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Common Fixed Point Theorem in Generalized $\mathcal{M}$-Fuzzy Metric Space

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Abstract: Our aim of this paper is to obtain a common fixed point theorem for three self mappings of generalized $\mathcal{M}$-fuzzy metric space. Now we prove common fixed point theorem for three self mapping of $\mathcal{M}$-fuzzy metric space.

***Mathematics Subject Classification*:** 47H10, 54H25.

***Keywords*:** $\mathcal{M}$-Fuzzy metric space, Complete $\mathcal{M}$-Fuzzy metric space, Common fixed point.

1. Introduction

Many authors have introduced the concept of fuzzy metric spaces in different way. Kramosil and Michalek is one of them. Recently S. Sedghi and N. Shobe developed a new concept of $\mathcal{M}$-fuzzy metric space and proved fixed point theorem in this newly developed space.

Definition 1.1: A mapping $* : [0, 1]\ x\ [0, 1] \to [0, 1]$ is called a triangular norm (shortly t-norm) if it satisfies the following conditions.

(i) $*(a, 1) = \text{a}$ for every $a \in [0, 1]$

(ii) $*(a, b) = *(b, a)$ for every $a, b \in [0, 1]$

(iii) $*(a, c) \geq *(b, d)$ for $a \geq b$; $c \geq d$

(iv) $*(a, *(b, c,)) = *(*(a, b), c)$ for all $a, b, c \in [0, 1]$

Example 1.2:

$$a * b = ab \quad \text{for} \quad a, b \in [0, 1].$$

Example 1.3:

$$a * b = \min\{a, b\} \quad \text{for} \quad a, b \in [0, 1].$$

Definition 1.4: The triple $(X, \mathcal{M}, *)$ is a $\mathcal{M}$-fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and $\mathcal{M}$ is a fuzzy set in $X^3 \times (0, \infty)$ satisfying the following conditions for each

$$x, y, z, a \in X \quad \text{and} \quad t, s > 0$$

1. $\mathcal{M}(x, y, z, t) > 0$, for all $x, y, z \in X$
2. $\mathcal{M}(x, y, z, t) = 1$ iff $x = y = z$, for all $t > 0$,
3. $\mathcal{M}(x, y, z, t) = \mathcal{M}(p\{x, y, z\}, t)$, where p is a permutation function,
4. $\mathcal{M}(x, y, a, t) * \mathcal{M}(a, z, z, s) \leq \mathcal{M}(x, y, z, t + s)$,
5. $\mathcal{M}(x, y, z, .) : (0, \infty) \to [0, 1]$ is continuous.

Example 1.5: Let X be a nonempty set and D is the D^*-metric on X.

Denote $a * b = a.b$ for all $a, b \in [0, 1]$. For each $t \in (0, \infty)$, Define

$$\mathcal{M}(x, y, z, t) = \frac{t}{t + D^*(x, y, z)}$$

Definition 1.6: A sequence $\{x_n\}$ in X converges to x if and only if $\mathcal{M}(x, x, x_n, t) \to 1$ as $n \to \infty$, for each $t > 0$.

Definition 1.7: A sequence $\{x_n\}$ is called a Cauchy sequence if for each $0 < \in < 1$ and $t > 0$, there exist $n_0 \in N$ such that $\mathcal{M}(x_n, x_n, x_m, t) > 1 - \in$ for each $n, m \geq n_0$.

Definition 1.8: A fuzzy $\mathcal{M}$-metric $(X, \mathcal{M}, *)$ is said to be complete if every Cauchy sequence is convergent.

Definition 1.9: Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$-fuzzy metric space, then $\mathcal{M}$ is called of first type if for every $x, y \in X$ we have $\mathcal{M}(x, x, y, t) \geq \mathcal{M}(x, y, z, t)$ for every $z \in X$.

Theorem 2.1: Let T_1, T_2 and T_3 be three mappings of a complete first type. $\mathcal{M}$-fuzzy metric space $(X, \mathcal{M}, *)$ satisfying the conditions.

$$\mathcal{M}(T_1x, T_2y, T_3z, t) \geq \min\{\mathcal{M}(x, y, z, t/r), \mathcal{M}(x, T_1x, T_2y, t/r), \mathcal{M}(y, T_2y, T_3z, t/r), \mathcal{M}(z, T_3z, T_1x, t/r)\} \quad (2.1.1)$$

Where $0 < r < 1$.Then T_1, T_2 and T_3 have a unique common fixed point.

