3 (like switching to mode 2 from mode 1) may be used. However, the system could not remain idle (mode 3) for long, and some cost should be applied, and more details are necessary. This mandatory change does form part of the control actions, and this case is not included in what follows. The uncontrolled process in our model, is continuous in probability, contrary to an automaton process, which may be discontinuous in probability.

Remark that the automatic switching process n_t cannot usually be constructed from a Markov chain $\{n_{\tau_i} : i = 0, 1, \ldots\}$ in $\mathbb{N}$, with $\{\tau_i\}$ being the sequence of times in between two consecutive jumps. Its requires the Markov chain $\{(n_{\tau_i}, \tau_i) : i = 0, 1, \ldots\}$ in $\mathbb{N} \times \mathbb{R}^+$. The reader may compare with typical models, e.g., see Yin and Zhang [38, 39], Yin and Zhu [40], among others. Thus, n_t is a simple semi-Markov process with values in $\mathbb{N}$, and its waiting time $\mathfrak{s}_t = t - \sup\{s \leq t : n_s \leq n_t\}$ is also defined in a similar way to our signal process y_t. Actually, we may have the functions r and p also depending on x', and the Markov property of z_t can be understood as conditioned to x'_t.

Let us first recall Example 2.1 with the above notations. We have $x_t = (x'_t, n_t)$, with x_t in E, n_t in $\{1, 2\}$ so that x_t is an 'ordinary' switching process. Without control, n_t is a Markov chain, which can go from 1 to 2 and remains in 2 (i.e., 2 is an absorbing state), and n_t is independent of x'_t

$$A\varphi(x, n) = \begin{cases} A^1_{x'}\varphi(x', 1) + a[\varphi(x', 2) - \varphi(x', 1)], & \text{if} \quad n = 1, \\ A^1_{x'}\varphi(x', 2), & \text{if} \quad n = 2. \end{cases}$$

Next, the only control action is to go from $n = 2$ to $n = 1$, and it can be applied only at the jump-times of y_t, which is independent of $x_t = (x'_t, n_t)$,

$$A_y \varphi(y) = \partial_y \varphi(y) + \lambda(y)[\varphi(0) - \varphi(y)].$$

Note that the times in between two consecutive jumps from $n = 1$ to $n = 2$ form an IID sequence with a common exponential distribution with parameter a, and if one desires a little more general model, say with a 'nice' distribution other than the exponential, some conceptual problems appear, i.e., the (uncontrolled) switching Markov process n_t could not be constructed anymore from a Markov chain with values in $\{1, 2\}$. This suggests that perhaps, a more suitable model would have a switching process modeled as a simple semi-Markov process n_t, instead of a Markov process. Thus, we could consider an example like Example 2.1 where the Markov process n_t becomes a Markov process $(\mathfrak{s}_t, n_t)$, derived from a Markov chain in $\{1, 2\} \times \mathbb{R}$, i.e., the infinitesimal generator would be $A\varphi(\varphi(x', \mathfrak{s}, 2) = A^1_{x'}(\varphi(x', \mathfrak{s}, 2)$ and

$$A\varphi(x', \mathfrak{s}, 2) = A^1_{x'}\varphi(x', \mathfrak{s}, 1) + \partial_{\mathfrak{s}}\varphi(x', \mathfrak{s}, 1) + a(\mathfrak{s})[\varphi(x', 0, 2) - \varphi(x', \mathfrak{s}, 1)].$$

Moreover, we could use $\tilde{n}$ in the general model to distinguish the uncontrolled switching process $\tilde{n}_t$ from the controlled switching process n_t.

4.2. The switching procedure. Now, a 'switching' (action) is to intervene in the system evolution to produce an instantaneous change of operating mode from the current mode n to another mode n' (usually different from n), and in most cases, there are some constraints on the modes that are allowed. Thus, assume given a subset $N(x)$ of $\mathbb{N}_0$, with $x = (x', \mathfrak{s}, n)$ in the reduced state-space $E = E' \times \mathbb{R}^+ \times \mathbb{N}$, but usually, $N(x)$ depends only on n in $\mathbb{N}_0$. Actually, in terms of the impulse control problem, instead of given $N(x)$, let us suppose, for instance, that the set $\Gamma(x)$ appearing in condition (3.10) satisfies

$$\Gamma(x) = \Gamma(x', \mathfrak{s}, n) = \{(x', 0, n') : n' \in N(x)\} \subset \{x'\} \times \{0\} \times \mathbb{N}_0, \tag{4.1}$$

in the switching control model, and with this notation, the arguments of the Controlled Process Section 3.3 can be applied.

A precise definition of $\Gamma(x', \mathfrak{s}, n)$ depends on the specific problem which is being considered. For instance, in Example 2.1 with n_t semi-Markov (as at the end of the previous subsection), one could imagine that the transition from mode 2 to mode 1 (using a control) is realized by the implementation of a new machine. Then there is some logic to consider that the result of the control for $(\mathfrak{s}_t, n_t)$ is $(0, 1)$, and in this case $\Gamma(x', \mathfrak{s}, n) = \{(x', 0, 1)\}$. Nevertheless, from the theoretical viewpoint, the only requirement is that $\Gamma(x', \mathfrak{s}, n)$ satisfies (3.10) and (3.12) , being understood that (in the present context) x' remains unchanged. Actually, more on this point is discussed a couple of paragraphs below.

Remark 4.1. Note that for the uncontrolled process, the process $\mathfrak{s}_t$ represents the time elapsed since the last 'automatic' (i.e., uncontrolled) switching. Hence, if a switching control is decided then it is necessary to specify how this 'auxiliary' (for the semi-Markov process n_t) process behaves. Among the many possibilities, there are two cases which have a particular interpretation: (1) the meaning of the process $\mathfrak{s}_t$ is retained after a switching control, i.e., $\mathfrak{s}_t$ is unchanged immediately after a switching control; or (2) the meaning of the process $\mathfrak{s}_t$ becomes 'time elapsed since the last (uncontrolled or controlled) switching', i.e., $\mathfrak{s}_t$ is reset to 0 immediately after a switching control and the expression $\mathfrak{s}_t = t - \sup\{s \le t : n_s \le n_t\}$ remains valid. All this is less important for the impulse control model, but it seems more relevant in the switching control model since n represents the current mode of operation for the system. This last option (2) is assumed with the above choice of $\Gamma(x) \subset \{x'\} \times \{0\} \times \mathbb{N}_0$. However, if the option (1) is preferred then

$$\Gamma(x) = \Gamma(x', \mathfrak{s}, n) = \{(x', \mathfrak{s}, n') : n' \in N(x)\} \subset \{x'\} \times \{\mathfrak{s}\} \times \mathbb{N}_0. \tag{4.2}$$

Certainly, there are other valid choices (depending on the desired switching model) and if (x'_t, n_t) is Markov (instead of semi-Markov) process then the process $\mathfrak{s}_t$ is not necessary and this discussion is irrelevant.

The operator M becomes

$$Mv(x) = \inf\big\{c(x,\xi) + v(\xi) : \xi \in \Gamma(x),\ n' \ne n\big\}, \tag{4.3}$$

where $c(x,\xi) = c(x;\xi) = c(x', \mathfrak{s}, n; x', 0, n')$. Certainly, to match Example 2.1, we have to replace $\mathbb{N}_0$ with $\{1,2\}$ and take $N(1) = \{1\}$ (which means that switching is certainly not optimal, since its cost is strictly positive).

4.3. Solving the HJB equation. We have seen in the previous subsections that our switching problem with constraint for $(x_t, y_t) = (x'_t, \mathfrak{s}_t, n_t, y_t)$ satisfies the assumptions of Section 3 . Now the results of [28, 29, 31] can be directly used to solve this switching control problem, i.e., to solve the HJB equation and to identify an optimal control. Thus, we only describe the method, referring to [28, 29, 31] for the proofs.

