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TRIANGULAR NUMBERS IN THE GENERALIZED PELL SEQUENCE AND GENERALIZED ASSOCIATED PELL SEQUENCE

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Abstract

A positive integer N is called a "Triangular Number" if it is of the form $N = \frac{m(m+1)}{2}$, where m is an integer greater than zero. A positive integer N is called Generalized Triangular Number for any integer m. For a fixed integer $\alpha > 0$, a new sequence is called Generalized Pell Sequence $\{P_n^{(\alpha)}\}$ is defined by

$$P_0^{(\alpha)} = 0,\ P_1^{(\alpha)} = 1 \text{ and } P_{n+2}^{(\alpha)} = (\alpha+1)P_{n+1}^{(\alpha)} + \frac{\alpha(3\alpha-1)}{2}P_n^{(\alpha)} \text{ for } n \geq 0$$

and Generalized Associated Pell Sequence $\{Q_n^{(\alpha)}\}$ is defined by

$$Q_0^{(\alpha)} = 1, Q_1^{(\alpha)} = 1 \text{ and } Q_{n+2}^{(\alpha)} = (\alpha+1)Q_{n+1}^{(\alpha)} + \frac{\alpha(3\alpha-1)}{2}Q_n^{(\alpha)} \text{ for n} \geq 0$$

We proved that there exists Generalized Triangular Numbers in the sequence $\{P_n^{(\alpha)}\}$ for n = 0, 1 and there exists Triangular Numbers in the sequence $\{Q_n^{(\alpha)}\}$ for n=0,1 and some other results relative to triangular numbers

INTRODUCTION

It is well known that a positive integer N is called a "Triangular Number" if it is of the form $N = \frac{m(m+1)}{2}$, where m is an integer greater than zero.

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A positive integer N is called a Generalized Triangular Number for any integer m.

Mc Daniel has shown that 1 is the only triangular number in the Pell sequence $\{P_n\}$ is defined by

$$P_0 = 0,\ P_1 = 1 \text{ and } P_{n+2} = 2P_{n+1} + P_n \text{ for } n \geq 0. \quad (1)$$

B. Srinivasa Rao proved that 1 and 3 are the only triangular numbers in the Associated Pell Sequence $\{Q_n\}$ is defined by

$$Q_0 = 1,\ Q_1 = 1 \text{ and } Q_{n+2} = 2Q_{n+1} + Q_n \text{ for } n \geq 0. \quad (2)$$

Now we define for a fixed integer $\alpha > 0$, a new sequence called the Generalized Pell sequence $\{P_n^{(\alpha)}\}$ by

$$P_0^{(\alpha)} = 0,\ P_1^{(\alpha)} = 1 \text{ and } P_{n+2}^{(\alpha)} = (\alpha+1)P_{n+1}^{(\alpha)} + \frac{\alpha(3\alpha-1)}{2}P_n^{(\alpha)} \text{ for } n \geq 0 \quad (3)$$

and a Generalized Associated pell sequence $\{Q_n^{(\alpha)}\}$ is defined by the recursive relation

$$Q_0^{(\alpha)} = 1, Q_1^{(\alpha)} = 1 \text{ and } Q_{n+2}^{(\alpha)} = (\alpha+1)\ Q_{n+1}^{(\alpha)} + \frac{\alpha(3\alpha-1)}{2}Q_n^{(\alpha)} \text{ for } n \geq 0 \quad (4)$$

Note that $P_n^{(1)} = P_n$ and $Q_n^{(1)} = Q_n$ for n = 0, 1, 2, 3, …..

In this paper, we proved that there exist triangular numbers in the sequence $\{P_n^{(\alpha)}\}$ and $\{Q_n^{(\alpha)}\}$ for n = 0, 1.

An integer N is triangular number if and only if 8N + 1 is a perfect square. To find triangular numbers in the Generalized Pell Sequence and Generalized Associated Pell Sequence we have to identify those n for which $8P_n^{(\alpha)} + 1$ and $8Q_n^{(\alpha)} + 1$ are perfect squares.

