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θ-CLOSURE AND δ-CLOSURE OPERATORS IN BITOPOLOGICAL SPACES

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Abstract: The purpose of this paper is to give the covering characterization of closure, θ-closure, δ-closure, interior, θ-interior and δ-interior operators in bitopological spaces.

Kewwords: Bitopology, Closure, Interior, Open covering.

1. INTRODUCTION

In this chapter closure, θ-closure, δ-closure, interior, θ-interior and δ-interior operators are studied in bitopological spaces by using a covering of X. Throughout the paper (X, τ_1, τ_2) is a bitopological space and $i, j = 1, 2$, $i \neq j$. In this paper $j\text{-}cl\,A$ and $j\text{-int}\,A$ denote the closure of A and interior of A in a topological space (X, τ_j) for any subset A of X. A sub set U of X is i-open in X if $U \in \tau_i$.

2. *ij* - θ - CLOSURE AND *ij* - δ - CLOSURE OPERATORS

Kariofillis[2] and Sanjay Tahiliani[4] studied q-closure operators in bitopolgical spaces. Banerjee[1], Khedr and Al-Areefi[3] extended the notion of δ-closure to bitopological spaces. Takashi Noiri and Valeriu Popa also studied the δ-closure and θ-closure operators in bitopological spaces. In this section the above operators are further investigated.

Definition 2.1: Let B be a subset of (X, τ_1, τ_2). A point $x \in X$ is said to be in the $ij\text{-}\theta clB$[2] if $B \cap j\text{-}clU \neq \varnothing$ for every i-open set U containing x.

A subset A of X is said to be ij-θ-closed if $A = ij\text{-}\theta clA$. The complement of an ij-θ-closed set is ij-θ-open in (X, τ_1, τ_2). The set $ij\text{-}\theta\text{int}B$ is defined as the union of all ij-θ-open sets contained in B. Hence $x \in ij\text{-}\theta\text{int}B$ if and only if there is an i-open set U such that $x \subseteq U \subseteq j\text{-}clU \subseteq B$.

The next two lemmas will be useful in sequel.

Lemma 2.2: If U is j-open in X then $ij\text{-}\theta\, cl\, U = i\text{-}cl\, U$. [2]

Lemma 2.3: (i) $X \setminus ij\text{-}\theta\, clB = ij\text{-}\theta\text{int}\,(X\setminus B)$.

(ii) $X \setminus ij\text{-}\theta\,\text{int}B = ij\text{-}\theta\, cl\,(X\setminus B)$. [4, 5]

Definition 2.4: Let B be a subset of (X, τ_1, τ_2). A point $x \in X$ is said to be in the set ij-δclB[1, 3] if $B \cap i\text{-int}(j\text{-}cl\,U) \neq \varnothing$ for every i-open set U containing x.

A subset A of X is said to be ij-δ-closed if $A = ij\text{-}\delta clA$. The complement of an ij-δ-closed set is ij-δ-open in (X, τ_1, τ_2). The set ij-$\delta\text{int}B$ is defined as the union of all ij-δ-open sets contained in B. Hence $x \in ij\text{-}\delta\text{int}B$ if and only if there is an i-open set U such that $x \in U \subseteq i\text{-int}(j\text{-}cl\,U) \subseteq B$.

Lemma 2.5: (i) $X \setminus ij\text{-}\delta\,cl\,B = ij\text{-}\delta\,\text{int}\,(X\setminus B)$.

(ii) $X \setminus ij\text{-}\delta\,\text{int}\,B = ij\text{-}\delta\,cl\,(X\setminus B)$. [6]

Lemma 2.6: Let B be a subset of (X, τ_1, τ_2). Then

(i) $i\text{-}clB \subseteq ij\text{-}\delta clB \subseteq ij\text{-}\theta clB$.

(ii) $ij\text{-}\delta\,cl\,B$, $ij\text{-}\theta\,cl\,B$ are i-closed sets.

(iii) if B is j-open then $i\text{-}cl\,B = ij\text{-}\delta\,cl\,B = ij\text{-}\theta\,cl\,B$.

Proof: Let $x \in i\text{-}clB$. Let U be an i-open set in X containing x. Then $U \cap B \neq \varnothing$.

Since U is i-open in X, $U \subseteq i\text{-int}(j\text{-}cl\,U)$ that implies $i\text{-int}(j\text{-}cl\,U) \cap B \neq \varnothing$.

Then by Definition 2.4, $x \in ij\text{-}\delta\,cl\,B$. This proves that $i\text{-}cl\,B \subseteq ij\text{-}\delta\,cl\,B$.

Now let $x \in ij\text{-}\delta\,cl\,B$. Let U be an i-open set in X containing x.