Starting with the HJB equation (3.17) of the auxiliary problem, we use the classical iterations (e.g., see Bensoussan and Lions [3]):

$$\begin{gathered} u_0^n(x) = \min\Big\{\mathbb{E}_{x0}\Big\{\int_0^\tau \mathrm{e}^{-\alpha t} f(x_t, y_t)\mathrm{d}t + \mathrm{e}^{-\alpha\tau} u_0^n(x_\tau)\Big\}, Mu_0^{n-1}(x)\Big\}, \\ \text{for any } n \ge 1 \text{ with } u_0^0(x) = \mathbb{E}_{x0}\Big\{\int_0^\infty \mathrm{e}^{-\alpha t} f(x_t, y_t)\mathrm{d}t\Big\}. \end{gathered} \tag{4.4}$$

This is a sequence of HJB equations corresponding to a sequence of optimal stopping problems with constraint, typically,

$$w(x) = \min\Big\{\mathbb{E}_{x0}\Big\{\int_0^\tau \mathrm{e}^{-\alpha t} f(x_t, y_t)\mathrm{d}t + \mathrm{e}^{-\alpha\tau} w(x_\tau)\Big\}, \psi(x)\Big\}, \tag{4.5}$$

with $\psi(x) = Mu_0^{n-1}(x)$, and $w^0(x) = u_0^0(x)$. Indeed, the problem (4.5) can be seen as a discrete time optimal stopping problem for the Markov chain (3.6), and its corresponding HJB equation can be written as

$$w(x) = \min\big\{f_\alpha(x) + Pw(x), \psi(x)\big\}, \tag{4.6}$$

where $Pw(x) = \mathbb{E}_{x0}\{\mathrm{e}^{-\alpha\tau} w(x_\tau)\}$ as in Remark 3.7.

Then, **under the assumptions of Section 3**, we have:

Theorem 4.2 ([28])**.** *If the terminal cost ψ belongs to $C_b(E)$ then the HJB equation (4.6) has a unique solution w in $C_b(E)$, which is the optimal cost of the* *corresponding optimal stopping problem. Moreover, the following stopping time* $\hat\theta = \tau_{\hat\eta}$ *corresponding to discrete stopping time*

$$\hat\eta = \inf\big\{n \geq 0 : w(X_n) = \psi(X_n)\big\}$$

is optimal, with the convention that $\tau_\infty = \infty$ *and that the infimum over an empty set is* ∞.

Certainly, in Theorem 4.2 the data corresponding to impulse/switching cost function $c(x, \xi)$ has been replaced by the assumption that ψ belongs to $C_b(E)$. Thus, because the operator M maps $C_b(E)$ into itself, this means that for each $n \geq 1$, the HJB equation (4.4) for $u_0^n(x)$ has a unique solution in $C_b(E)$, which is the optimal cost of the corresponding optimal stopping problem and the first exit time of the continuation region $\{x : u_0^n(x) < M u_0^{n-1}(x)\}$ is optimal, i.e., $\hat\theta = \tau_{\hat\eta}$ with the convention that $\theta_\infty = \infty$. Hence, for each $n \geq 1$ we have

Corollary 4.3 ([28])**.** *The HJB equation* (4.4) *has a unique solution* u_0^n *in* $C_b(E)$, *and the sequence* $\{u_0^n\}$ *is monotone decreasing and converges uniformly to the unique solution of* (3.17), *i.e.,*

$$u_0(x) = \min\big\{f_\alpha(x) + P u_0(x), M u_0(x)\big\}, \tag{4.7}$$

where M *is given by* (3.11), *i.e., like* (4.6) *with* $\psi = M u_0$. *Moreover, each function* $u_0^n(x)$ *is the optimal cost of an impulse control problem with at most n impulses, i.e.,*

$$u_0^n(x) = \inf\big\{J_{x,0}(\nu) : \nu = \{(\eta_i, \xi_i) : 1 \leq i \leq n\}\big\}, \quad \forall x \in E, \tag{4.8}$$

and $u_0(x)$ *is the optimal cost* (3.15) *with* $y = 0$. *Furthermore, the impulse control obtained from the continuation region is optimal, for each of the optimal impulse control costs* u_0^n *and* u_0.

Next, knowing $u_0(x) = u_0(x, 0)$, the relation (3.17) gives $u_0(x, y)$, for $y > 0$, and the expression (3.18) gives $u(x, y)$. Thus

Theorem 4.4 ([29, 31])**.** *The function*

$$u_0(x, y) = \mathbb{E}_{xy}\Big\{\int_0^\tau e^{-\alpha t} f(x_t, y_t) dt + e^{-\alpha\tau} u_0(x_\tau)\Big\}, \quad y > 0,$$

is the optimal cost (3.15), *i.e.,*

$$u_0(x, y) = \inf\big\{J_{xy}(\nu) : \nu \in \mathcal{V}_0\big\}$$

and there exists an optimal control $\hat\nu_0 = \{(\hat\theta_i, \hat\xi_i) : i \geq 1\}$, *with* $\hat\theta_i$ *obtained from* $\hat\eta$, *namely*

$$\hat\theta_1 = \inf\big\{t \geq 0 : u_0(x_t, y_t) = M u_0(x_t, y_t),\ y_t = 0\big\},$$

and the $\hat\theta_i$, $i > 1$ *are obtained by translations, and* $\hat\xi_i = \hat\xi(x_{\theta_i}^{i-1})$, *where* $\hat\xi(x)$ *realizes the infimum in* $M u_0(x, 0)$.

Theorem 4.5 ([29, 31])**.** *The function*

$$u(x, y) = \mathbb{E}_{xy}\Big\{\int_0^\tau e^{-\alpha t} f(x_t, y_t) dt + e^{-\alpha\tau} u_0(x_\tau)\Big\}, \quad y \geq 0,$$

belongs to the domain $\mathcal{D}(A_{xy}) \subset C_b(E \times \mathbb{R}^+)$ of the weak generator of the uncontrolled process, and the equation

$$-A_{xy}u(x,y) + \alpha u(x,y) + \lambda(x,y)[u(x,0) - Mu(x,0)]^+ = f(x,y)$$

is satisfied. Moreover, $u(x,y)$ is the optimal impulse cost (3.14)*, i.e.,*

$$u(x,y) = \inf\left\{J_{xy}(\nu) : \nu \in \mathcal{V}\right\}$$

and there exists an optimal control $\hat{\nu}$ which is obtained from $\hat{\nu}_0$ by translation with τ_1, the first jump of y_t.

Recall that $u(x,y) = u_0(x,y)$ for any $y > 0$ and that, the condition (3.12) ensures that in each step of the HJB solving the optimal stopping problems with stopping cost $\psi = Mu_0^n$ is a continuous and bounded functions and so, the results in [28] can be applied. However, the second part of condition (3.9) on the function $c(x,\xi)$ is needed to prove the uniform convergence of $u_0^n \to u_0$, and the condition (3.10) ensures that two simultaneous intervention is certainly not optimal. Note that comparing with the hybrid models, $D^\wedge = \emptyset$, $D^\vee = E \times \{0\}$.

5. Weaker Assumptions

For switching control models, it is often necessary to weaken condition (3.9) relative to the cost-per-impulse $c(x,\xi)$, i.e., to allow the possibility $c(x,\xi) = 0$ for some $x \in E$ and $\xi \in \Gamma(x)$. However, in this section, assumption (3.10) and condition (3.12) on the operator M –which really imposes conditions on the mapping $\Gamma(x)$– is certainly retained in our formulation.

All this involves reconsidering the arguments in [29, 31] in such a way that assumption (3.9) can be replaced by

$$\begin{aligned} &f : E \times \mathbb{R}^+ \to \mathbb{R}^+ \ \text{ bounded and continuous}, \quad \alpha > 0, \\ &c : E \times E \to [0, +\infty[\ \text{ bounded and continuous}, \end{aligned} \tag{5.1}$$

without changing the results.

5.1. An initial discussion. Our discussion goes as follows:

(1) Inventory management is a prototype of optimal impulse control, where x represents the inventory levels of the various items and a simple expression of $c(x,\xi)$ has the form $c_0 \mathbb{1}_{\{x \neq \xi\}} + c_1(x,\xi)$ with a constant $c_0 > 0$ and a non-negative function c_1. However, a prototype of optimal switching control is an energy and power system management, where x represents the various power levels provided with the current active configuration n (which is, for instance, a label from 1 to $N < \infty$ or a more complicate digital designation), and a simple expression of $c(x,n;\xi,\eta)$ could be $c_0 \mathbb{1}_{\{n \neq \eta,\, \eta \in \mathcal{N}\}} + c_1(x,n;\xi,\eta)$ with a constant $c_0 > 0$, a proper subset $\mathcal{N}$ of configurations, and a non-negative function c_1, in other words, switching costs may be zero.

(2) As mentioned earlier, condition (3.12) ensures that at each step in the iteration (4.4) the solution $u_0^n(x)$ is continuous, but (3.9) relative to $c(x,\xi) \geq c_0 > 0$, is used to establish the uniform convergence of $u_0^n(x)$ to $u_0(x)$ and the uniqueness of the solution of the HJB equation, similarly to the standard case of impulse control problem (i.e., without the constraint given by a signal process y_t).