For the first few values of n, the terms of Generalized Pell Sequence $\{P_n^{(\alpha)}\}$ and Generalized Associated Pell Sequence $\{Q_n^{(\alpha)}\}$ are given in

Table 1 (a)

n	$P_n^{(\alpha)}$	$Q_n^{(\alpha)}$
0	0	1
1	1	1
2	$(\alpha + 1)$	$\frac{1}{2}(3\alpha^2 + \alpha + 2)$
3	$\frac{1}{2}(5\alpha^2 + 3\alpha + 2)$	$\frac{1}{2}(3\alpha^3 + 7\alpha^2 + 2\alpha + 2)$
4	$\frac{1}{2}(8\alpha^3 + 10\alpha^2 + 4\alpha + 2)$	$\frac{1}{4}(15\alpha^4 + 20\alpha^3 + 23\alpha^2 + 6\alpha + 4)$
5	$\frac{1}{4}(31\alpha^4 + 40\alpha^3 + 31\alpha^2 + 10\alpha + 4)$	$\frac{1}{4}(24\alpha^5 + 53\alpha^4 + 42\alpha^3 + 33\alpha^2 + 8\alpha + 4)$

For certain values of α, the values of $P_n^{(\alpha)}$, $8P_n^{(\alpha)} + 1$, $Q_n^{(\alpha)}$ and $8Q_n^{(\alpha)} + 1$ are given in the following tables.

Table 1(b)

α	1	2	3	4	5	6	7	8	9	10
$P_0^{(\alpha)}$	0	0	0	0	0	0	0	0	0	0
$P_1^{(\alpha)}$	1	1	1	1	1	1	1	1	1	1
$P_2^{(\alpha)}$	2	3	4	5	6	7	8	9	10	11
$P_3^{(\alpha)}$	5	14	28	47	71	100	134	173	217	266
$P_4^{(\alpha)}$	12	57	160	345	636	1057	1632	2385	3340	4521
$P_5^{(\alpha)}$	29	241	976	2759	6301	12499	22436	37381	58789	88301
$8P_0^{(\alpha)} + 1$	1	1	1	1	1	1	1	1	1	1
$8P_1^{(\alpha)} + 1$	9	9	9	9	9	9	9	9	9	9
$8P_2^{(\alpha)} + 1$	17	25	33	41	49	57	65	73	81	89
$8P_3^{(\alpha)} + 1$	41	113	225	377	569	801	1073	1385	1737	2129
$8P_4^{(\alpha)} + 1$	97	457	1281	2761	5089	8457	13057	19081	26721	36169
$8P_5^{(\alpha)} + 1$	233	1929	7809	22073	50409	99993	179489	299049	470313	706409

Table 1(c)

α	11	12	13	14	15
$P_0^{(\alpha)}$	0	0	0	0	0
$P_1^{(\alpha)}$	1	1	1	1	1
$P_2^{(\alpha)}$	12	13	14	15	16
$P_3^{(\alpha)}$	320	379	443	512	586
$P_4^{(\alpha)}$	5952	7657	9660	11985	14656
$P_5^{(\alpha)}$	127744	179131	244661	326719	427876
$8P_0^{(\alpha)} + 1$	1	1	1	1	1
$8P_1^{(\alpha)} + 1$	9	9	9	9	9
$8P_2^{(\alpha)} + 1$	97	105	113	121	129
$8P_3^{(\alpha)} + 1$	2561	3033	3545	4097	4689
$8P_4^{(\alpha)} + 1$	47617	61257	77281	95881	117249
$8P_5^{(\alpha)} + 1$	102195	1433049	1957289	2613753	3423009

Table 1(d)