Proof: Consider an arbitrary point x_0 in X and define a sequence $\{x_n\}$ in X by $x_{3n+1} = T_1x_{3n}$; $x_{3n+2} = T_2x_{3n+1}$; $x_{3n+3} = T_3x_{3n+2}$ for all $n = 0, 1, 2\ldots$ on using (2.1.1)

$$\begin{aligned}
\mathcal{M}(x_1, x_2, x_3, t) &= \mathcal{M}(T_1x_0, T_2x_1, T_3x_2, t) \\
&\geq \min\{\mathcal{M}(x_0, x_1, x_2, t/r), \mathcal{M}(x_0, T_1x_0, T_2x_1, t/r), \mathcal{M}(x_1, T_2x_1, T_3x_2, t/r), \\
&\quad \mathcal{M}(x_2, T_3x_2, T_1x_0, t/r)\} \\
&\geq \min\{\mathcal{M}(x_0, x_1, x_2, t/r), \mathcal{M}(x_0, x_1, x_2, t/r), \mathcal{M}(x_1, x_2, x_3, t/r), \\
&\quad \mathcal{M}(x_2, x_3, x_1, t/r)\} \\
&= \min\{\mathcal{M}(x_0, x_1, x_2, t/r), \mathcal{M}(x_1, x_2, x_3, t/r)\} \\
&\geq \mathcal{M}(x_0, x_1, x_2, t/r)
\end{aligned}$$

Continuing this way

$$\begin{aligned}\mathcal{M}(x_2, x_3, x_4, t) &\geq \mathcal{M}(x_1, x_2, x_3, t/r)\\ &\geq \mathcal{M}(x_0, x_1, x_2, t/r^2)\end{aligned}$$

In general, we can define $\{x_n\}$ in X.

$$\begin{aligned}\mathcal{M}(x_n, x_{n+1}, x_{n+2}, t) &\geq \mathcal{M}(x_0, x_1, x_2, t/r^n)\\ &\to 1 \text{ as } n \to \infty\end{aligned}$$

Since $\mathcal{M}$ is first type

$$\begin{aligned}\mathcal{M}(x_n, x_n, x_{n+1}, t) &\geq \mathcal{M}(x_n, x_{n+1}, x_{n+2}, t)\\ &\to 1 \text{ as } n \to \infty\end{aligned}$$

Now we prove $\{x_n\}$ is a $\mathcal{M}$-fuzzy Cauchy sequence.

Let $m, n \geq n_0$ and $m > n$.

$$\begin{aligned}\mathcal{M}(x_n, x_n, x_m, t) &\geq \mathcal{M}(x_n, x_n, x_{m-1}, t/2) * \mathcal{M}(x_{m-1}, x_m, x_m, t/2)\\ &\geq \mathcal{M}(x_n, x_n, x_{m-2}, t/2^2) * \mathcal{M}(x_{m-2}, x_{m-1}, x_{m-1}, t/2^2)\\ &\quad * \mathcal{M}(x_m, x_{m-1}, x_{m-1}, t/2)\\ &\geq \mathcal{M}(x_n, x_n, x_n, t/2^{m-n}) * \mathcal{M}(x_n, x_n, x_{n+1}, t/2^{m-n-1})\\ &\quad * \ldots\ldots\ldots\ldots\ldots * \mathcal{M}(x_{m-1}, x_{m-1}, x_m, t/2)\\ &= 1 * 1 * \ldots * 1\\ &= 1.\end{aligned}$$

Thus $\quad \mathcal{M}(x_n, x_n, x_{n+1}) \to 1$ as $m, n \to \infty$.

Therefore $\{x_n\}$ is a Cauchy sequence in X and X is generalized complete $\mathcal{M}$-fuzzy metric space and we have $\{x_n\} \to x$ in X.

Hence $\{x_{3n}\}$, $\{x_{3n+1}\}$ and $\{x_{3n+2}\}$ are all converges to X.

Now, we prove that x is a fixed point of T_1.