(3) The condition $c(x,\xi) \geq c_0 > 0$ plays an essential role, it allows to deduce that the (Markov) optimal impulse control $\hat{\nu}$ constructed from the continuation region $\{x \in E : u_0(x) < Mu_0(x)\}$ is indeed an impulse control, i.e., the sequence $\{\hat{\theta}_i\}$ of impulse-times satisfies $\hat{\theta}_i \to \infty$ with probability one. For instance, if ν is an impulse control with a finite cost, i.e., $J_{x0}(\nu) < C_0 = 2\sup_x u_0^0(x)$, then

$$c_0 \mathbb{E}_{x0}\Big\{ \mathrm{e}^{-\alpha\theta_k} \Big\} \leq \mathbb{E}_{x0}\Big\{ \sum_{i \geq k} c(\xi_{i-1},\xi_i)\mathrm{e}^{-\alpha\theta_i} \Big\} \to 0 \;\text{ as }\; k \to \infty$$

since, $\mathbb{E}_{x0}\{\sum_{i\geq 1} c(\xi_{i-1},\xi_i)\mathrm{e}^{-\alpha\theta_i}\} \leq C_0$.

(4) Adding the restriction 'no simultaneous impulses' does not change the infimum of the cost $J_{xy}(\nu)$ over all (admissible) impulse controls, since condition (3.10) implies precisely, that any transition from state x to ξ obtained by two or more successive (simultaneous) impulses can also be achieved with just one impulse with equal or lower cost.

Therefore, for an impulse control model with a constraint given by a signal process, we can use the assertions in Remark 3.5 and Definition 3.6 to deduce that $\mathbb{E}_{xy}\{\mathbb{1}_{\tau_n<T}\} \to 0$ as $n \to \infty$, uniformly in (x,y) within $E \times \mathbb{R}^+$, for any fixed $T > 0$. Hence, because the condition (3.10) is retained, simultaneous impulses are ruled-out (since they are not better than single impulses), and thus, 'zero-admissible discrete time impulse controls' $\{(\eta_i,\xi_i) : i \geq 1\}$ have the property $\eta_i < \eta_{i+1}$ whenever $\eta_i < \infty$. Nevertheless, under the condition (3.10) and *without* assuming $c(x,\xi) \geq c_0 > 0$, it not so clear that the optimal impulse control obtained from the continuation region satisfies $\hat{\theta}_i \to \infty$, since it is not clear that all optimal impulses form the stopping region $\{x : u_0(x) = Mu_0(x)\}$ moves the state to the continuation region $\{x : u_0(x) < Mu_0(x)\}$, and so, an infinite number of simultaneous impulses may occur, i.e., moving the state from ξ to $\xi' \neq \xi$ forward and backward with zero-cost $c(\xi,\xi') = c(\xi',\xi) = 0$ (certainly, this makes more sense for a switching model than an impulse model, but the possibility is there).

5.2. No simultaneous impulses. As mentioned above, (3.10) implies that impulse (or switching) controls with simultaneous impulses are not necessary, since a strictly lower cost is not obtained by allowing simultaneous impulses (or switchings). But, this does not ensure that if $v(x) \leq Mv(x)$ for every x in E then the selector $\hat{\xi}(x) = \hat{\xi}(x,v)$ realizing the infimum in $Mv(x)$ –i.e., satisfying $v(\hat{\xi}(x,v)) + c(x,\hat{\xi}(x,v)) = Mv(x)$, as given in assumption (3.12)–, can be chosen (i.e., can be modified) so that $v(\hat{\xi}(x,v)) < Mv(\hat{\xi}(x,v))$, unless the 'continuation region' $\{x : v(x) < Mv(x)\}$ is empty. Such a minimizer would imply that the optimal impulse control associated with the continuation region does not have any simultaneous impulses. This difficulty can be overcome by assuming

$$\begin{aligned} &\emptyset \neq \Gamma(x) \text{ is closed } \forall x \in E, \quad \Gamma(\xi) \subset \Gamma(x) \quad \forall \xi \in \Gamma(x), \\ &\text{and } \; c(x,\xi) + c(\xi,\xi') > c(x,\xi'), \quad \forall \xi \in \Gamma(x),\ \forall \xi' \in \Gamma(\xi), \end{aligned} \tag{5.2}$$

which is a stronger (or strict) version of condition (3.10).

Lemma 5.1. *Let us assume* (5.1) *on* $c(x,\xi)$*,* (5.2) *in lieu of* (3.10)*, and suppose that the impulse operator* M *defined by* (3.11) *satisfies* (3.12)*. If* v *is a function in*

$B^+(E)$ and $v(x) = Mv(x)$ for some x in E, then its continuation region $C_v = \{x \in E : v(x) < Mv(x)\}$ is nonempty and any selector $\hat{\xi}(x) = \hat{\xi}(x,v)$ as in condition (3.12) *satisfies $v(\hat{\xi}(x,v)) < Mv(\hat{\xi}(x,v))$.*

Proof. Indeed, if $v(x) = Mv(x)$ with x in E then $v(x) = c(x, \hat{\xi}(x)) + v(\hat{\xi}(x))$. Now, let make an argument by contradiction, i.e., suppose that the $\hat{\xi}(x)$ does not belong to the continuation region. Therefore, if $v(\hat{\xi}(x)) = Mv(\hat{\xi}(x))$ then deduce that $v(\hat{\xi}(x)) = c(x, \hat{\xi}(\hat{\xi}(x))) + v(\hat{\xi}(\hat{\xi}(x)))$. Hence, combine these two equalities to get

$$v(x) = c(x, \hat{\xi}(x)) + v(\hat{\xi}(x)) = c(x, \hat{\xi}(x)) + c(\hat{\xi}(x), \hat{\xi}(\hat{\xi}(x))) + v(\hat{\xi}(\hat{\xi}(x)))$$

and in view of (5.2), it follows

$$v(x) > c(x, \hat{\xi}(\hat{\xi}(x))) + v(\hat{\xi}(\hat{\xi}(x))) \geq Mv(x),$$

which is a contradiction. □

Remark 5.2. The operator M given by (3.11) is 'almost' the effective impulse (or switching) operator, since an actual intervention should move the state, i.e., if some x belongs to $\Gamma(x)$ then the correct expression for M should be

$$Mv(x) = \inf \big\{ c(x,\xi) + v(\xi) : \xi \in \Gamma(x),\, \xi \neq x \big\}.$$

and because the set $\{\xi \in \Gamma(x),\, \xi \neq x\}$ is not necessarily closed, the compactness of $\Gamma(x)$ does not ensure that a minimizer should always exist. This is usually overcome by assuming that $c(x,x) > 0$ for every $x \in E$. In any case, if

$$\bar{M}v(x) = \inf \big\{ c(x,\xi) + v(\xi) : \xi \in \Gamma(x),\, \xi \neq x,\, Mv(\xi) \neq v(\xi) \big\}$$

then under condition (3.10) it follows that $\bar{M}v = Mv$, and therefore, condition (5.2) implies that the minimizer $\hat{\xi}(x,\cdot)$ of M as in assumption (3.12) can be used also for $\bar{M}$, and the requirement $x \neq \hat{\xi}(x,v)$ (for x in the stopping region) since this would force $\hat{\xi}(x,v) = x$ to remain in the stopping region instead of moving to the continuation region; and clearly, $c(x, \hat{\xi}(x,v)) + v(\hat{\xi}(x,v)) = v(x)$ and $\hat{\xi}(x,v) = x$ implies $c(x,x) = 0$. Also, under condition (3.10), if the mapping $\Xi : (x,\xi) \mapsto \xi'$ from $\{(x,\xi) : x \in E,\, \xi \in \Gamma(x)\}$ into $\Gamma(x) \subset E$ is measurable then an impulse (or switching) control composed (a finite number of times) with Ξ is transformed into an impulse (or switching) control with a possible lower cost and without simultaneous impulses, e.g., $\Xi(\hat{\xi}(x,v), v)$ for any minimizer $\hat{\xi}(x,v)$ as in (3.12). However, this does not ensure that the impulse control obtained from the continuation region satisfies $\hat{\theta}_i \to \infty$, an infinite number of simultaneous impulses may occur.

5.3. Markov impulses and results. Therefore, Lemma 5.1 implies that the (Markov) optimal impulse (or switching) control $\hat{\nu}$ constructed from the continuation region does not have simultaneous impulses (or switchings). To clarify this last point, let us give more details on the construction of a feedback impulse control. Indeed, the 'admissible Markov' impulse (or switching) control $\{\eta_i, \theta_i, \xi_i : i \geq 1\}$ corresponds to a stopping region $S = S_0 \times \{0\}$ with $S_0 \subset E$, and an impulse function satisfying $\xi(x,0) \in \Gamma(x)$, for any $x \in S_0$.