α	1	2	3	4	5	6	7	8	9	10
$Q_0^{(\alpha)}$	1	1	1	1	1	1	1	1	1	1
$Q_1^{(\alpha)}$	1	1	1	1	1	1	1	1	1	1
$Q_2^{(\alpha)}$	3	8	16	27	41	58	78	101	127	156
$Q_3^{(\alpha)}$	7	29	76	157	281	457	694	1001	1387	1861
$Q_4^{(\alpha)}$	17	127	496	1379	3121	6157	11012	18301	28729	43091
$Q_5^{(\alpha)}$	41	526	2896	10349	28561	66406	136676	281801	449569	743846
$8Q_0^{(\alpha)} + 1$	9	9	9	9	9	9	9	9	9	9
$8Q_1^{(\alpha)} + 1$	9	9	9	9	9	9	9	9	9	9
$8Q_2^{(\alpha)} + 1$	25	65	129	217	329	465	625	809	1017	1249
$8Q_3^{(\alpha)} + 1$	57	233	609	1257	2249	3656	5553	8009	11097	14889
$8Q_4^{(\alpha)} + 1$	137	1017	3969	11033	24969	49257	88097	146409	229833	344729
$8Q_5^{(\alpha)} + 1$	329	4209	23169	82793	228489	531249	1093409	2254409	3596553	5950769

Table 1(e)

α	11	12	13	14	15
$Q_0^{(\alpha)}$	1	1	1	1	1
$Q_1^{(\alpha)}$	1	1	1	1	1
$Q_2^{(\alpha)}$	188	223	261	302	346
$Q_3^{(\alpha)}$	2432	3109	3901	4817	5866
$Q_4^{(\alpha)}$	62272	87247	119081	158929	208036
$Q_5^{(\alpha)}$	1175296	1787101	2630681	3766414	5264356
$8Q_0^{(\alpha)} + 1$	9	9	9	9	9
$8Q_1^{(\alpha)} + 1$	9	9	9	9	9
$8Q_2^{(\alpha)} + 1$	1505	1785	2089	2417	2769
$8Q_3^{(\alpha)} + 1$	19457	24873	31209	38537	46929
$8Q_4^{(\alpha)} + 1$	498177	697977	952649	1271433	1664289
$8Q_5^{(\alpha)} + 1$	9402369	14296809	21045449	30131313	42114849

The following properties of the sequences { P_n } and { Q_n }given as

$$P_{-n} = (-1)^{n+1} P_n \text{ and } Q_{-n} = (-1)^n Q_n \tag{5}$$

$$P_{-n} = \frac{a^n - b^n}{2\sqrt{2}} \text{ and } Q_{-n} = \frac{a^n + b^n}{2} \tag{6}$$

where $a = 1 + \sqrt{2}$ and $b = 1 - \sqrt{2}$

$$P_{m+n} = 2P_m Q_n - (-1)^n P_{m-n} \tag{7}$$

$$P_{m+n} = P_m P_{n+1} + P_{m-n} P_n \tag{8}$$

$$Q_n^2 = 2 P_n^2 + (-1)^n \tag{9}$$

$$Q_{2n} = 2Q_n^2 - (-1)^n \tag{10}$$

As a direct consequence of (6) we have

$$Q_{m+n} = 2Q_m Q_n - (-1)^n Q_{m-n} \text{ for all integers m and n.} \tag{11}$$

Lemma 1 : *If n, k and t are integers then* $P_{n+2kt} \equiv (-1)^{t(k-1)} P_m \pmod{Q_k}$.

Proof: If $t = 0$, the lemma is trivial.

We prove this lemma for $t > 0$ by using induction hypothesis on t. By using (7)

$$P_{2k+n} = 2P_{n+k} Q_k - (-1)^k P_{(n+k)-k} = 2P_{n+k} Q_k - (-1)^k P_n$$

$$\equiv - (-1)^k P_n \pmod{Q_k}$$

$$\equiv (-1)^{k+1} P_n \pmod{Q_k}.$$

Proving the lemma for $t = 1$.

