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ON MEROMORPHIC MULTIVALENT FUNCTION USED BY LINEAR OPERATOR

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Abstract: In the present paper, we introduce two classes of meromorphically multivalent functions and application of linear operators on these classes. We study various properties and coefficients bounds, the concept of neighbourhood also investigated.

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1. INTRODUCTION

Let $T^*(p)$ the class of functions $f(z)$ of the form

$$f(z) = z^{-p} + \sum_{n=1}^{\infty} a_n z^{n-p}, p \in I\!N = \{1, 2, ...\} \tag{1}$$

which are analytic and multivalent in the punctured unit disk $U^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\}$.

The Hadamard product of f and g where f definedd by (1) and $g(z) = z^{-p} + \sum_{n=1}^{\infty} b_n z^{n-p}$ denote by $f * g$ defined as:

$$(f * g)(z) = z^{-p} + \sum_{n=1}^{\infty} a_n b_n z^{n-p}. \tag{2}$$

Now let

$$\varphi_p(a, c; z) = z^{-p} + \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} z^{n-p} \tag{3}$$

($z \in U^*$, $a \in I\!R$, $c \in I\!R$, $c \neq 0, -1, -2, ...$) $(a)_0 = 1$ and $(a)_n = a(a+1) \ldots (a+n-1)$, $n \in I\!N$ which is called shifted factorial.

Consider the class $K_{a,c}(p; A, B, \delta)$, a function $f \in T^*(p)$ belongs to $K_{a,c}(p; A, B, \delta)$ if it satisfies the following condition

$$\left|\frac{z(L_p(a,c)' f(x)) + pL_p(a,c) f(z)}{Bz(L_p(a,c) f(z))' + p[B+(A-B)(1-\delta)] L_p(a,c) f(z)}\right| < 1 \tag{4}$$

where $(-1 \le B < A \le 1,\ 0 \le \delta < 1,\ p \in I\!N,\ z \in U,\ a \in I\!R,\ c \in I\!R,\ c \ne 0, -1, -2, \ldots)$ and

$$L_p(a, c) f(z) = \varphi_p(a, c, z) * f(z), f \in T^*(p). \tag{5}$$

The Definition of $L_p(a, c) f(z)$ is motivated by Carlson – Shaffer [2] and the class $K_{a,c}(p; A, B, \delta)$ is generalized to the class studied by Liu and Srivastava [5].

The function $f(z) \in K_{a,c}(p; A, B, \delta)$ is in the class $K^+_{a,c}(p; A, B, \delta)$ such that

$$f(z) = z^{-p} + \sum_{n=p}^{\infty} |a_n| z^n, \ (p \in I\!N). \tag{6}$$

Special cases of the classes $K^+_{a,c}(p; A, B, \delta)$ and $K_{a,c}(p; A, B, \delta)$

(1) If $a = c = 1$; $\delta = 0$ we get the class $K^+_{1,1}(p; A, B)$ was investigated by Mogra [6].

(2) If $\delta = 0$ we get the class $K_{a,c}(p; A, B)$ was studied by Liu and Srivastava [5].

2. INCLUSION PROPERTIES OF THE CLASS $K_{a,c}(p; A, B, \delta)$

In order to prove our results we need the following Lemma.

Lemma (Jack [4]): Let $w(z)$ be analytic non constant function in U with $w(0) = 0$. If $w(z)$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in U$, then

$$z_0 w'(z_0) = \mu w(z_0), \quad \text{where} \quad \mu \in I\!R \text{ and } \mu \ge 1. \tag{7}$$

Theorem 2.1: Let $a \ge \frac{p(1-\delta)(A-B)}{B+1}$, then $K_{a+1,c}(p; A, B, \delta) \subset K_{a,c}(p; A, B, \delta)$ where $(-1 < B < A \le 1;\ 0 \le \delta < 1,\ p \in I\!N)$.