Thus $\theta_i = \infty$ and $\eta_i = \infty$ are possible with the understanding that $\theta_\infty = \infty$, $\eta_\infty = \infty$ and $\theta_i = \infty$ if and only if $\eta_i = \infty$. Therefore, beginning with $\tau_0^0 = 0$,

$\eta_0 = 0$, $\theta_0 = 0$, $\xi_0 = x$, and uncontrolled Markov evolution (of the state) $\{(x_t^0, y_t^0) : t \geq 0\}$, let us iterate (for $i = 0, 1, \ldots$) the expression

$$\tau_k^i = \inf\{t > \tau_{\eta_i}^0 : y_t^i = 0\},\ \forall\, k > \eta_i, \qquad \eta_{i+1} = \inf\{k > \eta_i : x_{\tau_k^i}^i \in S_0\},$$
$$\theta_{i+1} = \tau_{\eta_{i+1}}^i, \quad \tau_{\eta_{i+1}}^{i+1} = \theta_{i+1}, \quad \xi_{i+1} = \xi(x_{\theta_{i+1}}^i, 0), \quad x_t^{i+1}, y_t^{i+1},\ t \geq \theta_i,$$

where $\{(x_t^i, y_t^i) : t \geq \theta_i\}$ is the Markov evolution after the i-impulse (or switching), see Definition 3.9 (iv). This shows that $\tau_{\eta_{i+1}}^{i+1} = \theta_{i+1} = \tau_{\eta_{i+1}}^i \geq \tau_{\eta_i}^i$ and $\theta_{i+1} - \theta_i \geq \tau_{\eta_i+1}^i - \tau_{\eta_i}^i \sim \tau_1$ (i.e., equal distribution), which is also valid for any admissible impulse control, not necessarily Markov. Since there exists a constant $0 < a < 1$ such that $E_{x0}\{\mathrm{e}^{-\alpha\tau_1}\} \leq a < 1$, for any $x \in E$, obtained in Definition 3.6, all these arguments and the Markov property yield the estimate

$$E_{x0}\{\mathrm{e}^{-\alpha\theta_i}\} \leq a^i, \quad \forall x \in E,\ \forall i, \tag{5.3}$$

for some constant a in $]0, 1[$ (which may depend on α). Hence, we deduce that $\theta_i \to \infty$ as $i \to \infty$, uniformly in x.

The argument used in [29, Theorem 4.2, (4.13)] to show the uniform convergence of $u_0^n(x) \to u_0(x)$ on E –which is indeed the same 'analytic' argument as in Hanouzet and Joly [14]–, cannot be used now (because $c(x, \xi) \geq c_0 > 0$ is not retained, only $c(x, \xi) \geq 0$ is assumed). Thus, we propose to apply an argument similar to [24, 26], i.e., directly from

$$0 \leq u_0^n(x) - u_0(x) \leq \mathbb{E}_{x0}\Big\{\int_{\theta_n}^{\infty} f(x_t^n, y_t^n)\mathrm{e}^{-\alpha t}\mathrm{d}t\Big\} \leq$$
$$\leq \frac{1}{\alpha}\big(\sup_{x,y}\{|f(x, y)|\}\big)\mathbb{E}_{x0}\{\mathrm{e}^{-\alpha\theta_n}\}, \quad \forall n,\ \forall x \in E.$$

and estimate (5.3), we obtain the desired uniform convergence. Actually, considering $u_0^n(x, y)$ and $u_n(x, y)$, we may also argue as follow: If T_i denotes the first jump time of $s \mapsto y_{\theta_i+s}$ (for a control $\nu = \{(\theta_i, \xi_i) : i \geq 1\}$ zero-admissible and not necessarily Markov), then $\theta_n \geq \theta_{n-1} + T_{n-1}$, $\forall n \geq 2$, so $\theta_n \geq \theta_1 + T_1 + T_2 + \cdots + T_{n-1}$. From the construction of the probabilities P_{xy} corresponding to ν, we have $\mathbb{E}_{xy}^1\{\mathrm{e}^{-\alpha T_1}\} = \mathbb{E}_{xy}\{\mathbb{E}_{\xi 0}[\mathrm{e}^{-\alpha T_1}]\}$, and from the law of T_1, we get $\mathbb{E}_{xy}^1\{\mathrm{e}^{-\alpha T_1}\} \leq a < 1$ and more generally $\mathbb{E}_{xy}^{n-1}\{\mathrm{e}^{-\alpha T_{n-1}}\} \leq a$, i.e., $\mathbb{E}_{xy}\{\mathrm{e}^{-\alpha\theta_n}\} \leq a^{n-1}$. Hence, the convergence $u_0^n(x, y) \to u_0(x, y)$, uniformly on $E \times \mathbb{R}^+$, has been proved.

In any case, with the previous arguments we can show the following

Theorem 5.3. *The results of* Theorem 4.4 *and* Theorem 4.5 *remain valid (with minimal changes as discussed above) if the assumptions* (5.1) *and* (5.2) *are used in lieu of* (3.9) *and* (3.10)*, i.e., a discounted impulse/switching control model with a constraint given by a signal process can be solved.*

Note that the substitution of assumptions mentioned in Theorem 5.3 does not apply for the usual impulse control models without a constraint given by a signal process. Again, remark that under the weaker assumption (5.1), if the condition (3.10) is not assumed and a discrete time setting is considered, then multiple simultaneous impulses are logically possible in the definition of the auxiliary with optimal cost $u_0(x)$ given by (3.15) with $y = 0$. However, for the costs $u(x, y)$ and $u_0(x, y)$ given by (3.14) and (3.15) in a continuous time setting, we may argue that multiple simultaneous impulses should not be allowed, because a constraint like 'only one intervention at any given time' make sense from the modeling point of view. Nevertheless, if such a constraint is also assumed then we may solve the HJB equation for $u_0(x)$ and therefore we can obtain the costs $u(x, y)$ and $u_0(x, y)$, but the continuity on x would follow from imposing a maximum finite number of possible multiple simultaneous impulses, and even more, there may not be an optimal impulse control, since the expression of the optimal impulse (or switching) control does not necessary produces an impulse (or switching) control without multiple simultaneous interventions.

Let us clarify the meaning of zero-admissible impulse controls as the context of an extension of Corollary 4.3 without assuming the condition (3.10), i.e., when simultaneous impulses may occur, namely, the possibility $\eta_i = \eta_{i-1} < \infty$ with a positive probability for some $i \geq 0$, and this class of impulse controls is denoted by $\mathcal{V}_0' \supset \mathcal{V}_0$. In this section, the optimal costs $u_0^n(x)$ and $u_0(x)$ are defined without this possibility, i.e., a zero-admissible impulse control includes the condition $\eta_{i-1} < \eta_i$ whenever $\eta_{i-1} < \infty$, for any i. If optimal costs $\bar{u}_0^n(x)$ and $\bar{u}_0(x)$ were defined similarly, but allowing simultaneous impulses, i.e., $\eta_i = \eta_{i-1} < \infty$ is permitted then certainly, $\bar{u}_0^n(x) \leq u_0^n(x)$ and $\bar{u}_0(x) \leq u_0(x)$ and the equalities hold whenever assumption (3.10) is retained. With this in mind, we have

Theorem 5.4. *If the assumption* (3.9) *is replaced with* (5.1) *then the HJB equation* (4.4) *has a unique solution u_0^n in $C_b(E)$, and the sequence $\{u_0^n\}$ is monotone decreasing and converges uniformly to the* maximum *solution of* (3.17)*, i.e., u_0 solves* (4.7) *and any other solution u satisfies $u \leq u_0$. Moreover, each function $u_0^n(x)$ is the optimal cost of an impulse control problem with at most n impulses, i.e.,* (4.8) *holds, and $u_0(x)$ is the optimal cost* (3.15) *with $y = 0$, i.e., the minimization uses impulse controls satisfying $\eta_{i-1} < \eta_i$ whenever $\eta_{i-1} < \infty$, for any i. Furthermore, if $\bar{u}_0^n(x)$ denotes the optimal cost with possible simultaneous impulses then the impulse control $\{(\hat{\eta}_i, \hat{\xi}_i) : 1 \leq i \leq n\}$ obtained from the continuation region is optimal, but of course, it may have simultaneous interventions, i.e., $\hat{\eta}_i = \hat{\eta}_{i-1} < \infty$ with a positive probability for some $i \leq n$. However, if condition* (3.10) *with the measurable selector $\Xi(x, \xi, v)$ (see* Remark 5.2*) is retained then for any ν' in $\mathcal{V}_0'$ there exists another ν in $\mathcal{V}_0$ such that $J_{x,0}(\nu') \geq J_{x,0}(\nu)$, and so $\bar{u}_0^n(x) = u_0^n(x)$, and hence, the impulse control obtained from the continuation regions (i.e., $\{u_0^k < \psi_k = M u_0^{k-1}\}$, $k = 1, \ldots, n$ and the operator $\Xi(x, \xi, v)$) can be modified to be optimal, i.e., without having simultaneous interventions.*

This does not prove that the impulse control obtained from the continuation region corresponding to u_0 or $\bar{u}_0$ is optimal since we do not know that $\hat{\eta}_i \to \infty$ as $i \to \infty$ (which would need additional assumptions).