Now, Assume that the lemma holds all integers $\leq$ t.Then again by (7) and the induction hypothesis, we get

$$\begin{aligned} P_{2k(t+1)n} &= P_{(2kt+n)+2k} \\ &\equiv (-1)^{k+1} P_{2kt+n} \pmod{Q_k} \\ &\equiv (-1)^{k+1} (-1)^{t(k+1)} P_n \pmod{Q_k} \\ &\equiv (-1)^{(t+1)(k+1)} P_n \pmod{Q_k}. \end{aligned}$$

If $t < 0$, say $t = -m$, where $m > 0$ by (5) we have

$$\begin{aligned} P_{n+2kt} &= P_{n-2km} = P_{n+2(-k)t} \\ &\equiv (-1)^{t(-k+1)} P_n \pmod{Q_{(-k)}} \end{aligned}$$

$$\equiv (-1)^{-t\,(-k+1)}\, P_n \pmod{Q_k}$$

$$\equiv (-1)^{t\,(k-1)}\, P_n \pmod{Q_k}$$

which completes the proof of the lemma.

Lemma 2 : *If m is even and n, k are any integers then* $Q_{n+2km} \equiv (-1)^k\, Q_n \pmod{Q_m}$

Proof: For k = 0, the lemma is trivial.

We prove this lemma for k > 0 by using induction on k, by (11)

$Q_{n+2m} = 2Q_{n+m}\, Q_m - (-1)^m\, Q_n$.

Because m is even, this gives the lemma for k = 1.

Assume that the lemma holds all integers ≤ k. By (11) and the induction hypothesis, we get

$$Q_{n+2(k+1)m} = 2Q_{n+2km}\, Q_{2m} - Q_{n+2(k-1)m}$$

$$\equiv 2(-1)^k\, Q_n\, Q_{2m} - (-1)^{k-1}\, Q_n \pmod{Q_m}$$

$$\equiv (-1)^k\, (2Q_{2m} - 1)\, Q_n \pmod{Q_m} \qquad (12)$$

But since m is even it follows from (10) that

$$2Q_{2m} + 1 \equiv -1 \pmod{Q_m} \qquad (13)$$

By (12) and (13) together prove the lemma for k + 1.

Hence by induction the lemma holds for k > 0.

If k < 0, say k = - r, where r > 0, we have

$$Q_{n+2km} = Q_{n-2rm} = 2\, Q_n\, Q_{2rm} - (-1)^{2rm}\, Q_{n+2rm}$$

$$= 2\, Q_n\, Q_{2rm} - Q_{n+2rm}$$

$$\equiv 2\, Q_n (-1)^r - (-1)^r\, Q_n \pmod{Q_m}$$

$$\equiv (-1)^r\, Q_n \pmod{Q_m}$$

$$\equiv (-1)^k\, Q_n \pmod{Q_m}$$

which completes the proof of the Lemma.

First we prove those n for which $8P_n^{(1)} + 1$ and $8Q_n^{(1)} + 1$ be perfect squares, i.e., $8P_n^{(1)} + 1$ is perfect square only when n = 0 or 1 and $8Q_n^{(1)} + 1$ is perfect square only when n = 0, 1 or 2. So, $P_1^{(1)}$, $Q_0^{(1)}$, $Q_1^{(1)}$ and $Q_2^{(1)}$ are the only triangular numbers.

To prove above results we present the period k of the sequence $\left\{Q_t^{(1)}\right\}_{t=0}^{\infty}$ modulo certain integer M > 0. That is for all integers u ≥ 0, $Q_{t+ku}^{(\alpha)} \equiv Q_t^{(\alpha)}$ (mod M). Also if modulo M, the sequence { $Q_t^{(\alpha)}$ } has period k, we have R_t and U_t for t = 0, 1, 2, ….. k – 1, where $Q_t^{(1)} \equiv R_t$ (mod M) and $8Q_t^{(1)} + 1 \equiv U_t$ (mod M), For certain values of M > 0, the period k of $\{Q_t^{(1)}\}$, the numbers R_t (t = 0, 1, 2, ….. k – 1) and the numbers U_t (t = 0, 1, 2, ….. k – 1) are given in

Table 2(a)