Now, we have

$$\begin{aligned}\mathcal{M}(T_1x, x, x, t) &\geq \lim_{n\to\infty} \mathcal{M}(T_1x, x_{3n+2}, x_{3n+3}, t)\\ &\geq \lim_{n\to\infty} \mathcal{M}(T_1x, T_2x_{3n+1}, T_3x_{3n+2}, t)\\ &\geq \lim_{n\to\infty} \min\{\mathcal{M}(x, x_{3n+1}, x_{3n+2}, t/r), \mathcal{M}(x, T_1x, T_2x_{3n+1}, t/r),\\ &\quad \mathcal{M}(x_{3n+1}, T_2x_{3n+1}, T_3x_{3n+2}, t/r), \mathcal{M}(x_{3n+2}, T_3x_{3n+2}, T_1x, t/r)\}\end{aligned}$$

$$= \lim_{n\to\infty} \min\{\mathcal{M}(x, x_{3n+1}, x_{3n+2}, t/r), \mathcal{M}(x, T_1x, x_{3n+2}, t/r),$$

$$\mathcal{M}(x_{3n+1}, x_{3n+2}, x_{3n+3}, t/r), \mathcal{M}(x_{3n+2}, x_{3n+3}, T_1x, t/r)\}$$

$$\geq \lim_{n\to\infty} \min\{1, \mathcal{M}(x, T_1x, x, t/r), \mathcal{M}(x, x, x, t/r), \mathcal{M}(x, x, T_1x, t/r)\}$$

$$\geq \mathcal{M}(x, x, T_1x, t/r)$$

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$$\geq \mathcal{M}(x, x, T_1x, t/r^n)$$

$$\to 1 \text{ as } n \to \infty$$

Thus $\mathcal{M}(T_1x, x, x, t) = 1$ for all $t > 0$

Hence $T_1x = x$.

Similarly we can prove for $T_2x = x$ & $T_3x = x$.

Therefore x is common fixed point of T_1, T_2 and T_3

Uniqueness: Suppose y is another common fixed point of T_1, T_2 and T_3.

Then

$$\mathcal{M}(x, x, y, t/r) = \mathcal{M}(T_1x, T_2x, T_3y, t)$$

$$\geq \min\{\mathcal{M}(x, x, y, t/r), \mathcal{M}(x, T_1x, T_2x, t/r),$$

$$\mathcal{M}(x, T_2x, T_3y, t/r), \mathcal{M}(y, T_3y, T_1x, t/r)\}$$

$$\geq \min\{\mathcal{M}(x, x, y, t/r), 1, \mathcal{M}(x, x, y, t/r), \mathcal{M}(y, y, x, t/r)\}$$

$$= \mathcal{M}(y, y, x, t/r)$$

$$\geq \mathcal{M}(x, x, y, t/r^2)$$

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$$\geq \mathcal{M}(y, y, x, t/r^n)$$

$$\to 1 \text{as } n \to \infty$$

Therefore $\mathcal{M}(x, x, y, t) = 1$

Hence $x = y$.

This completes the proof of the theorem.

Corollary 2.2: Let T_1, T_2 and T_3 be three mappings of a complete first type $\mathcal{M}$-fuzzy metric space $(X, \mathcal{M}, *)$ satisfying the conditions.

$$\mathcal{M}(T_1x, T_2y, T_3z, t) \geq \{\mathcal{M}(x, y, z, t/r) * \mathcal{M}(x, T_1x, T_2y, t/r) * \mathcal{M}(y, T_2y, T_3z, t/r) * \mathcal{M}(z, T_3z, T_1x, t/r)\} \quad (2.2.1)$$

Where $0 < r < 1$.Then T_1, T_2 and T_3 have a unique common fixed point.

Theorem 2.3: Let $(X, \mathcal{M}, *)$ be a generalized complete fuzzy metric space and $T : X \to X$ be a mappings such that

$$4\,\mathcal{M}(Tx, Ty, Tz, t) \geq \{\mathcal{M}(x, y, z, t_1/ka) + \mathcal{M}(x, Tx, Ty, t_2/kb) + \mathcal{M}(x, y, Ty, t_3/kc) + \mathcal{M}(Tx, z, Tz, t_4/kd)\} \quad (2.3.1)$$

for all $x, y, z \in X$, and $t = t_1 + t_2 + t_3 + t_4$ and $a + b + c + d = 1$. Then T has a unique fixed point.