Since $x \in ij\text{-}\delta\,cl\,B$, by Definition 2.4, $i\text{-int}(j\text{-}cl\,U) \cap B \neq \varnothing$.

This implies that $(j\text{-}cl\,U) \cap B \neq \varnothing$. Then by Definition 2.1, $x \in ij\text{-}\theta\,cl\,B$.

This proves $ij\text{-}\delta\,cl\,B \subseteq ij\text{-}\theta\,cl\,B$. Thus (i) is proved.

Suppose $x \in i\text{-}cl\,(ij\text{-}\delta cl\,B)$. Let U be an i-open set containing x. Then $U\,(ij\text{-}\delta\,cl\,B) \neq \varnothing$. Let $y \in U \cap (\,ij\text{-}\delta\,cl\,B)$. Since $y \in ij\text{-}\delta\,cl\,B$, by Definition 2.4, $i\text{-int}\,(j\text{-}cl\,U) \cap B \neq \varnothing$. This proves that $x \in ij\text{-}\delta\,cl\,B$ that implies $ij\text{-}\delta\,cl\,B$ is i-closed. Now let $x \in i\text{-}cl\,(ij\text{-}\theta\,cl\,B)$. Let U be an i-open set containing x. Then $U\,(ij\text{-}\theta\,cl\,B) \neq \varnothing$. Let $y \in U \cap ij\text{-}\theta\,cl\,B)$ that implies $y \in ij\text{-}\theta\,cl\,B$. Then by Definition 2.1, $(j\text{-}cl\,U) \cap B \neq \varnothing$. This proves that $x \in ij\text{-}\theta\,cl\,B$. Therefore $ij\text{-}\theta\,cl\,B$ is i-closed. We thus proved (ii).

(iii) Suppose B is j-open. Then by Lemma 2.2, $i\text{-}cl\,B = ij\text{-}\theta\,cl\,B$. By applying (i) it follows that $i\text{-}cl\,B = ij\text{-}\delta\,cl\,B = ij\text{-}\theta\,cl\,B$.

Lemma 2.7: Let B be a subset of (X, τ_1, τ_2). Then

(i) $i\text{-int}\,B \supseteq ij\text{-}\delta\,\text{int}\,B \supseteq ij\text{-}\theta\,\text{int}\,B$.

(ii) $ij\text{-}\delta\,\text{int}\,B$, $ij\text{-}\theta\,\text{int}\,B$ are i-open sets.

(iii) if B is j-closed then $i\text{-int}\,B = ij\text{-}\delta\,\text{int}\,B = ij\text{-}\theta\,\text{int}\,B$.

Proof: Follows from Lemma 2.5 and Lemma 2.6.

Lemma 2.8: If B is j-open then

$$j\text{-int}\,(i\text{-}cl\,B) = ji\text{-}\delta\,\text{int}\,(i\text{-}cl\,B) = ji\text{-}\theta\,int\,(i\text{-}cl\,B)$$
$$= j\text{-int}\,(ij\text{-}\delta\,cl\,B) = ji\text{-}\delta\,\text{int}\,(ij\text{-}\delta\,cl\,B) = ji\text{-}\theta\,\text{int}\,(ij\text{-}\delta\,cl\,B)$$
$$= j\text{-int}\,(ij\text{-}\theta\,cl\,B) = ji\text{-}\delta\,\text{int}\,(ij\text{-}\theta\,cl\,B) = ji\text{-}\theta\,\text{int}\,(ij\text{-}\theta\,cl\,B).$$

Proof: Suppose B is j-open. Then, by Lemma 7.2.6 (iii), we have

$$i\text{-}cl\,B = ij\text{-}\delta\,cl\,B = ij\text{-}\theta\,cl\,B \tag{7.1}$$

This implies that $i\text{-}cl\,B$, $ij\text{-}\delta\,cl\,B$, $ij\text{-}\theta\,cl\,B$ are all i-closed.

Since $i\text{-}cl\,B$ is i-closed, using Lemma 2.7 (iii), we get

$$j\text{-int}\,(i\text{-}cl\,B) = ji\text{-}\delta\,\text{int}\,(i\text{-}cl\,B) = ji\text{-}\theta\,int(\,i\text{-}cl\,B). \tag{7.2}$$

Since $ij\text{-}\delta\,cl\,B$ is i-closed, again using Lemma 2.7 (iii), we get

$$j\text{-int}\,(ij\text{-}cl\,B) = ji\text{-}\delta\,int\,\,(ij\text{-}\delta\,cl\,B) = ji\text{-}\theta\,\text{int}\,(ij\text{-}\delta\,cl\,B) \qquad (7.3)$$

Since $ij\text{-}\theta\,cl\,B$ is i-closed, by using Lemma 2.7 (iii), we get

$$j\text{-int}\,(ij\text{-}\theta\,cl\,B) = ji\text{-}\delta\,\text{int}\,(ij\text{-}\theta\,cl\,B) = ji\text{-}\theta\,\text{int}\,(ij\text{-}\theta\,cl\,B) \qquad (7.3)$$

Then using Eqn. (7.1), Eqn. (7.2), Eqn. (7.3) and Eqn. (7.4) the lemma follows.