I *Mod* *M*	*II* *Period* *K*	*III* R_t *(t = 0, 1, 2, …., k – 1)* $Q_t^{(1)} \equiv R_t$ *(mod M)*	*IV* U_t *(t = 0, 1, 2, …., k –1)* $8Q_t^{(1)} + 1 \equiv U_t$ *(mod M)*
7	6	1, 1, 3, 0, 3, 6.	2, 2, 4, 1, 4, 0.
9	24	1, 1, 3, 7, 8, 5, 0, 5, 1, 7, 6, 1, 8, 8, 6, 2, 1, 4, 0, 4, 8, 2, 3, 8.	0, 0, 7, 3, 2, 5, 1, 5, 0, 3, 4, 0, 2, 2, 4, 8, 0, 6, 1, 6, 2, 8, 7, 2.
10	12	1, 1, 3, 7, 7, 1, 9, 9, 7, 3, 3, 9	9, 9, 5, 7, 7, 9, 3, 3, 7, 5, 5, 3.
23	22	1, 1, 3, 7, 6, 8, 0, 8, 5, 7, 8, 1, 10, 10, 8, 4, 5, 3, 0, 3, 6, 4, 3, 10.	9, 9, 2, 11, 22, 7, 11, 4, 17, 13, 18, 1, 18, 12, 17, 21, 11, 18, 22, 14, 2, 16.

Lemma 3 : Suppose n ≡ 0 or 1 (mod 4). Then $8Q_n^{(1)} + 1$ is a perfect square if and only if *n* = 0 or 1.

Proof: We know that $Q_n^{(1)} = Q_n$.

If n = 0 or 1 then $8Q_n + 1 = 3^2$, by table 1(d).

Conversely, suppose n ≡ 0 or 1 (mod 4) and n ∉ {0, 1}. Then n can be written as n = 2km + ε, m = 2^r , r ≥ 1, k is odd and ε = 0 or 1.

Therefore by (11) and by table 1(d), we get

$Q_n = Q_{2km+\varepsilon} \equiv (-1)^k Q_\varepsilon \pmod{Q_m} \equiv -1 \pmod{Q_m}$

So that $8Q_n + 1 \equiv -7 \pmod{Q_m}$

Hence the Jacobi Symbol

$$\left(\frac{8Q_n+1}{Q_m}\right)=\left(\frac{-7}{Q_m}\right)=\left(\frac{-1}{Q_m}\right)\cdot\left(\frac{7}{Q_m}\right)$$

$$=\left(\frac{-1}{Q_m}\right)\cdot\left(\frac{Q_m}{7}\right)\cdot\left(\frac{-1}{Q_m}\right)=\left(\frac{Q_m}{7}\right) \tag{14}$$

Now note that for modulo 7, the sequence $\{Q_n\}$ is periodic with period 6. In fact by (11) and Table 1(d), we have

$$Q_{n+6} = 2\, Q_3 \,.\, Q_{n+3} + Q_n = 2(7)\, Q_{n+3} + Q_n$$

$$\equiv Q_n \pmod 7.$$

Also, since $m = 2^r \equiv \pm 2 \pmod 6$, we have from (5) and Table 2(a) that

$$Q_m \equiv Q_{\pm 2} \pmod 7 \equiv Q_2 \pmod 7 \equiv 3 \pmod 7$$

$$\therefore \qquad \left(\frac{Q_m}{7}\right)=\left(\frac{3}{7}\right)=\left(\frac{-4}{7}\right)=\left(-\frac{1}{7}\right)\left(\frac{4}{7}\right)=-1 \tag{15}$$

Now (14) and (15) gives $\left(\frac{8Q_n+1}{Q_m}\right) = -1.$

Proving that $8Q_n + 1$ cannot be a square,

which completes the proof of the lemma.

Lemma 4 : *Suppose $n \equiv \pm 2 \pmod{36}$. Then $8Q_n^{(1)} + 1$ is a perfect square if and only if $n = \pm 2$.*

Proof: We know that $Q_n^{(1)} = Q_n$.

If $n = \pm 2$ then $8Q_n^{(1)} + 1 = 5^2$ by (5) and Table 1(d).

Conversely, suppose $n \equiv \pm 2 \pmod{36}$ and $n \notin \{-2, 2\}$.

