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Commutative Associative Binary Operations on a Set with Four Elements

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Abstract: The main goal of this paper is to count commutative as well as associative binary operation on four element set, by using partition and composition of mapping. This is achieved using algorithm given by Sehgal et al. [2013] on associative binary operations on a set with five elements and Yogesh et al. [2013] on commutative as well as associative binary operations on a set with three elements.

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1. Introduction

There is easy calculation to counting binary operation on a set with n element which is n^{n^2}. If set has only four elements, then number of binary operations is as very large 4,294,967,296. Similarly, there is easy calculation to counting commutative binary operation on a set with n element which is $n^{n(n+1)/2}$. If set has only four elements, then number of binary operations is also large 1,048,576. In [7] authors count associative binary operations on four element set which comes as 3492. But no easy calculation and no educated guess, seems to give the answer to following question:

> "How many binary operations on a four elements are associative as well as commutative?"

The objective of this paper is to answer this question. In other words, how many of the 1,048,576 different commutative binary operations on four-element set are associative or how many of the 3,492 different associative binary operations on four-element set are commutative.

2. Operations on a Set with Two Elements

Form [2], we have following results

- Binary operation on a set of two elements is 16.
- Associative binary operation on a set of two elements is 8.

- Commutative binary operation on a set of two elements is 8.
- Associative as well as commutative binary operation on a set of two elements is 6

3. Operations on a Set with Three Elements

From [2, 3, 10], we have following results.

- Binary operation on a set of three elements is 19683.
- Associative binary operation on a set of three elements is 113.
- Commutative binary operation on a set of three elements is 729.
- Associative as well as commutative binary operation on a set of three elements is 63.

4. Operations on a Set with Four Elements

As mentioned in the introduction, the number of possible binary operations on a set of four elements is 4,294,957,296. Of these 1,048,576 are commutative. We now proceed to answer the question: How many binary operations on a set of four elements are associative as well as commutative?

5. Algorithm for Finding Number of Associative Binary Operation on N-Element

Algorithm given below is taken from [3, 4, 10]. The analysis of the commutative associative binary operations on n-element set S will now divide into 3 steps:

1. Partition the set of n^n mappings in such a way that element of same partition can be obtained by using one-one and onto mapping from S onto S.
2. Rearrange the partition according to their order of any element of the partition. (Say order as k).
3. Calculate the contribution towards number of associative as well as commutative binary operations when one row is fill by the first element of i^{th} partition which can also fill k-1 more rows and remaining row n-k can be filled by i^{th} and onwards partitions (if any) with two conditions given below.

Conditions: Before starting calculation, firstly we insured that no commutative associative table counted twice. For this, we make some rules:

(i) If we fixed r^{th} row from any element i^{th} partition (which has order k) then we can fill at least k-1 more rows. Remaining unfilled rows can by filled with element of i^{th} and onwards partition (if any) however selected element of i^{th} partition cannot fill the unfilled rows before selected row.

(ii) If table contains n different entry of i^{th} partition then contribution towards number of commutative associative operations counted is 1/n.

A. Step 1st

If set S has n elements, then total number of mapping possible from set S to S = n^n.

In our problem $n = 4$, then total number of mapping possible from set S to S = 256.

Here we consider S = {0, 1, 2, 3} and first digit of each mapping comes from 0, second comes from 1, third comes from 2 and last comes from 3.

Partition No	*First element of Partition*	*Total number of element in partition*
1	0000	4
2	0001	24
3	0003	12
4	0011	12
5	0012	24
6	0013	24
7	0022	12
8	0023	12
9	0032	12
10	0123	1
11	0132	6
12	0211	24
13	0231	8
14	1000	12
15	1001	12
16	1002	24
17	1032	3
18	1200	24
19	1230	6

B. Step 2nd

Partition No	*First element of Partition*	*Order of an element of partition*	*Generated elements belongs to partition*
1	1230	4	1,14,1,19
2	0012	3	2,6,18
3	1002	3	3,16,12
4	1200	3	4,4,17

Partition No	*First element of Partition*	*Order of an element of partition*	*Generated elements belongs to partition*
5	0231	3	5,5,19
6	0001	2	6,18
7	0013	2	7,15
8	0211	2	8,17
9	0011	2	9,18
10	0032	2	10,17
11	1000	2	11,15
12	1001	2	12,16
13	0132	2	13,19
14	1032	2	14,19
15	0003	1	15
16	0022	1	16
17	0023	1	17
18	0000	1	18
19	0123	1	19

C. Step 3rd

1. When $i = 1$(assume element from partition is 1230)

   Number of elements in Ist Partition are 6

Assumed row for 1230	*Contribution towards number of associative as well as commutative*
Ist	$0.5 \times 6 = 3$
2nd	$0.5 \times 6 = 3$
3rd	$0.5 \times 6 = 3$
4th	$0.5 \times 6 = 3$
Total	12

2. When $i = 2$(assume element from partition is 0012)

   Number of elements in 2nd Partition are 24

Assumed row for 0012	*Contribution towards number of associative as well as commutative*
Ist	$0 \times 24 = 0$
2nd	$0 \times 24 = 0$
3rd	$1 \times 24 = 24$
4th	$1 \times 24 = 24$
Total	48

3. When $i = 3$(assume element from partition is 1002)

   Number of elements in 3rd Partition are 24

Assumed row for 1002	*Contribution towards number of associative as well as commutative*
Ist	0×24=0
2nd	0×24=0
3rd	1×24=24
4th	1×24=24
Total	48