Proof. Assume that $f \in K_{a+1,c}(p; A, B, \delta)$ and suppose that

$$\frac{z(L_p(a,c) f(z))'}{L_p(a,c) f(z)} = -p\left(\frac{1+[B+(A-B)(1-\delta)]\, w(z)}{1+Bw(z)}\right) \tag{8}$$

for $w(z)$ is analytic or meromorphic in U, with $w(0) = 0$. From (3) and (5) we have

$$z(L_p(a, c) f(z))' = aL_p(a+1, c) f(z) - (a+p) L_p(a, c) f(z). \tag{9}$$

Now from (9) and (8), we get

$$\frac{aL_p(a+1,c) f(z)}{L_p(a,c) f(z)} = \frac{a + [aB - p(A-B)(1-\delta)]\, w(z)}{1+Bw(z)} \tag{10}$$

then

$$\frac{(L_p(a+1,c)f(z))'}{L_p(a+1,c)f(z)} = \frac{z(L_p(a,c)f(z))'}{L_p(a,c)f(z)} + \frac{[aB-p(A-B)(1-\delta)]zw'(z)}{a+[aB-p(A-B)(1-\delta)]w(z)} - \frac{Bzw'(z)}{1+Bw(z)}. \quad (11)$$

The last expression obtained by differentiating logarithmically with respect to z of (10), so

$$\frac{z(L_p(a+1,c)f(z))'}{L_p(a+1,c)f(z)} = -p\left[\frac{1+[B+(A-B)(1-\delta)]w(z)}{1+Bw(z)}\right]$$

$$-\frac{p(1-\delta)(A-B)zw'(z)}{(1+Bw(z))[a+(aB-p(A-B)(1-\delta))w(z)]}. \quad (12)$$

Now suppose that there exists $z_0 \in U$ such that $\max_{|z|\le|z_0|}|w(z)| = |w(z_0)| = 1$, then by Jack's lemma we have $z_0w'(z_0) = \mu w(z_0)$; $(\mu \geq 1)$.

Let $w(z_0) = e^{i\theta}(0 \le \theta < 2\pi)$ in (12), we get after setting $z = z_0$

$$\left|\frac{z_0(L_p(a,c)f(z_0))' + pL_p(a,c)f(z_0)}{Bz_0(L_p(a,c)f(z_0))' + [Bp+(A-B)(1-\delta)p]L_p(a,c)f(z_0)}\right|^2 - 1$$

$$= \left|\frac{-p(a+\mu)+[(aB-p(A-B)(1-\delta))]e^{i\theta}}{a+[aB-\mu B-p(A-B)(1-\delta)]e^{i\theta}}\right|^2 - 1$$

$$\geq \left|\frac{a+\mu+[aB-p(A-B)(1-\delta)]e^{i\theta}}{a+[aB-\mu-p(A-B)(1-\delta)]e^{i\theta}}\right|^2 - 1$$

$$= \frac{2\mu(1+\cos\theta)[a(B+1)-p(A-B)(1-\delta)]}{|a+[aB-\mu-p(A-B)(1-\delta)]e^{i\theta}|^2} \geq 0,$$

since $a \geq \frac{p(A-B)(1-\delta)}{1+B}$.

This is a contradiction with our hypothesis that $f \in K_{a+1,c}(p; A, B, \delta)$, then $|w(z)| < 1$, $(z \in U)$ and we have $f \in K_{a,c}(p; A, B, \delta)$.

Theorem 2.2: Let $f(z) \in K_{a,c}(p; A, B, \delta)$, then $g(z)$ definedd by

$$L_p(a,c)g(z) = \left(\frac{k-p\alpha}{z^k}\int_0^z t^{k-1}[L_p(a,c)f(t)]^\alpha dt\right)^{1/\alpha} \quad (13)$$

where $\alpha > 0$, $R(k) \geq p\alpha\left(\frac{1+[B+(A-B)(1-\delta)]}{1+B}\right) > 0$, $p \in I\!N$ is also in the class $K_{a,c}(p; A, B, \delta)$.