6. Possible Simultaneous Switchings

The assumption (3.10) should be partially dropped (i.e., retaining only the condition that $\Gamma(x)$ is a non-empty closed subset of E) to allow simultaneous impulses or switchings. This means that within the Definition 3.9, every impulse (or switching) control satisfied $\theta_i \leq \theta_{i+1}$, instead of (implicitly) assuming $\theta_i < \theta_{i+1}$ whenever $\theta_i < \infty$, and in both cases, it is also assumed that $\theta_i \to \infty$ as $i \to \infty$. In any case, impulse (or switching) controls are sequential decisions and therefore, $\theta_{i+1} = \theta_i < \infty$ means an impulse (or switching) time with two simultaneous impulses (or switchings), i.e., from $x_i = x^{i-1}_{\theta_i}$ the state is moved first to $\xi_i \in \Gamma(x_i)$ and then from ξ_i to the state $\xi_{i+1} \in \Gamma(\xi_i)$. With this setting, the expression of cost $J_{xy}(\nu)$ is the same as given by (3.13). There may be a finite number of impulses (or switchings) at a particular instant, but not an infinite number since this would violate the condition that $\theta_i \to \infty$ as $i \to \infty$ (and would not be realistic). This class of (zero-)admissible impulse or switching controls is denoted by $\mathcal{V}' \supset \mathcal{V}$ (and $\mathcal{V}'_0 \supset \mathcal{V}_0$), and referred to as 'simultaneous (time) interventions' since the time of interventions may satisfy $\theta_{i+1} = \theta_i < \infty$ for some i. As explained later on, this notation is modified to better understand of our setting.

6.1. Sequential construction. As explained earlier, these controls are sequential decisions and by convenience, $(\theta_0, \eta_0, \xi_0) = (0, 0, x)$ is added as well as $\theta_\infty = \eta_\infty = \infty$ and the condition $\theta_i = \infty$ if and only if $\eta_i = \infty$, and certainly, these cases are not considered interventions. Therefore, if the equality $\theta_i = \theta_{i+1}$ is allowed then necessarily the relation $\theta_{\eta_i} = \tau_{\eta_i}$ should be reconsidered. Indeed, using the fact that a signal occurs if and only if $y_t = 0$, the sequence of signals is now defined sequentially together with an impulse (or switching) control. As described in Subsection 3.3, θ_i and η_i are stopping times and ξ_i is 'adapted with values in $\Gamma(\cdot)$', and thus, the sequential procedure to define interventions can be expressed as follows:

• First, if $\{(x^0_t, y^0_t) : t \geq 0\}$ denotes the uncontrolled Markov evolution (of the state) and $\tau^0_k = \inf\{t > \tau^0_{k-1} : y^0_t = 0\}$ $(k \geq 1)$ is the sequence of signals with $\tau^0_0 = 0$, then $\theta_1 \geq 0$, $\eta_1 \geq 0$ and $\xi_1 \in \Gamma(x)$ are chosen satisfying $\theta_1 = \tau^0_{\eta_1}$, and the state is moved from $(x^0_{\theta_1}, y^0_{\theta_1})$ to $(\xi_1, y^0_{\theta_1})$ so that the Markov evolution $\{(x^1_t, y^1_t) : t \geq \theta_1\}$ continues anew (on the region $\theta_i < \infty$) with the initial condition $(x^1_{\theta_1}, y^1_{\theta_1}) = (\xi_1, 0)$, since $y^0_{\tau^0_k} = 0$ for every $k \geq 0$.

• Next, iterate for every $i \geq 1$, **if** $\theta_i = \infty$ (and $\eta_i = \infty$) **then** $\theta_{i+1} = \infty$, $\eta_{i+1} = \infty$ and ξ_{i+1} is irrelevant (i.e., no more interventions), **otherwise** (i.e., **if** $\theta_i < \infty$ and $\eta_i < \infty$) and if $\tau^i_k = \inf\{t > \tau^i_{k-1} : y^i_t = 0\}$ denotes the sequence $(k > \eta_i)$ of signals after $\tau^i_{\eta_i} = \theta_i$ **then** $\theta_{i+1} \geq \theta_i$, $\eta_{i+1} \geq \eta_i$ and $\xi_{i+1} \in \Gamma(x^{i-1}_{\theta_i})$ are chosen satisfying $\theta_{i+1} = \tau^i_{\eta_{i+1}}$, and the state is moved from $(x^i_{\theta_{i+1}}, y^i_{\theta_{i+1}})$ to $(\xi_{i+1}, y^i_{\theta_{i+1}})$, so that the Markov evolution $\{(x^{i+1}_t, y^{i+1}_t) : t \geq \theta_{i+1}\}$ continue anew (on the region $\theta_{i+1} < \infty$) with the initial condition $(x^{i+1}_{\theta_{i+1}}, y^{i+1}_{\theta_{i+1}}) = (\xi_{i+1}, 0)$ since $y^i_{\tau^i_k} = 0$ for every $k \geq i$, and now, $\tau^{i+1}_k = \inf\{t > \tau^{i+1}_{k-1} : y^{i+1}_t = 0\}$ denotes $(k > \eta_{i+1})$ the signals after $\tau^{i+1}_{\eta_{i+1}} = \theta_{i+1}$.

• Hence, for every $i \geq 1$, it should be understood that $\{(x^i_t, y^i_t) : t \geq \theta_i\}$ denotes the Markov evolution after the i-intervention applied at 'time' θ_i (or η_i in a discrete time model), with the signals (after $\tau^i_0 = \theta_i$) given by $\tau^i_k = \inf\{t > \tau^i_{k-1} : y^i_t = 0\}$, $k > \eta_i$. They satisfy $\theta_i \geq \theta_{i-1}$, $\eta_i \geq \eta_{i-1}$ and $\theta_i = \tau^i_{\eta_i}$ for every $i \geq 1$. For any $i = 1, 2, \ldots$, recall that $\theta_i < \infty$ if and only if $\eta_i = \infty$, that $\theta_\infty = \eta_\infty = \infty$, that

$\theta_i = \infty$ may have a non-zero probability (an intervention requires $\theta_i < \infty$, and so, the impulse ξ_i may not be defined when $\theta_i = \infty$), that the intervention i occurs when the stopping time $\theta_i < \infty$, and that the upper index i of the process and the signals, $\{(x_t^i, y_t^i) : t \geq \theta_i\}$ and $\{\tau_k^i : k \geq \eta_i\}$, correspond to notation 'after' the i intervention.

This procedure used to define (and choose) an impulse (or switching) control includes the possibility that $\theta_i = \theta_{i-1} < \infty$ (or equivalently, $\eta_i = \eta_{i-1} < \infty$) for some $i \geq 1$, and to actually produce a 'valid' impulse (or switching) control the choice should ensure that $\theta_i \to \infty$ (and $\eta_i \to \infty$) as $i \to \infty$, since possible simultaneous impulses (or switchings) are allowed and regarded as simultaneous interventions. Clearly, requesting that at each step in the iteration $i \geq 1$ the choice of θ_i (and η_i) satisfies $\theta_i < \theta_{i-1}$ whenever $\theta_{i-1} < \infty$ (and $\eta_i < \eta_{i-1}$ whenever $\eta_{i-1} < \infty$) will overrule any possible simultaneous interventions, and because the signals $\tau_k^i \to \infty$ as $k \to \infty$, this implies that $\theta_i \to \infty$ (and $\eta_i \to \infty$) as $i \to \infty$ (as in previous sections). With this in mind, it seems unnecessary to change the initial notation of our impulse (or switching) control model, since $\theta_{i+1} = \theta_i < \infty$ means that the $i+1$ and i interventions are applied simultaneously.