Proof: Let x_0 be an arbitrary fixed element in X and define a sequence $\{x_n\}$ in X as

$$x_{n+1} = Tx_n \quad \text{for all} \quad n = 0, 1, 2, \ldots$$

Putting $t_1 =$ at, $t_2 = bt$, $t_3 = ct$, $t_4 = dt$ in (2.3.1)

we get

$$\begin{aligned}
4\,\mathcal{M}(x_n, x_n, x_{n+1}, t) &= 4\,\mathcal{M}(Tx_{n-1}, Tx_{n-1}, Tx_n, t)\\
&\geq \{\mathcal{M}(x_{n-1}, x_{n-1}, x_n, t/k) + \mathcal{M}(x_{n-1}, Tx_{n-1}, Tx_{n-1}, t/k)\\
&\quad + \mathcal{M}(x_{n-1}, x_{n-1}, Tx_{n-1}, t/k) + \mathcal{M}(Tx_{n-1}, x_n, Tx_n, t/k)\}\\
&\geq \{\mathcal{M}(x_{n-1}, x_{n-1}, x_n, t/k) + \mathcal{M}(x_{n-1}, x_n, x_n, t/k)\\
&\quad + \mathcal{M}(x_{n-1}, x_{n-1}, x_n, t/k) + \mathcal{M}(x_n, x_n, x_{n+1}, t/k)\}\\
3\,\mathcal{M}(x_n, x_n, x_{n+1}, t) &\geq 3\,\mathcal{M}(x_{n-1}, x_{n-1}, x_n, t/k)\\
\mathcal{M}(x_n, x_n, x_{n+1}, t) &\geq \mathcal{M}(x_{n-1}, x_{n-1}, x_n, t/k) \text{ for all } n \geq 0
\end{aligned}$$

Thus

$$\mathcal{M}(x_n, x_n, x_{n+1}, t) \geq \mathcal{M}(x_0, x_0, x_1, t/k^n)$$

Hence $\{x_n\}$ is $\mathcal{M}$-fuzzy Cauchy sequence in X.

Since X is complete $\mathcal{M}$-fuzzy metric space and $\{x_n\}$ is a fuzzy Cauchy sequence in X. We have $x_n \to x$ in X.

Now, we prove that x is a fixed point of T.

Suppose $x \neq Tx$

Now, we have

$$\begin{aligned}
4\,\mathcal{M}(Tx, x, x, t) &= \lim_{n\to\infty} 4\,\mathcal{M}(Tx, x_{n+1}, x_{n+2}, t)\\
&\geq \lim_{n\to\infty} 4\,\mathcal{M}(Tx, Tx_n, Tx_{n+1}, t)
\end{aligned}$$

$$\geq \lim_{n\to\infty} \{\mathcal{M}(x, x_n, x_{n+1}, t/k) + \mathcal{M}(x, Tx, Tx_n, t/k) +$$

$$\mathcal{M}(x, x_n, Tx_n, t/k) + \mathcal{M}(Tx, x_{n+1}, Tx_{n+1}, t/k)\}$$

$$\geq \lim_{n\to\infty} \{\mathcal{M}(x, x_n, x_{n+1}, t/k) + \mathcal{M}(x, Tx, x_{n+1}, t/k)$$

$$+ \mathcal{M}(x, x_n, x_{n+1}, t/k) + \mathcal{M}(Tx, x_{n+1}, x_{n+2}, t/k)\}$$

$$\geq 2 + \mathcal{M}(x, Tx, x, t/k) + \mathcal{M}(Tx, x, x, t/k)$$

$$\mathcal{M}(Tx, x, x, t) \geq \mathcal{M}(x, Tx, x, t/k)$$

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$$\geq M(x, Tx, x, t/k^n)$$

$$\to 1 \text{ as } n \to \infty$$

Thus $\mathcal{M}(Tx, x, x, t) = 1$ for all t > 0

Hence $Tx = x$.

Therefore x is fixed point of T.

Uniqueness: Suppose $x \neq y$ such that $x = Tx$ and $Ty = y$.

Now consider

$$4\,\mathcal{M}(x, x, y, t) = 4\,\mathcal{M}(Tx, Tx, Ty, t)$$

$$\geq \{\mathcal{M}(x, x, y, t/k) + \mathcal{M}(x, Tx, Tx, t/k) + \mathcal{M}(x, x, Tx, t/k)$$

$$+ \mathcal{M}(Tx, y, Ty, t/k)\}$$

$$\geq \mathcal{M}(x, x, y, t/k) + \mathcal{M}(x, x, x, t/k) + \mathcal{M}(x, x, x, t/k)$$

$$+ \mathcal{M}(x, y, y, t/k)$$

$$\geq 2 + \mathcal{M}(x, x, y, t/k) + \mathcal{M}(x, y, y, t/k)$$

$$\geq 2 + 2\,\mathcal{M}(x, y, y, t/k)$$

$$\mathcal{M}(x, x, y, t) \geq \mathcal{M}(x, y, y, t/k)$$

Which is contradiction.