Proof: Consider $f(z) \in K_{a,c}(p; A, B, \delta)$ and by using (13), we have

$$[L_p(a, c)\, g(z)]^{\alpha} = \frac{k - p\alpha}{z^k} \int_0^z t^{k-1} [L_p(a, c)\, f(t)]^{\alpha}\, dt\,. \tag{14}$$

After differentiating logarithmically both sides of (14), we get

$$\frac{z(L_p(a,c)\, g(z))'}{L_p(a,c)\, g(z)} = -\frac{k}{\alpha} + \frac{k - p\alpha}{\alpha}\left[\frac{L_p(a,c)\, f(z)}{L_p(a,c)\, g(z)}\right]^{\alpha}. \tag{15}$$

Let

$$\frac{z(L_p(a,c)\, g(z))'}{L_p(a,c)\, g(z)} = -p\left(\frac{1 + [B + (A-B)(1-\delta)]\, w(z)}{1 + Bw(z)}\right). \tag{16}$$

then from (15) and (16), we get

$$\frac{k(L_p(a,c) f(z))^{\alpha} + (\alpha p - k)(L_p(a,c) f(z))^{\alpha}}{(L_p(a,c) g(z))^{\alpha}} = \frac{\alpha p + \alpha p[B + (A-B)(1-\delta)]w(z)}{1 + Bw(z)}. \tag{17}$$

Differentiating both sides of (17), we have

$$\frac{z(L_p(a,c)\, f(z))'}{L_p(a,c)\, f(z)} = \frac{p(1 + [B + (A-B)(1-\delta)]\, w(z))}{\alpha p(1 + [B + (A-B)(1-\delta)]\, w(z)) - k(1 + Bw(z))}$$

$$\times\left[k - \alpha p\left\{\frac{1 + [B + (A-B)(1-\delta)]\, w(z) + Bzw'(z)}{1 + Bw(z)}\right\} + \frac{[B + (A-B)(1-\delta)]\, zw'(z)}{1 + [B + (A-B)(1-\delta)]\, w(z)}\right]. \tag{18}$$

By making necessary changes in previous theorem and suppose that $\max_{|z| \le |z_0|} |w(z)| = |w(z_0)| = 1$, we find $z_0 w'(z_0) = \mu w(z_0)$ by applying Jack's Lemma, where $z_0 \in U$, $\mu \ge 1$ and $\mu \in \mathbb{R}$. Let $w(z_0) = e^{i\theta}(\theta \ne p)$, in (18), we have

$$\left|\frac{z_0(L_p(a,c)\, f(z_0))' + L_p(a,c)\, f(z_0)}{Bz_0(L_p(a,c)\, f(z_0))' + p[B + (A-B)(1-\delta)]\, L_p(a,c)\, f(z_0)}\right|^2 - 1$$

$$= \left|\frac{k + \mu - \alpha p + [Bk - \alpha p(B + (A-B)(1-\delta)]\, e^{i\theta}}{k - \alpha p + [Bk - B\mu - \alpha p(B + (A-B)(1-\delta)]\, e^{i\theta}}\right|^2 - 1$$

$$= \frac{h(\theta)}{|(k - p\alpha) + [Bk - B\mu - \alpha p(B + (A-B)(1-\delta)]\, e^{i\theta}|^2}$$

where

$$h(\theta) = \mu^2(1 - B^2) + 2\mu\,[(1 + B^2)\,k - \alpha p\,(1 + B\,(B + (A - B)(1 - \delta)))]$$
$$+ 2\mu\,[2BRe\,(k) - p\alpha\,(2B + (A - B)\,(1 - \delta))]\cos\theta$$

where

$$0 \le \theta < 2\pi,\ -1 \le B < A \le 1,\ \mu \ge 1,\ 0 \le \delta < 1.$$

By hypothesis we have $Re\,(k) \ge p\alpha\left(\frac{1+[B+(A-B)(1-\delta)]}{1+B}\right)$ thus $h(0) \ge 0$ and $h(\pi) \ge 0$ which shows that $h(\theta) \ge 0\,(0 \le \theta < 2\pi)$. So we get contradiction with our hypothesis. Therefore, $|w(z)| < 1$, $z \in U$, then $g(z) \in K_{a,c}(p; A, B, \delta)$.

3. COEFFICIENT BOUNDS

To investigate the coefficient bounds and some other results we assume that $a > 0$, $c > 0$ and $A + B \le 0$, $(-1 \le B < A \le 1)$.