6.2. k-simultaneous impulses. Nevertheless, we may think that 'an intervention' (at a given time) means a finite number of simultaneous impulses (or switchings) that transfer the state successively from $x = x_1$ to $\xi_1 = x_2$ in $\Gamma(x_1)$, next from x_2 to $\xi_2 = x_3$ in $\Gamma(x_2)$, and so forth, to end with a transfer from x_k to $\xi_k = \xi$ in $\Gamma(x_k)$. Thus a 'natural' cost due to this intervention is $\bar{c}(x,\xi) = c(x_1,\xi_1) + \cdots + c(x_k,\xi_k)$, and clearly, the running cost function $f(x,y)$ is 'non sensible' to this multiple impulses (i.e., for f only the states x before and ξ after simultaneous impulses are relevant). As mentioned in the previous sections, there is no need to consider possible simultaneous impulses (or switchings), when condition (3.10) is assumed, so this condition is partially dropped in this section. A formal setting allowing multiple simultaneous impulses may be formulate as follows:

Definition 6.1. If $\Gamma(x)$ is as given in Section 3.2, with condition (3.10) partially dropped, i.e., assuming only that $\emptyset \neq \Gamma(x)$ is closed, then a *k-simultaneous impulse* (or in short, *k-impulse*) from x to ξ is a k-uple $\xi = (\xi_1, \ldots, \xi_k)$ such that $x = x_1$, $\xi_1 \in \Gamma(x_1)$, $x_2 = \xi_1$, $\xi_2 \in \Gamma(x_2)$, $\ldots$, $x_k = \xi_{k-1}$, $\xi_k \in \Gamma(x_k)$, ξ_k, e.g., a (single) impulse as used in previous sections is an 1-impulse from x to ξ. A 'multiple simultaneous impulses' is used as alternative name when the integer k of simultaneous impulses is not necessarily mentioned, and implicitly understood that only a finite number of simultaneous impulses is used. Also, define the iterates of $\Gamma(x)$ as

$$\Gamma_k(x) = \{(\xi_1, \ldots, \xi_k) : \xi_1 \in \Gamma(x), \xi_2 \in \Gamma(\xi_1), \ldots, \xi_k \in \Gamma(\xi_{k-1})\} \subset E^k,$$

for any $k \geq 1$. Clearly, $\Gamma(x)$ is identified with $\Gamma_1(x)$ and for any integer $\kappa \geq 1$ define

$$\Gamma^\kappa(x) = \Gamma_1(x) \cup \Gamma_1(x) \cup \cdots \cup \Gamma_{\kappa-1}(x) \cup \Gamma_\kappa(x),$$

the set of possible k-impulses with $1 \leq k \leq \kappa$. Similarly, if the set $\{x \in E : x \in \Gamma(x)\}$ is non-empty then it may be useful to define the set $\Gamma_k'(x) \subset \Gamma_k(x)$ of all

k-impulses with $\Gamma'(x) = \{\xi \in \Gamma(x) : \xi \neq x\}$ in lieu of $\Gamma(x)$, which are referred to as *strict* k-simultaneous impulse. The function $c(x,\xi)$ is initially defined for any $x \in E$ and $\xi \in \Gamma(x)$ and therefore, extended to any $(\xi_1,\ldots,\xi_k) \in \Gamma_k(x)$ by linearity, i.e.,

$$c(x,\xi_1,\ldots,\xi_k) = c(x,\xi_1) + c(\xi_1,\xi_2) + \cdots + c(\xi_{k-1},\xi_k),$$

and eventually conveniently extended to $E \times E^k$, fro any $k \geq 1$. Thus, the same notation $c(x,\xi)$ for any $x \in E$ and $\xi \in \Gamma_k(x)$ can still be used, with the previous meaning, i.e., for ξ belongs to $\Gamma_k(k) \subset E^k$ the expression of $c(x,\xi)$ changes accordingly.

Hence, an alternative and equivalent way to describe a simultaneous impulse (or switching) control $\{(\theta_i,\xi_{i,1},\ldots,\xi_{i,k_i}) : i \geq 1\}$ is to impose that $\theta_i < \theta_{i+1}$ whenever $\theta_i < \infty$ and $(\xi_{i,1},\ldots,\xi_{i,k_i})$ has values in $\Gamma_{k_i}(x)$, i.e., at time θ_i the state is moved successively from x to $\xi_{i,1}$, from $\xi_{i,1}$ to $\xi_{i,2}$, and so forth, to reach ξ_{i,k_i} in k_i instantaneous impulses (or switching). Thus, the notation $\{(\theta_i,\xi^i) : i \geq 1\}$ with $\xi^i \in \Gamma_{k_i}(x)$ makes a subtle difference with notation used in previous sections, and with this in mind, Definition 3.9 for admissible and zero-admissible impulse controls is meaningful. Certainly, $\theta_0 = 0$ and $\xi_0 = x$ is added by convenience; and a 'simple impulse' from x to ξ is 'strict' if $x \neq \xi$, and a k-impulse $\xi = (\xi_1,\ldots,\xi_k)$ from x to ξ (actually to x_k) is 'strict' if $\xi \neq x = x_1 \neq \xi_1 = x_2 \neq \xi_2 = x_3,\ldots,x_{k-1} \neq \xi_{k-1} = x_k \neq \xi_k$, i.e., it belongs to $\Gamma'_k(x)$. Therefore:

- The class $\mathcal{V}_0^\star \supset \mathcal{V}_0$ of all zero-admissible impulse controls with possible simultaneous *impulses* can be represented as $\nu = \{(\theta_i,\xi_{i,1},\ldots,\xi_{i,k_i}) : i \geq 1\}$ or simply $\nu = \{(\theta_i,\xi^i) : i \geq 1\}$ or the discrete time version $\nu = \{(\eta_i,\xi^i) : i \geq 1\}$, satisfying $\theta_i < \theta_{i+1}$ whenever $\theta_i < \infty$, $\eta_i < \eta_{i+1}$ whenever $\eta_i < \infty$, and $(\xi_{i,1},\ldots,\xi_{i,k_i})$ has values in $\Gamma_{k_i}(x)$, as previously. This class is the same $\mathcal{V}'_0$ described at the beginning of this section, usually referred to as zero-admissible impulse control with possible simultaneous *interventions*, i.e., $\{(\theta_n,\xi_n) : n \geq 1\}$ satisfying $\theta_n \leq \theta_{n+1}$ and $\xi_n \in \Gamma(x)$, besides the zero-admissible condition $y_{\theta_n} = 0$, and also the necessary requirement that $\theta_n \to \infty$ as $n \to \infty$.
- Similarly, for a given positive integer κ, denote by $\mathcal{V}_0^\kappa$ the class of impulse controls ν in $\mathcal{V}_0$ such that $k_i \leq \kappa$ for every $i \geq 1$, and by convenience, we set $\theta_0 = 0$, $\xi^0 = x$, and $\xi^i = (\xi_{i,1},\ldots,\xi_{i,k_i})$, for any $i \geq 1$. Clearly, $\mathcal{V}_0^\kappa \subset \mathcal{V}_0^{\kappa+1} \subset \mathcal{V}_0^\star$ and if $\kappa = 1$ then $\mathcal{V}_0^1$ is the class of $\mathcal{V}_0$ of zero-admissible impulse controls. Similarly, the class $\mathcal{V}^\kappa$ is defined, which corresponds to the class of admissible impulse controls $\mathcal{V}$ *without* possible simultaneous impulses, see in Definition 3.9.

6.3. Impulse operator M. Thus, besides the operator M as given by (3.11), if simultaneous impulses (or switchings) are allowed then iterations of M are useful, and it is convenient to define the operator

$$\begin{aligned} M_k v(x) = \inf\big\{&c(x_1,\xi_1) + \cdots + c(x_k,\xi_k) + v(\xi_k) : x = x_1 \in E, \\ &x_1 \neq x_2 = \xi_1 \in \Gamma(x_1),\ldots,\, x_{k-1} \neq x_k = \xi_{k-1} \in \Gamma(x_{k-1}), \\ &\qquad x \neq \xi = \xi_k \in \Gamma(x_k)\big\}, \end{aligned} \tag{6.1}$$

which agrees with the (power) expression $M^k v = M(M^{k-1})v$. As implicitly understood, the impulse (or switching) cost $c(x,\xi)$ is continuously defined (and used only) for x in E and ξ in $\Gamma(x)$, and for convenience, it is extended to the whole $E \times E$ and $E \times E^k$ for a k-impulse. Setting up an impulse (or switching) control model imposes (implicitly) the restriction $x \notin \Gamma(x)$, i.e., an impulse (or switching) that does not actually move the state is not allowed (and unnecessary). Thus, because an optimal control is desired, it is natural to assume (at least) that the set $\Gamma(x) \subset E$ is closed. This combined with what follow, justify the assertion that the extension of the function $c(x,\xi)$ to the whole product space $E \times E$ could be not necessarily continuous on the diagonal $x = \xi$, unless the diagonal is an isolated region, like in an usual switching control model.

Also, it may be expected that a positive cost should be associated with any intervention (impulse or switching), i.e., $c(x,\xi) > 0$ for any $x \neq \xi \in \Gamma(x)$ (in an inventory control problem this includes a positive fixed cost per order), unless a switching control model is in mind, e.g., the cost-per-switching may be associated with beginning some operation (i.e., starting a machine), and therefore, stopping the operation, may have no cost (i.e., a zero cost), which yields $c(x,\xi) = 0$ for some $\xi \neq x$. For instance, if there is no cost for two interventions then switching forward and backward between them, produces an undesired situation: the system may get trap at a finite time, and this situation could be possible even when a constraint given by a signal process is enforced.