Then n can be written as $n = 2.3^2 \,.\, 2^r \,.\, g \pm 2$, where $r \geq 1$ and g is odd.

$$m = \begin{cases} 3^2 \,.\, 2^r & \text{if } r \equiv 3 \pmod{10} \\ 3.2^r & \text{if } r \equiv 1 \text{ (or) } 6 \pmod{10} \\ 2^r & \text{otherwise.} \end{cases} \tag{16}$$

So that $n = 2km \pm 2$, where k is odd (in fact , $k = g$, $3g$ or $3^2 g$). Also, since

$2^{t+10} \equiv 2^t \pmod{22}$ for $t \geq 1$, it follows that m, defined in (16) is such that

$$m \equiv \pm 4, \pm 6, \pm 10 \pmod{22} \tag{17}$$

For instance, if $r \equiv 6 \pmod{10}$, then $r = 10u+6$ for some integer u and in this case (16)

$$m = 3.2^r = 3.2^{10u+6} \equiv 3.2^6 \pmod{22} \equiv 6 \pmod{22}.$$

Now by Lemma (2), (5) and Table 1(d),

we have $Q_n = Q_{2km \pm 2} \equiv (-1)^k Q_2 \pmod{Q_m} \equiv -3 \pmod{Q_m}$.

So that $8Q_n + 1 \equiv -23 \pmod{Q_m}$. Therefore

$$\left(\frac{8Q_n+1}{Q_m}\right) = \left(\frac{-23}{Q_m}\right) = \left(\frac{-1}{Q_m}\right) \cdot \left(\frac{23}{Q_m}\right)$$

$$= \left(\frac{-1}{Q_m}\right) \cdot \left(\frac{Q_m}{23}\right) \cdot \left(\frac{-1}{Q_m}\right) = \left(\frac{Q_m}{23}\right) \tag{18}$$

Note that for modulo 23, the sequence $\{Q_j\}$ is periodic with period 22. That is

$$Q_{j+22i} \equiv Q_j \pmod{23} \text{ for all integers } i \geq 0 \tag{19}$$

Now (17), (18) and (5) imply that $Q_m \equiv Q_4$, Q_6 or $Q_{10} \pmod{23}$.

i.e., $Q_m \equiv 17, 7$ or $5 \pmod{23}$ by Table 2(a).

Therefore (18) gives

$$\left(\frac{8Q_n+1}{Q_m}\right) = \left(\frac{17}{23}\right), \left(\frac{7}{23}\right) \text{ or } \left(\frac{5}{23}\right) \text{ showing} \left(\frac{8Q_n+1}{Q_m}\right) = -1$$

Therefore $8Q_{n+1}$ is not a perfect square.

Lemma 5 : Suppose $n \equiv \pm 1 \pmod{2^2 . 3}$. Then $8P_n^{(1)} + 1$ is a perfect square if and only if $n = \pm 1$.

Proof: Note that $P_n^{(1)} = P_n$.

If $n = \pm 1$, then by (5) and by Table 1(b)

We have $8P_n+1 = 8P_{\pm 1} + 1 = 3^2$.

Conversely, suppose $n \equiv \pm 1 \pmod{2^2 . 3}$ and $n \notin \{-1, 1\}$. Then n can be written as

$n = 2km \pm 1$ then $m = 2^r$, $r \geq 1$, k is odd.

Therefore, by (11) and Table 1(d),

we get $Q_n = Q_{2km+\varepsilon} \equiv (-1)^k Q_\varepsilon \pmod{Q_m} \equiv -1 \pmod{Q_m}$.