Theorem 3.1: If $f(z) \in T^*(p)$ definedd by (6), then $f \in K^+_{a,c}(p; A, B, \delta)$ if and only if

$$\sum_{n=p}^{\infty} [(1 - B)(n + p) - p\,(A - B)\,(1 - \delta)]\,\frac{(a)_{n+p}}{(c)_{n+p}}\,|a_n| \le p\,(1 - \delta)\,(A - B). \qquad (19)$$

The result is sharp for $f(z)$ given by

$$f(z) = z^{-p} + \left(\frac{p(1-\delta)(A-B)}{n(1-B)+p(1-B-(A-B)(1-\delta))}\right)\frac{(c)_{n+p}}{(a)_{n+p}}\,z^n,\ n = p, p+1, \ldots \qquad (20)$$

Proof: Let $f \in K^+_{a,c}(p; A, B, \delta)$ given by (6), then

$$\left|\frac{z(L_p(a,c)\,f(z))' + pL_p(a,c)\,f(z)}{Bz(L_p(a,c)\,f(z))' + p(B+(A-B)(1-\delta))\,L_p(a,c)\,f(z)}\right|$$

$$= \left|\frac{\sum_{n=p}^{\infty}(n+p)\,|a_n|\,\frac{(a)_{n+p}}{(c)_{n+p}}\,z^{n+p}}{p(A-B)(1-\delta) + \sum_{n=p}^{\infty}(B(n+p)+p(A-B)(1-\delta))\,|a_n|\,\frac{(a)_{n+p}}{(c)_{n+p}}\,z^{n+p}}\right| < 1.$$

choose z to be real and $z \to 1^-$, we obtain

$$\sum_{n=p}^{\infty}\frac{(a)_{n+p}}{(c)_{n+p}}\,(n+p)|a_n| \le p(A-B)(1-\delta) + \sum_{n=p}^{\infty}(B(n+p)+p(A-B)(1-\delta)|a_n|\,\frac{(a)_{n+p}}{(c)_{n+p}}$$

then,

$$\sum_{n=p}^{\infty} [(1-B)(n+p) - p(A-B)(1-\delta)] \frac{(a)_{n+p}}{(c)_{n+p}} |a_n| \le p(A-B)(1-\delta).$$

Conversely, assume that the inequality (10) holds true then

$$\left| \frac{z(L_p(a,c)f(z))' + pL_p(a,c)f(z)}{Bz(L_p(a,c)f(z))' + p(B+(A-B)(1-\delta))L_p(a,c)f(z)} \right|$$

$$\le \left| \frac{\sum_{n=p}^{\infty} (n+p)|a_n| \frac{(a)_{n+p}}{(c)_{n+p}} |a_n|}{p(A-B)(1-\delta) + \sum_{n=p}^{\infty} (B(n+p) + p(A-B)(1-\delta)) \frac{(a)_{n+p}}{(c)_{n+p}} |a_n|} \right| < 1$$

$(z \in U; z \in \mathbb{C}; |z| = 1)$.

Here, by Maximum Modulus Theorem we get $f(z) \in K^{+}_{a,c}(p; A, B, \delta)$. Finally, we observe that the function given by (20) is an extremal function.

Next we investigate the extreme points of the class $K^{+}_{a,c}(p; A, B, \delta)$.

Theorem 3.2: $f(z) \in K^{+}_{a,c}(p; A, B, \delta)$ of the form (6) if and only if it can be expressed of the form

$$f(z) = \sum_{n=p-1}^{\infty} \lambda_n f_n(z), \lambda_n \ge 0, n = p-1, p, \ldots \tag{21}$$

where $f_{p-1}(z) = z^{-p}$, $f_n(z) = z^{-p} + \frac{p(1-\delta)(A-B)}{n(1-B)+p(1-B-(A-B)(1-\delta))} \frac{(c)_{n+p}}{(a)_{n+p}} z^n$, $n = p, p+1, \ldots$ and $\sum_{n=p-1}^{\infty} \lambda_n = 1$.