Still, there may be other reasons to decide for simultaneous impulses, e.g., when $\xi_1 \in \Gamma(x_1)$, $\xi_2 \in \Gamma(\xi_1)$, and either $\xi_2 \notin \Gamma(x)$ or $c(x_1,\xi_1) + c(\xi_1,\xi_2) < c(x_1,\xi_2)$ with $\xi_2 \in \Gamma(x_1)$. Actually, to include all these possibilities, involves to drop the assumption (3.10) partially, i.e., retaining only the condition that $\Gamma(x)$ is a nonempty closed subset of E. For instance, if we recall that $\mathcal{V}_0^\star$ denotes the set of zero-admissible impulses having at most finite number of simultaneous impulses then the dynamic programming applied to the optimal cost $u_0(x) = \inf\{J_{x0}(\nu) : \nu \in \mathcal{V}_0^\star\}$ yields (4.7) as the HJB equation with $Mv(x) = \inf_{k\geq 1} M_k v(x)$ instead of M being given by (3.11), i.e., u_0 is expected to be the maximum solution of $w(x) = \min\big\{f_\alpha(x) + Pw(x), \inf_{k\geq 1} M_k w(x)\big\}$,within a suitable class of functions w. This analysis could be worked out under the condition: there exists a positive integer κ and a constant $c_\kappa > 0$ such that

$$c(x,\xi_1,\ldots,\xi_k) \geq c_\kappa, \quad \forall (\xi_1,\ldots,\xi_k) \in \Gamma_k(x),\ \forall k > \kappa, \tag{6.2}$$

since this would implies that any impulse control $\{\theta_n,\xi_n\}$ in $\mathcal{V}_0'$ cannot have a finite cost and also an infinite number of simultaneous k-interventions with $k > \kappa$, namely, an impulse control model with possible κ-simultaneous impulses (i.e., in the class $\mathcal{V}_0^\kappa$). Conceptually, this situation works fine also for the standard switching control model, where some costs may be zero but no loop may show up.

6.4. Manageable situations. Now, if the number of possible simultaneous impulses is a priori bounded by a positive integer κ, i.e.,

$$u_0(x) = \inf\{J_{x0}(\nu) : \nu \in \mathcal{V}_0^\kappa\}, \tag{6.3}$$

then the HJB becomes

$$w(x) = \min\big\{f_\alpha(x) + Pw(x), M_1 w(x), \ldots, M_\kappa w(x)\big\}. \tag{6.4}$$

This HJB equation can be solved almost as in Subsection 5, where simultaneous impulses were ruled out. Indeed, it suffices to use impulse (or switching) controls $\{(\theta_i, \xi_{i,1}, \ldots, \xi_{i,k_i}) : i \geq 1\}$ with $k_i \leq \kappa$ and the sequence of VIs as (4.6) with $\psi = \min\{M_1 u_0^{n-1}(x), \ldots, M_\kappa u_0^{n-1}(x)\}$ has a unique solution at each step n, and $u_0^n(x) \to u_0(x)$ uniformly in x, since the estimate (5.3) remains valid for any impulse control in $\mathcal{V}_0^\kappa$. However, this does not ensure that the optimal impulse (or switching) control $\hat{\nu}$ obtained from the continuation region $u_0(x) < \min\{M_1 u_0(x), \ldots, M_\kappa u_0(x)\}$ has at most κ impulses. Moreover, each $u_0^n(x)$ corresponds to the optimal cost on the class of impulse controls $\nu = \{(\eta_i, \xi^i) : i \geq 1\}$ in $\mathcal{V}_0^\kappa$ satisfying $\eta_{n+1} = \infty$, and if $\hat{\nu}^n$ is the impulse control obtained from the continuation regions associated with $u_0^1, \ldots, u_0^n$, then $\hat{\nu}^n$ may belong to the class $\mathcal{V}_0^\star \smallsetminus \mathcal{V}_0^\kappa$, i.e., may have simultaneous interventions. This is because the cost-per-impulse is not necessarily strictly positive or the condition (3.10) is not enforced.

Hence, to deal with simultaneous impulses (or switchings), an 'iterated' condition (5.2) could be reformulated as follows: besides $\Gamma(x)$ being a non-empty closed subset of E, suppose that there exists a positive integer κ such that for every $k > \kappa$, $x \in E$ and $(\xi_1, \ldots, \xi_k) \in \Gamma_k(x)$ there exists $k' \leq \kappa$ and $(\xi'_1, \ldots, \xi'_{k'}) \in \Gamma_{k'}(x)$ such that

$$\xi'_{k'} = \xi_k \quad \text{and} \quad c(x, \xi_1, \ldots, \xi_k) > c(x, \xi'_1, \ldots, \xi'_{k'}). \tag{6.5}$$

since $\kappa = 1$ becomes essentially condition (5.2), even if it may happen that $\Gamma(\xi) \not\subset \Gamma(x)$ for some $\xi \in \Gamma(x)$. An interpretation of this assumption is as follows: 'strictly more that κ simultaneous impulses (or switchings) are irrelevant', because, any action to transfer the state from x to ξ_k with a k-simultaneous impulse having $k > \kappa$ can be achieved with k'-simultaneous impulses with $k' \leq \kappa$ and a strictly lower cost.

A version of Lemma 5.1 with M replaced by M_k given by (6.1) can be proved, i.e., under the condition (6.5), if $x \in E$, $k \leq \kappa$, $v(x) \leq M_{k'} v(x)$, for any k', and

$$\begin{aligned} v(x) &= M_k v(x) = c(x, \hat{\xi}_1(x,v), \ldots, \hat{\xi}_k(x,v)) + v(\hat{\xi}_k(x,v)), \\ v(x) &< M_{k+1} v(x) = c(x, \hat{\xi}'_1(x,v), \ldots, \hat{\xi}'_{k+1}(x,v)) + v(\hat{\xi}'_{k+1}(x,v)), \end{aligned}$$

then $v(\hat{\xi}_k(x,v)) < \inf\{M_k v(\hat{\xi}_k(x,v)) : 1 \leq k \leq \kappa\}$.

Therefore, under the assumption (6.5) instead of (3.10) for an impulse control model with possible simultaneous impulses (or switching), we are essentially back to conditions of the Section 5, and so, it would be not so hard to complete the above arguments to deduce:

Theorem 6.2. *The results of* Theorem 4.4 *and* Theorem 4.5 *remain valid (with minimal changes as discussed above) if the conditions* (3.9), (3.10) *are replaced with* (5.1), (6.5)*, i.e., a discounted impulse/switching control model with a constraint given by a signal process and possible simultaneous impulses/switchings can be solved, namely, using $\mathcal{V}_0^\kappa$ instead of $\mathcal{V}_0$, and similarly with $\mathcal{V}$.*

Also, similar to Theorem 5.4, we have

Theorem 6.3. *If the assumptions* (3.9) *and* (3.10) *are replaced with* (5.1) *and an iterated condition* (3.10) *or a non-strict version of condition* (6.5)*, namely,*

$$\xi'_{k'} = \xi_k \quad \textit{and} \quad c(x, \xi_1, \ldots, \xi_k) \geq c(x, \xi'_1, \ldots, \xi'_{k'}), \tag{6.6}$$

then $M_k v(x) \geq \inf\{M_{k'} v(x) : 1 \leq k \leq \kappa\}$ *for any* x*, the HJB equation*

$$u_0^n(x) = \min\big\{f_\alpha(x) + Pu_0^n(x), M_1 u_0^{n-1}(x), \ldots, M_\kappa u_0^{n-1}(x)\big\}.$$

has a unique solution u_0^n *in* $C_b(E)$*, and the sequence* $\{u_0^n\}$ *is monotone decreasing and converging uniformly to the* maximum *solution* u_0 *of* (6.4)*, i.e.,* u_0 *solves* (6.4) *and any other solution* u *satisfies* $u \leq u_0$*. Moreover, each function* $u_0^n(x)$ *is the optimal cost of an impulse control problem with at most* n *impulses, i.e.,* (4.8) *holds, and* $u_0(x)$ *is the optimal cost within the class* $\mathcal{V}_0^\star$*, i.e., the minimization uses impulse controls satisfying* $\xi^i \in \Gamma_{k_i}(x)$ *and* $\eta_{i-1} < \eta_i$ *whenever* $\eta_{i-1} < \infty$*, for any* i*. Furthermore, if for the optimal cost* u_0^n *we define the continuation region*

$$\{x \in E : u_0^n(x) < \psi_n(x)\}, \quad \psi_n = \min\{M_k u_0^{n-1} : 1 \leq \kappa\},$$

as well as the optimal multiple impulse $(\hat{\xi}_1, \ldots, \hat{\xi}_k)$*, with* $0 \leq k \leq \kappa$*, depending on* x *and* u_0^{n-1} *and satisfying*

$$\psi_n(x) = M_k u_0^{n-1}(x) = c(x, \hat{\xi}_1, \ldots, \hat{\xi}_k) + u_0^{n-1}(\hat{\xi}_k),$$

then the impulse control $\{(\hat{\eta}_i, \hat{\xi}^i) : 1 \leq i \leq n\}$ *obtained from the continuation region is optimal for* u_0^n*, but it may have simultaneous interventions (certainly, if the condition* (6.5) *is retained then no simultaneous impulses may occur, in the sense of* $\eta_{i-1} < \eta_i$ *whenever* $\eta_{i-1} < \infty$*), i.e.,* $\hat{\eta}_i = \hat{\eta}_{i-1} < \infty$ *with a positive probability for some* $i \leq n$*.*