This completes the proof of the theorem

Corollary 2.4: Let $(X, \mathcal{M}, *)$ be a generalized complete fuzzy metric space and $T : X \to X$ be a mapping such that

$$\mathcal{M}(Tx, Ty, Tz, t) \geq \min\{\mathcal{M}(x, y, z, t_1/ka), \mathcal{M}(x, Tx, Ty, t_2/kb), \mathcal{M}(x, y, Ty, t_3/kc), \mathcal{M}(y, z, Tz, t_4/kd)\} \tag{2.4.1}$$

for all $x, y, z \in X$, and $t = t_1 + t_2 + t_3 + t_4$ and $a + b + c + d = 1$.

Then T has a unique fixed point.

Putting $t_4 = 0$, $t_1 =$ at, $t_2 = bt$, $t_3 = ct$, in (2.4.1). We get the following

Corollary 2.5: Let $(X, \mathcal{M}, *)$ be a generalized complete fuzzy metric space and $T : X \to X$ be a mapping such that $\mathcal{M}(Tx, Ty, Tz, t) \geq \min\{\mathcal{M}(x, y, z, t/k), \mathcal{M}(x, Tx, Ty, t/k), \mathcal{M}(x, y, Ty, t/k)\}$ for all $x, y, z \in X$, and $t = t_1 + t_2 + t_3$ and $a + b + c = 1$. Then T has a unique fixed point.

Putting $t_4, t_3 = 0$, $t_1 =$ at, $t_2 = bt$, in (2.4.1). We get the following

Corollary 2.6: Let $(X, \mathcal{M}, *))$ be a generalized complete fuzzy metric space and $T : X \to X$ be a mapping such that $\mathcal{M}(Tx, Ty, Tz, t) \geq \min\{\mathcal{M}(x, y, z, t/k), \mathcal{M}(x, Tx, Ty, t/k)\}$ for all $x, y, z \in X$, and $t = t_1 + t_2$ and $a + b = 1$. Then T has a unique fixed point.

Putting $t_4, t_3, t_2 = 0$, $t_1 =$ at, in (2.4.1). We get the following

Corollary 2.7: Let $(X, \mathcal{M}, *))$ be a generalized complete fuzzy metric space and $T : X \to X$ be a mapping such that $\mathcal{M}(Tx, Ty, Tz, t) \geq \mathcal{M}(x, y, z, t/k)$ for all $x, y, z \in X$, and $t = t_1$ and a $= 1$. Then T has a unique fixed point.

Putting $t_3 = 0$, $t_1 =$ at, $t_2 = bt$, $t_4 = dt$ in (2.4.1). We get the following

Corollary 2.8: Let $(X, \mathcal{M}, *))$ be a generalized complete fuzzy metric space and $T : X \to X$ be a mapping such that $\mathcal{M}(Tx, Ty, Tz, t) \geq \min\{\mathcal{M}(x, y, z, t/k), \mathcal{M}(x, Tx, Ty, t/k), \mathcal{M}(y, z, Tz, t/k)\}$ for all $x, y, z \in X$, and $t = t_1 + t_2 + t_4$ and $a + b + d = 1$. Then T has a unique fixed point.

Putting $t_2 = 0$, $t_1 =$ at, $t_3 = ct$, $t_4 = dt$ in (2.4.1). We get the following

Corollary 2.9: Let $(X, \mathcal{M}, *))$ be a generalized complete fuzzy metric space and $T : X \to X$ be a mapping such that $\mathcal{M}(Tx, Ty, Tz, t) \geq \min\{\mathcal{M}(x, y, z, t/k), \mathcal{M}(x, y, Ty, t/k), \mathcal{M}(y, z, Tz, t/k)\}$ for all $x, y, z \in X$, and $t = t_1 + t_3 + t_4$ and $a + c + d = 1$. Then T has a unique fixed point.

Putting $t_1 = 0$, $t_2 = bt$, $t_3 = ct$, $t_4 = dt$ in (2.4.1). We get the following

Corollary 2.10: Let $(X, \mathcal{M}, *))$ be a generalized complete fuzzy metric space and $T : X \to X$ be a mapping such that $\mathcal{M}(Tx, Ty, Tz, t) \geq \min\{\mathcal{M}(x, Tx, Ty, t/k), \mathcal{M}(x, y, Ty, t/k), \mathcal{M}(y, z, Tz, t/k)\}$ for all $x, y, z \in X$, and $t = t_2 + t_3 + t_4$ and $b + c + d = 1$. Then T has a unique fixed point.

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