Proof: Let $f(z)$ of the form (21), then

$$f(z) = \lambda_{p-1} z^{-p} + \sum_{n=p}^{\infty} \lambda_n \left[z^{-p} + \frac{p(1-\delta)(A-B)(c)_{n+p}}{n(1-B)+p(1-B-(A-B)(1-\delta))(a)_{n+p}} z^n \right]$$

$$= \left[z^{-p} + \sum_{n=p}^{\infty} \frac{p(1-\delta)(A-B)(c)_{n+p}}{[n(1-B)+p(1-B-(A-B)(1-\delta))](a)_{n+p}} \lambda_n z^n \right]$$

then by Theorem 3.1 we have $f(z) \in K^{+}_{a,c}(p; A, B, \delta)$.

Conversely, let $f(z) \in K^{+}_{a,c}(p; A, B, \delta)$ where $f(z)$ given by (6) then

$$\sum_{n=p}^{\infty} \frac{[n(1-B)+p(1-B-(A-B)(1-\delta))]\,(a)_{n+p}}{p(1-\delta)(A-B)\,(c)_{n+p}}\,|a_n| \le 1$$

so we obtain $\lambda_{p-1} = 1 - \sum_{n=p}^{\infty} \lambda_n$ where

$$\lambda_n = \frac{[n(1-\beta)+p(1-B-(A-B)(1-\delta))]\,(a)_{n+p}}{p(1-\delta)(A-B)\,(c)_{n+p}}\,|a_n|, \;\; n = p, p+1, \ldots$$

then

$$f(z) = \lambda_{p-1} z^{-p} + \sum_{n=p}^{\infty} \lambda_n f_n(z) = \sum_{n=p-1}^{\infty} \lambda_n f_n(z).$$

Theorem 3.3: Let $f_i(z) = z^{-p} + \sum_{n=p}^{\infty} |a_{n,i}| z^n$ for $i = 1, \ldots,$ belongs to $K^{+}_{a,c}(p; A, B, \delta)$ then $G(z) = \sum_{i=1}^{\ell} g_i f_i(z) \in K^{+}_{a,c}(p; A, B, \delta)$ where $\sum_{i=1}^{\ell} g_i = 1$.

Proof: By Theorem 3.1 and for every $i = 1, \ldots, \ell$ we have

$$\sum_{n=p}^{\infty} [(1-B)(n+p) - p(A-B)(1-\delta)] \frac{(a)_{n+p}}{(c)_{n+p}} |a_{n,i}| \le p(A-B)(1-\delta)$$

then

$$G(z) = \sum_{i=1}^{\ell} g_i \left(z^{-p} + \sum_{n=p}^{\infty} |a_{n,i}| z^n \right) = Z^{-}P_{+} \sum_{n=p}^{\infty} \left(\sum_{i=1}^{\ell} g_i\, |a_{n,i}| \right) z^n$$

Since

$$\sum_{n=p}^{\infty} \left(\frac{(1-B)(n+p) - p(A-B)(1-\delta)}{P(A-B)(1-\delta)} \right) \left(\sum_{i=1}^{\ell} g_i\, |a_{n,i}| \right) \frac{(a)_{n+p}}{(c)_{n+p}}$$

$$= \sum_{i=1}^{\ell} g_i \left(\sum_{n=p}^{\infty} \left(\frac{(1-B)(n+p) - p(A-B)(1-\delta)}{P(A-B)(1-\delta)} \right) \frac{(a)_{n+p}}{(c)_{n+p}} |a_{n,i}| \right).$$

4. NEIGHBOURHOODS

Definition 4.1: Let $a > 0$, $c > 0$, $-1 \le B < A \le 1$ and $\gamma \ge 0$, we defined γ-neighbourhood of a function $f \in T^*(p)$ and denote by $N_\gamma(f)$ contains all functions $g(z) = z^{-p} + \sum_{n=1}^{\infty} b_n z^{n-p} \in T^*(p)$ satisfying

$$\sum_{n=1}^{\infty} \frac{[(1+|B|)n + p(A-B)(1-\delta)]}{P(A-B)(1-\delta)} \frac{(a)_n}{(c)_n} |a_n - b_n| \le \gamma . \qquad (22)$$

Theorem 4.1: Let $f \in K_{a,c}(p; A, B, \delta)$, then $N_\gamma(f) \subset K_{a,c}(p; A, B, \delta)$ for every $\mu \in \mathbb{C}$ with $|\mu| < \gamma$, $\gamma > 0$, $\frac{f(z)+\mu z^{-p}}{1+\mu} \in K_{a,c}(p; A, B, \delta)$.