Lemma 7.2.9: If B is j-closed then

$$j\text{-}cl\,(i\text{-int}\,B) = ji\text{-}\delta\,cl\,(i\text{-int}\,B) = ji\text{-}\theta\,cl\,(i\text{-int}\,B)$$
$$= j\text{-}cl\,(ij\text{-}\delta\,\text{int}\,B) = ji\text{-}\delta\,cl\,(ij\text{-}\delta\,\text{int}\,B) = ji\text{-}\theta\,cl\,(ij\text{-}\delta\,\text{int}\,B)$$
$$= j\text{-}cl\,(ij\text{-}\theta\,\text{int}\,B) = ji\text{-}\delta\,cl\,(ij\text{-}\theta\,\text{int}\,B) = ji\text{-}\theta\,cl\,(ij\text{-}\theta\,\text{int}\,B).$$

Proof: Suppose B is j-closed. Then by using Lemma 2.7(iii), we see that

$$i\text{-int}\,B = ij\text{-}\delta\,\text{int}\,B = ij\text{-}\theta\,int\,B \tag{7.5}$$

Therefore $ij\text{-}\delta\,\text{int}\,B$ and $ij\text{-}\theta\,\text{int}\,B$ are all i-open.

Since $i\text{-int}\,B$ is i-open, by using Lemma 2.6 (iii) we see that

$$j\text{-}cl\,(i\text{-int}\,B) = ji\text{-}\delta\,cl\,(i\text{-int}\,B) = ji\text{-}\theta\,cl\,(i\text{-int}\,B) \tag{7.6}$$

Since $ij\text{-}\delta\,int$B is i-open , by using Lemma 2.6 (iii) we see that

$$j\text{-}cl\,\,(ij\text{-}\delta\,\text{int}\,B) = ji\text{-}\delta\,cl\,\,(ij\text{-}\delta\,\text{int}\,B) = ji\text{-}\theta\,cl(ij\text{-}\delta\,\text{int}\,B) \tag{7.7}$$

Since $ij\text{-}\theta\,int$B is i-open, again by using Lemma 2.6 (iii) we see that

$$j\text{-}cl\,(ij\text{-}\theta\,\text{int}\,B) = ji\text{-}dcl(ij\text{-}\theta\,\text{int}\,B) = ji\text{-}qcl(ij\text{-}\theta\,\text{int}\,B) \tag{7.8}$$

Now using Eqn. (7.5), Eqn. (7.6), Eqn. (7.7) and Eqn. (7.8) the lemma follows.

3. CHARACTERIZATIONS

Theorem 3.1: For any bitopological space (X, τ_1, τ_2), the following are equivalent.

(i) For every i-open cover $\{U_\alpha : \alpha \in \Delta\}$ of X there is a countable subset N of Δ such that $X = \bigcup_{n \in N} j\text{-}cl(U_n)$.

(ii) For every i-open cover $\{U_\alpha : \alpha \in \Delta\}$ of X there is a countable subset N of Δ such that $X = \bigcup_{n \in N} (ji\text{-}\delta cl U_n)$.

(iii) For every i-open cover $\{U_\alpha : \alpha \in \Delta\}$ of X there is a countable subset N of Δ such that $X = \bigcup_{n \in N} (ji\text{-}\theta cl U_n)$.

Proof: Suppose (i) holds. Let $\{U_\alpha : \alpha \in \Delta\}$ be an i-open cover for X. By using (i), there is a countable subset N of Δ such that $X = \bigcup_{n \in N} (j\text{-}cl U_n)$. By using Lemma 2.6 (iii), it follows that $j\text{-}cl\, U_n = ji\text{-}\Delta cl\, U_n = ji\text{-}\theta\, cl\, U_n$. This proves that $X = \bigcup_{n \in N} (ji\text{-}\delta cl U_n)$ and $X = \bigcup_{n \in N} (ji\text{-}\theta cl U_n)$. Thus we have established that (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii). Again using the above reasons, the implications (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i) can be established.

Theorem 3.2: For any bitopological space (X, τ_1, τ_2), the following are equivalent.