4. When $i = 4$(assume element from partition is 1200)

   Number of elements in 4th Partition are 24

Assumed row for 1200	*Contribution towards number of associative as well as commutative*
Ist	1×24=24
2nd	0.5×24=12
3rd	0.5×24=12
4th	0×24=0
Total	48

5. When $i = 5$(assume element from partition is 0231)

   Number of elements in 5th Partition are 8

Assumed row for 0231	*Contribution towards number of associative as well as commutative*
Ist	0×8=0
2nd	0.5×8=4
3rd	0.5×8=4
4th	0.5×8=4
Total	12

6. When $i = 6$(assume element from partition is 0001)

   Number of elements in 6th Partition are 24

Assumed row for 0001	*Contribution towards number of associative as well as commutative*
Ist	0×24=0
2nd	4.5×24=108
3rd	1.5×24=36
4th	2.5×24=60
Total	204

7. When $i = 7$(assume element from partition is 0013)

   Number of elements in 7th Partition are 24

Assumed row for 0013	*Contribution towards number of associative as well as commutative*
Ist	$0 \times 24 = 0$
2nd	$2 \times 24 = 48$
3rd	$2 \times 24 = 48$
4th	$0 \times 24 = 0$
Total	96

8. When $i = 8$ (assume element from partition is 0211)

   Number of elements in 8th Partition are 24

Assumed row for 0211	*Contribution towards number of associative as well as commutative*
Ist	$0 \times 24 = 0$
2nd	$2 \times 24 = 48$
3rd	$1 \times 24 = 24$
4th	$0 \times 24 = 0$
Total	72

9. When $i = 9$ (assume element from partition is 0011)

   Number of elements in 9th Partition are 12

Assumed row for 0011	*Contribution towards number of associative as well as commutative*
Ist	$0 \times 12 = 0$
2nd	$6 \times 12 = 72$
3rd	$1 \times 12 = 12$
4th	$0 \times 12 = 0$
Total	84

10. When $i = 10$ (assume element from partition is 0032)

    Number of elements in 10th Partition are 12

Assumed row for 0032	*Contribution towards number of associative as well as commutative*
Ist	$0 \times 12 = 0$
2nd	$0 \times 12 = 0$
3rd	$2 \times 12 = 24$
4th	$2 \times 12 = 24$
Total	48

11. When $i = 11$ (assume element from partition is 1000)

    Number of elements in 11^{th} Partition are 12

Assumed row for 1000	*Contribution towards number of associative as well as commutative*
Ist	10×12=120
2^{nd}	1×12=12
3^{rd}	0×12=0
4^{th}	0×12=0
Total	132

12. When $i = 12$ (assume element from partition is 1001)

    Number of elements in 12^{th} Partition are 12

Assumed row for 1001	*Contribution towards number of associative as well as commutative*
Ist	3×12=36
2^{nd}	3×12=36
3^{rd}	0×12=0
4^{th}	0×12=0
Total	72

13. When $i = 13$ (assume element from partition is 0132)

    Number of elements in 13^{th} Partition are 6

Assumed row for 0132	*Contribution towards number of associative as well as commutative*
Ist	0×6=0
2^{nd}	0×6=0
3^{rd}	2×6=12
4^{th}	2×6=12
Total	24

14. When $i = 14$ (assume element from partition is 1032)

    Number of elements in 14^{th} Partition are 3

Assumed row for 1032	*Contribution towards number of associative as well as commutative*
Ist	0.3333×3=1
2^{nd}	0.3333×3=1
3^{rd}	0.3333×3=1
4^{th}	0.3333×3=1
Total	4

15. When $i = 15$ (assume element from partition is 0003)

Number of elements in 15th Partition are 12

Assumed row for 0003	*Contribution towards number of associative as well as commutative*
Ist	8×12=96
2nd	0×12=0
3rd	0×12=0
4th	6.3333×12=76
Total	172

16. When $i = 16$ (assume element from partition is 0022)

Number of elements in 16th Partition are 12

Assumed row for 0022	*Contribution towards number of associative as well as commutative*
Ist	0×12=0
2nd	0×12=0
3rd	0×12=0
4th	0×12=0
Total	000

17. When $i = 17$ (assume element from partition is 0023)

Number of elements in 17th Partition are 12

Assumed row for 0023	*Contribution towards number of associative as well as commutative*
Ist	17.8333×12=214
2nd	0×12=0
3rd	7.5×12=90
4th	5.5×12=66
Total	370

18. When $i = 18$ (assume element from partition is 0000)

Number of elements in 18th Partition are 4

Assumed row for 0000	*Contribution towards number of associative as well as commutative*
Ist	1×4=4
2nd	0×4=0
3rd	0×4=0
4th	0×4=0
Total	4

19. When $i = 19$ (assume element from partition is 0123)

    Number of elements in 19^{th} Partition are 1

Assumed row for 0123	*Contribution towards number of associative as well as commutative*
Ist	0×1=0
2^{nd}	0×1=0
3^{rd}	0×1=0
4^{th}	0×1=0
Total	000

Summary of Section

Partition-No	*Total contribution towards number of associative as well as commutative*
1	12
2	48
3	48
4	48
5	12
6	204
7	96
8	72
9	84
10	48
11	132
12	72
13	24
14	4
15	172
16	60
17	0
18	4
19	0
Total	1140

6. Conclusions

The conclusion of this paper is that among the 1048576 different commutative binary operations on a Four-element set, S = {0, 1, 2, 3}, there are exactly 1140 operations which are associative. In other words, there exist exactly 1140 four-element commutative semi groups.

7. Future Work

One can find out Associative binary Operations on a n-Element Set by using same Algorithm, which we have used for Four-element and also verify the one of the result for five element set 30730 in [6].

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