Proof: Let $g \in K_{a,c}(p; A, B, \delta)$, then by (4) we have

$$\left| \frac{z(L_p(a,c)\,g(z))' + pL_p(a,c)\,g(z)}{Bz(L_p(a,c)\,g(z))' + p(B+(A-B)(1-\delta))L_p(a,c)\,g(z)} \right| \ne \zeta \qquad (23)$$

$(\zeta \in \mathbb{C}; |\zeta| = 1)$, equivalently we must have $\frac{(f*\varphi)(z)}{z^{-p}} \ne 0$, $z \in U^*$, where

$$\begin{aligned} \phi(z) &= z^{-p} + \sum_{n=1}^{\infty} d_n z^{n-p} \\ &= z^{-p} + \sum_{n=1}^{\infty} \left(\frac{n(1-\zeta B) - p\zeta(A-B)(1-\delta)}{p\zeta(A-B)(1-\delta)} \right) \frac{(a)_n}{(c)_n} z^{n-p} . \end{aligned}$$

So that

$$|d_n| \le \frac{n(1+|B|) + p(A-B)(1-\delta)}{p(A-B)(1-\delta)} \frac{(a)_n}{(c)_n}, \quad n = p, p+1, \ldots \cdot$$

Hence we have $\left(z^p \left(\frac{f(z)+\mu z^{-p}}{1+\mu} * \varphi(z) \right) \right) \ne 0$, then

$$\frac{1}{1+\mu} \frac{(f*\varphi)z)}{z^{-p}} + \frac{\mu}{1+\mu} \ne 0 \qquad (24)$$

then

$$\left| \frac{1}{1+\mu} \frac{(f*\varphi)(z)}{z^{-p}} + \frac{\mu}{1+\mu} \right| \ge \frac{1}{|1+\mu|} \left| \frac{(f*\varphi)(z)}{z^{-p}} \right| \frac{|\mu|}{|1+\mu|} > \frac{1}{1+\gamma} \frac{(f*\varphi)(z)}{z^{-p}} - \frac{\gamma}{1+\gamma}$$

to hold (24) we must have $\frac{1}{1+\gamma}\frac{(f*\varphi)(z)}{z^{-p}}-\frac{\gamma}{1+\gamma}\geq 0$ then $\left|\frac{(f*\varphi)(z)}{z^{-p}}\right|\geq\gamma$. Now

$$\gamma-\left|\frac{(g*\varphi)(z)}{z^{-p}}\right|\leq\left|\frac{((f-g)*\varphi)(z)}{z^{-p}}\right|\leq\sum_{n=1}^{\infty}|a_n-b_n||d_n||z|^n$$

$$<\sum_{n=1}^{\infty}\frac{n(1+|B|)+p(A-B)(1-\delta)}{p(A-B)(1-\delta)}\frac{(a)_n}{(c)_n}|a_n-b_n|\leq\gamma$$

thus $\frac{(g*\varphi)(z)}{z^{-p}}\neq 0$ and $g\in K_{a,c}(p;A,B,\delta)$.

Theorem 4.2: Let $f\in T^*(p)$ and let $s_1(z)=z^{-p}$ and $s_\ell(z)=z^{-p}+\sum_{n=1}^{\ell-1}a_n z^{n-p}$; $\ell=2,3,\ldots$, suppose that $\sum_{n=1}^{\infty}d_n|a_n|\leq 1$ where

$$d_n=\frac{n(1+|B|)+p(A-B)(1-\delta)}{p(A-B)(1-\delta)}\frac{(a)_n}{(c)_n}$$

(i) if $a>0$, $c>0$ then $f\in K_{a,c}(p;A,B,\delta)$, and

(ii) if $a>c>0$ then

$$\operatorname{Re}\left(\frac{f(z)}{s_\ell(z)}\right)>1-\frac{1}{d_\ell},\ \operatorname{Re}\frac{s_\ell(z)}{f(z)}>\frac{d_\ell}{1-d_\ell},\ z\in U,\ \ell\in\mathbb{N}.$$