(i) For every j-open cover $\{U_\alpha : \alpha \in \Delta\}$ of X there is a countable subset N of Δ such that $X = \bigcup_{n \in N} (j\text{-}\mathrm{int}(i\text{-}cl(U_n)))$.

(ii) For every j-open cover $\{U_\alpha : \alpha \in \Delta\}$ of X there is a countable subset N of Δ such that $X = \bigcup_{n \in N} (ji\text{-}\delta\,\mathrm{int}(i\text{-}cl U_n))$.

(iii) For every j-open cover $\{U_\alpha : \alpha \in \Delta\}$ of X there is a countable subset N of Δ such that $X = \bigcup_{n \in N} (ji\text{-}\theta\,\mathrm{int}(i\text{-}cl U_n))$.

(iv) For every j-open cover $\{U_\alpha : \alpha \in \Delta\}$ of X there is a countable subset N of Δ such that $X = \bigcup_{n \in N} (j\text{-}\mathrm{int}(ij\text{-}\delta cl U_n))$.

(v) For every j-open cover $\{U_\alpha : \alpha \in \Delta\}$ of X there is a countable subset N of Δ such that $X = \bigcup_{n \in N} (ji\text{-}\delta\,\mathrm{int}(ij\text{-}\delta cl U_n))$.

(vi) For every j-open cover $\{U_\alpha : \alpha \in \Delta\}$ of X there is a countable subset N of Δ such that $X = \bigcup_{n \in N} (ji\text{-}\theta\,\mathrm{int}(ij\text{-}\delta cl U_n))$.

(vii) For every j-open cover $\{U_\alpha : \alpha \in \Delta\}$ of X there is a countable subset N of Δ such that $X = \bigcup_{n \in N} (j\text{-int}(ij\text{-}\theta cl\, U_n))$.

(viii) For every j-open cover $\{U_\alpha : \alpha \in \Delta\}$ of X there is a countable subset N of Δ such that $X = \bigcup_{n \in N} (ji\text{-}\delta\, \text{int}(ij\text{-}\theta cl\, U_n))$.

(ix) For every j-open cover $\{U_\alpha : \alpha \in \Delta\}$ of X there is a countable subset N of Δ such that $X = \bigcup_{n \in N} (ji\text{-}\theta\, \text{int}(ij\text{-}\theta cl\, U_n))$.

Proof: Let $\{U_\alpha : \alpha \in \Delta\}$ be a j-open cover for X. Let N be a countable sub set of Δ.

Let $n \in N$. Since U_n is j-open, from Lemma 2.8, it follows that

$$\begin{aligned} j\text{-int}\,(i\text{-}cl\ U_n) &= ji\text{-}\delta\,\text{int}\,(i\text{-}cl\ U_n) = ji\text{-}\theta\,\text{int}\,(i\text{-}cl\ U_n) \\ &= j\text{-int}\,(ij\text{-}\delta\, cl\ U_n) = ji\text{-}\delta\,\text{int}\,(ij\text{-}\delta\, cl\ U_n) = ji\text{-}\theta\,\text{int}\,(ij\text{-}\delta\, cl\ U_n) \\ &= j\text{-int}\,(ij\text{-}\theta\, cl\ U_n) = ji\text{-}\delta\,\text{int}\,(ij\text{-}\theta\, cl\ U_n) = ji\text{-}\theta\,\text{int}\,(ij\text{-}\theta\, cl\ U_n). \end{aligned}$$

Then the theorem follows.

Theorem 3.3. For any bitopological space (X, τ_1, τ_2), the following are equivalent.

(i) For every i -open cover $\{U_\alpha : \alpha \in \Delta\}$ of X there is a countable subset N of Δ such that $X = j\text{-}cl\left(\bigcup_{n \in N} U_n\right)$.

(ii) For every i-open cover $\{U_\alpha : \alpha \in \Delta\}$ of X there is a countable subset N of Δ such that $X = ji\text{-}\delta cl\left(\bigcup_{n \in N} U_n\right)$.

(iii) For every i-open cover $\{U_\alpha : \alpha \in \Delta\}$ of X there is a countable subset N of Δ such that $X = ji\text{-}\theta cl\left(\bigcup_{n \in N} U_n\right)$.

Proof: Suppose (i) holds Let $\{U_\alpha : \alpha \in \Delta\}$ be an i-open cover for X. Then there is a countable subset N of Δ such that $X = j\text{-}cl\left(\bigcup_{n \in N} U_n\right)$. Let $U = \bigcup_{n \in N} U_n$. Since U is i-open, by using Lemma 7.2.6 (iii), $j\text{-}cl\text{U} = ji\text{-}\delta\ cl\ U = ji\text{-}\theta\ cl\ U$. This proves (ii) and (iii). By the same technique the reverse implications can be proved.

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