So, that

$$\begin{aligned} 8P_{n+1} &= 8P_{2km\pm 1} + 1 \\ &\equiv 8(-1)^{m(k+1)} P_{\pm 1} + 1 \pmod{Q_m} \\ &\equiv 8(-1) + 1 \pmod{Q_m} \\ &\equiv -7 \pmod{Q_m}. \end{aligned}$$

$$\left(\frac{8P_n+1}{Q_m}\right) = \left(\frac{-7}{Q_m}\right) = \left(\frac{-1}{Q_m}\right) \cdot \left(\frac{7}{Q_m}\right)$$

$$= \left(\frac{-1}{Q_m}\right) \cdot \left(\frac{Q_m}{7}\right) \cdot \left(\frac{-1}{Q_m}\right) = \left(\frac{Q_m}{7}\right)$$

By using Lemma 3,We can write $\left(\frac{Q_m}{7}\right) = -1$.

$$\therefore \quad \left(\frac{8P_n+1}{Q_m}\right) = -1.$$

Proving that $8P_n + 1$ cannot be a square,

which completes the proof of the lemma.

Lemma 6 : Suppose $n \equiv 0, 1, \pm 2 \pmod{72}$. Then $8Q_n^{(1)} + 1$ is a perfect square if and only if $n = 0, 1, \pm 2$.

Proof: We know that $Q_n^{(1)} = Q_n$.

If $n \equiv 0$ or $1 \pmod{72}$, then $n \equiv 0$ or $1 \pmod 4$ and $n \equiv \pm 2 \pmod{36}$.

The proof follows from Lemma (3) and Lemma (4).

Lemma 7: $8Q^{(1)}_{n+1}$ is not a perfect square if $n = 0, 1, \pm 2 \pmod{72}$.

Proof : We know that $Q_n^{(1)} = Q_n$.

We prove this in different steps eliminating at each stage certain integers n modulo 72 for which $8Q_n + 1$ is not a square. In each step we choose an integer m such that the period k (of the sequence $\{Q_n\}$ mod m) is a divisor of 72 and there by eliminating certain residue class modulo k.

Step I : Note that modulo 10, the sequence $\{Q_n\}$ is periodic with period 12. That is, $Q_{n+12u} \equiv Q_n$ (mod 10) for all integers $u \geq 0$. Therefore, if $n \equiv 3, 4, 6, 7, 8$ or 11 (mod 12) then we respectively have $8Q_n + 1 \equiv 8Q_3+1, 8Q_4 + 1, 8Q_6 + 1, 8Q_7 + 1, 8Q_8 + 1$ or $8Q_{11} + 1$ (mod 10) so that by periodic table, $8Q_n + 1 \equiv 3$ or 7 (mod 10) for these values of n, showing $8Q_n + 1$ is not a square, since $m^2 \equiv 0, 1, 4, 5, 6$ or 9 (mod 10) for any integer $m \geq 1$. Therefore, for the sequence in the form $8Q_n + 1$ we have to search those n for which $n \equiv 0, 1, 2, 5, 9$ or 10 (mod 12) or equivalently among $n \equiv 0, 1, 2, 5, 9, 10, 12, 13, 14, 17, 21$ or 22 (mod 24).

Step II : Modulo 9, the sequence $\{Q_n\}$ is periodic with period 24 that is $Q_{n+12u} \equiv Q_n$ (mod 9) for all integers $u \geq 0$. So that when $n \equiv 5, 9, 12, 13, 17$ or 21 (mod 24) we respectively have $Q_n \equiv Q_5, Q_9, Q_{12}, Q_{13}, Q_{17}$ or Q_{21} (mod 9) and therefore, in view of periodic table, $8Q_n + 1 \equiv 2, 3, 5, 6$ or 8 (mod 9), showing $8Q_n + 1$ is not a square, since $m^2 \equiv 0, 1, 4$ or 7 (mod 9) for any integer m or ≥ 1.

Thus, there remain $n \equiv 0, 1, 2, 10, 14$ or 22 (mod 24).

Step III : Modulo 11, also the sequence $\{Q_n\}$ is periodic with period 24, so that for $n \equiv 0$ or 14 (mod 24) we have $Q_n \equiv Q_{10}$ or Q_{14} (mod 11), showing $8Q_n+1 \equiv 2$ or 10 (mod 11), by periodic table. Therefore $8Q_n+1$ is not a square if n $\equiv 10$ or 14 (mod 24), since 2 and 10 are quadratic nonresidues modulo 11.