Proof: It is clear that $N_1(z^{-p})\subset K_{a,c}(p;A,B,\delta)$, since

$$\left(\frac{z^{-p}+z^{-p}\mu}{1+\mu}\right)=z^{-p}\in K_{a,c}(p;A,B,\delta)$$

then we have $f\in K_{a,c}(p;A,B,\delta)$, also $d_{n+1}>d_n>1$ thus $\sum_{n=1}^{\ell-1}|a_n|+d_\ell\sum_{n=\ell}^{\infty}|a_n|\leq 1$. Consider $G(z)=d_\ell\left[\frac{f(z)}{s_\ell(z)}-\left(1-\frac{1}{d_\ell}\right)\right]$ and use the last expression we get

$$\left|\frac{G(z)-1}{G(z)+1}\right|\leq\frac{d_\ell\sum_{n=\ell}^{\infty}|a_n|}{2-\sum_{n=1}^{\infty}|a_n|-d_\ell\sum_{n=\ell}^{\infty}|a_n|}\leq 1$$

then (i) is complete, to prove (ii).

Let $F(z) = (1+d_\ell)\left[\frac{s_\ell(z)}{f(z)} - \frac{d_\ell}{1-d_\ell}\right]$ so we have

$$\left|\frac{F(z)-1}{F(z)+1}\right| \le \frac{(1-d_\ell)\sum_{n=\ell}^{\infty}|a_n|}{2-2\sum_{n=1}^{\ell-1}|a_n| + (1-d_\ell)\sum_{n=\ell}^{\infty}|a_n|} \le 1$$

then the proof is complete.

Definition 4.2: Let $f(z) \in T^*(p)$ given by (6), then γ-neighbourhood of f and is denoted by $N_\gamma^+(f)$ contains all functions $g(z) = z^{-p} + \sum_{n=p}^{\infty} b_n z^n$ satisfying

$$\sum_{n=p}^{\infty} \frac{(1-B)(n+p) - p(A-B)(1-\delta)}{p(A-B)(1-\delta)} \frac{(a)_{n+p}}{(c)_{n+p}} |a_n - b_n| \le \gamma,$$

where $a > 0$, $c > 0$, $-1 \le B < A \le 1$, $0 \le \delta < 1$, $\gamma \ge 0$.

Theorem 4.3: If $f \in K^+_{a+1,\,c}(p; A, B, \delta)$ then $N_\gamma^+(f) \subset K^+_{a,\,c}(p; A, B, \delta)$ where $A + B \le 0$ and $\gamma = \frac{2p}{a+2p}$. The result is sharp.

Proof: By using the same procedure as in the proof of Theorem 4.1, with

$$h(z) = z^{-p} + \sum_{n=p}^{\infty} e_n z^n = z^{-p} + \sum_{n=p}^{\infty}\left[\frac{(1-\zeta B)(n+p) - p\zeta(A-B)(1-\delta)}{\zeta p(A-B)(1-\delta)} \frac{(a)_{n+p}}{(c)_{n+p}} z^n\right]$$

where $A + B \le 0$ and $f \in K^+_{a+1,\,c}(p; A, B, \delta)$, we have $\left|\frac{(f*h)(z)}{z^{-p}}\right| \ge \frac{2p}{a+2p} = \gamma$.

For sharpness, let

$$f(z) = z^{-p} + \left(\frac{(A-B)(1-\delta)}{2-2B-(A-B)(1-\delta)}\right)\frac{(c)_{2p}}{(a+1)_{2p}} z^p \in K^+_{a+1,\,c}(p; A, B, \delta)$$

and

$$g(z) = z^{-p} + \left[\frac{(A-B)(1-\delta)}{2-2B-(A-B)(1-\delta)} \cdot \frac{(c)_{2p}}{(a+1)_{2p}} + \frac{\gamma'(A-B)(1-\delta)}{2-2B-(A-B)(1-\delta)} \frac{(c)_{2p}}{(a)_{2p}}\right] z^p$$

where $\gamma' > \gamma = \frac{2p}{a+2p}$, we get $g(z) \in N_\gamma^+(f)$ but not in $K^+_{a+1,\,c}(p; A, B, \delta)$.

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