Note that it was not necessary to use the notation with multiple impulses (see Definition 6.1) within Theorem 5.4 and the optimal impulse control was denoted by $\{(\hat{\eta}_i, \hat{\xi}_i) : 1 \leq i \leq n\}$, but as a recall of this subtle difference, the notation $\{(\hat{\eta}_i, \hat{\xi}^i) : 1 \leq i \leq n\}$ was preferred in Theorem 6.3.

References

1. Bensoussan, A.,*Dynamic programming and inventory control*, IOS Press, Amsterdam, 2011.
2. Bensoussan, A. and Lions, J.L.: *Applications des inéquations variationnelles en contrôle stochastique*, Dunod, Paris, 1978.
3. Bensoussan, A. and Lions, J.L.: *Contrôle impulsionnel et inéquations quasi-variationnelles*, Gauthier-Villars, Paris, 1982.
4. Bensoussan, A. and Menaldi, J.L.: Hybrid control and dynamic programming, *Dynam. Contin. Discrete Impuls. Systems* **3** (1997), no. 4, 395–442.
5. Bensoussan, A. and Menaldi, J.L.: Stochastic hybrid control, *J. Math. Anal. Appl.* **249** (2000), no. 1, 261–288, Special issue in honor of Richard Bellman.
6. Brémaud, P.: *Point processes and queues*, Springer-Verlag, New York, 1981, Martingale dynamics.
7. Brémaud, P.: *Markov chains*, Springer-Verlag, New York, 1999, Gibbs fields, Monte Carlo simulation, and queues.
8. Çinlar, E. and Jacod, J.: *Representation of semimartingale Markov processes in terms of Wiener processes and Poisson random measures*, Seminar on Stochastic Processes (Evanston, Illinois, 1981), vol. 1, pp. 159–242, Birkhäuser, Boston, Massachussetts, 1981.
9. Costa, O.L.V., Dufour, F. and Piunovskiy, A.B.: Constrained and unconstrained optimal discounted control of piecewise deterministic Markov processes, *SIAM J. Control Optim.* **54** (2016), no. 3, 1444–1474.
10. Davis, M.H.A.: *Markov models and optimization*, Chapman & Hall, London, 1993.
11. Dupuis, P. and Wang, H.: Optimal stopping with random intervention times, *Adv. in Appl. Probab.* **34** (2002), no. 1, 141–157.

12. Egloff, D. and Leippold, M.: The valuation of American options with stochastic stopping time constraints, *Appl. Math. Finance* **16** (2009), no. 3-4, 287–305.
13. Ethier, S.N. and Kurtz, T.G.: *Markov processes*, John Wiley & Sons Inc., New York, 1986, Characterization and convergence.
14. Hanouzet, B. and Joly, J.L.: Convergence uniforme des itérés définissant la solution d'une inéquation quasi variationnelle abstraite, *C. R. Acad. Sci. Paris Sér. A-B* **286** (1978), no. 17, A735–A738.
15. Hernández-Lerma, O. and Lasserre, J.B.: *Discrete-time Markov control processes*, Springer-Verlag, New York, 1996, Basic optimality criteria.
16. Jasso-Fuentes, H., Menaldi, J.L. and Prieto-Rumeau, T.: Discrete-time hybrid control in Borel spaces, *Appl. Math. Optim.* (2018), 1–33, Link at https://doi.org/10.1007/s00245-018-9503-z.
17. Jasso-Fuentes, H., Menaldi, J.L. and Prieto-Rumeau, T.: Discrete-time control with non-constant discount factor, To appear (2019).
18. Jasso-Fuentes, H., Menaldi, J.L., Prieto-Rumeau, T., and Robin, M.: Discrete-time hybrid control in Borel spaces: Average cost optimality criterion, *J. Math. Anal. Appl.* **462** (2018), no. 2, 1695–1713.
19. Jasso-Fuentes, H., Menaldi, J.L. and Robin, M.: *Hybrid control for Markov-Feller processes*, Book to appear, 2020.
20. Lempa, J.: Optimal stopping with information constraint, *Appl. Math. Optim.* **66** (2012), no. 2, 147–173.
21. Lepeltier, J.P. and Marchal, B.: Théorie générale du contrôle impulsionnel markovien, *SIAM J. Control Optim.* **22** (1984), no. 4, 645–665.
22. Liang, G.: Stochastic control representations for penalized backward stochastic differential equations, *SIAM J. Control Optim.* **53** (2015), no. 3, 1440–1463.
23. Liang, G. and Wei, W.: Optimal switching at poisson random intervention times, *Discrete and Continuous Dynamical Systems* **Series B**, To appear, see arXiv:1309.5608v2 [math.PR].
24. Menaldi, J.L.: Sur le problème de contrôle impulsionnel et l'inéquation quasi variationnelle dégénérée associée, *C. R. Acad. Sci. Paris Sér. A-B* **284** (1977), no. 23, A1499–A1502.
25. Menaldi, J.L.: On the optimal impulse control problem for degenerate diffusions, *SIAM J. Control Optim.* **18** (1980), no. 6, 722–739.
26. Menaldi, J.L.: Optimal impulse control problems for degenerate diffusions with jumps, *Acta Appl. Math.* **8** (1987), no. 2, 165–198.
27. Menaldi, J.L.: *Stochastic hybrid optimal control models*, Stochastic models, II (Guanajuato, 2000), Aportaciones Mat. Investig., vol. 16, Soc. Mat. Mexicana, México, 2001, pp. 205–250.
28. Menaldi, J.L., and Robin, M.: On some optimal stopping problems with constraint, *SIAM J. Control Optim.* **54** (2016), no. 5, 2650–2671.
29. Menaldi, J.L., and Robin, M.: On some impulse control problems with constraint, *SIAM J. Control Optim.* **55** (2017), no. 5, 3204–3225.
30. Menaldi, J.L., and Robin, M.: On some ergodic impulse control problems with constraint, *SIAM J. Control Optim.* **56** (2018), no. 4, 2690–2711.
31. Menaldi, J.L., and Robin, M.: *On optimal stopping and impulse control with constraint*, Modeling, Stochastic Control, Optimization, and Applications (Q. Zhang and G. Yin, Eds.), Springer, Switzerland, 2019, pp. 427-450.
32. Palczewski, J. and Stettner, L.: Finite horizon optimal stopping of time-discontinuous functionals with applications to impulse control with delay, *SIAM J. Control Optim.* **48** (2010), no. 8, 4874–4909.
33. Peskir, G. and Shiryaev, A.: *Optimal stopping and free-boundary problems*, Birkhäuser Verlag, Basel, 2006.
34. Robin, M.: *Contrôle impulsionnel des processus de Markov*, Thèse d'état, 1978, 353 pp. Link at https://hal.archives-ouvertes.fr/tel-00735779/document.
35. Stettner, L.: Discrete time adaptive impulsive control theory, *Stochastic Process. Appl.* **23** (1986), no. 2, 177–197.
36. Stettner, L.: On ergodic impulsive control problems, *Stochastics* **18** (1986), no. 1, 49–72.

37. Wang, H.: Some control problems with random intervention times, *Advances in Applied Probability* **33** (2001), no. 2, 404–422.
38. Yin, G.G. and Zhang, Q.: *Discrete-time Markov chains*, Springer-Verlag, New York, 2005, Two-time-scale methods and applications, Stochastic Modelling and Applied Probability.
39. Yin, G.G. and Zhang, Q.: *Continuous-time Markov chains and applications*, 2nd Ed., Springer-Verlag, New York, 2013, Two-time-scale Approach.
40. Yin, G.G. and Zhu, C.: *Hybrid switching diffusions*, Springer, New York, 2010, Properties and applications.

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