Thus there remain $n \equiv 0, 1, 2$ or 22 (mod 24) or equivalently, $n \equiv 0, 1, 2, 22, 24, 25, 26, 46, 48, 49, 50$ or 70 (mod 72).

Step IV : Modulo 199, the sequence $\{Q_n\}$ has period 18, so that if $n \equiv 4, 11, 13, 14$ or 17 (mod 18) then by periodic table, we respectively have $8Q_n + 1 \equiv 137, 78, 71, 37, 192$ (mod 199) giving $8Q_n + 1$ is not a square, since 71, 78, 137 and 192 are quadratic nonresidues modulo 199.

Hence we eliminate $n \equiv 22, 49$ and 50 (mod 72).

Step V : Modulo 197, the sequence$\{Q_n\}$ has period 36, if $n \equiv \pm 10, \pm 12$ (mod 36) then by periodic table, we respectively have $8Q_n + 1 \equiv 113$ or 194 (mod 197), showing these n can be eliminated. Thus we can eliminate $n \equiv 24, 26, 46$ and 48 (mod 72).

Step VI : Modulo 73 the sequence $\{Q_n\}$ is periodic with period 72. Therefore if $n \equiv 25$ (mod 72), then $8Q_n + 1 \equiv 56$ (mod 73) by periodic table, showing $8Q_n + 1$ is not a square, Since $\left(\frac{56}{73}\right) = -1$

Finally, there remain n = 0, 1, 2 (mod 72).

Theorem 1: $Q_n^{(1)}$ is triangular number if and only if $n = 0, 1, \pm 2$.

Proof : From Lemma 6 and 7 we have $8Q_n^{(1)} + 1$ is a perfect square if n = 0, 1, ± 2.

Therefore we have $Q_n^{(1)}$ is triangular number.

Observation : (i) From table 1(d) observe that $Q_n^{(2)}$, $Q_n^{(4)}$, $Q_n^{(5)}$, $Q_n^{(6)}$, $Q_n^{(8)}$, $Q_n^{(9)}$ and $Q_n^{(10)}$ are triangular numbers if n = 0, 1.

(ii) For $Q_n^{(3)}$ when n = 0, 1 or 4 and $Q_n^{(7)}$ when n = 0, 1 or 2 be triangular numbers.

Theorem 2: (i) $P_n^{(1)}$ is triangular number if and only if $n = 1$.

(ii) $P_n^{(2)}$ is a generalized triangular number if $n = 0$, 1 or 2.

Proof : (i) From Lemma 5 the result is proved.

we have $8P_n^{(1)} + 1$ is perfect square if $n = 1$.

(ii) If N is triangular number then 8N + 1 must be perfect square.

i.e., $N = \frac{m(m+1)}{2}$

We know that when $8P_n^{(1)} + 1$ is perfect square then $P_n^{(\alpha)}$ is a generalized triangular number for any integer m.

By table 1(b) , $P_2^{(2)} = 3$.

We have $8P_2^{(2)} + 1 = 25 = 5^2$.

Note that zero is not a triangular number for any integer $m > 0$.

Therefore $P_n^{(2)}$ becomes generalized triangular number if $n = 0$, 1 or 2.

Hence the theorem.

***Observation*:** From table 1(b) & 1(c) observe that $P_n^{(5)}$, $P_n^{(9)}$ and $P_n^{(14)}$ are generalized triangular numbers when n = 0, 1 or 2.

Theorem 3 : $Q_n^{(\alpha)}$ is triangular number if $n = 0, 1$.

Proof : We prove the theorem by proving $8Q_n^{(\alpha)} + 1$ is a perfect square.

i.e.,we have to prove $8Q_n^{(\alpha)} + 1$ is a perfect square if $n = 0,1$ for all integers $\alpha > 0$.

To show this we use the Principle of Mathematical Induction on α.

From Lemmas 6 and 7, We know that $8Q_n^{(\alpha)} + 1$ is a perfect square if $n = 0, 1$.

For $\alpha = 1$ the result is true.

Assume that it is true for $\alpha = m$.

Observing that $8Q_0^{(m)} + 1$ and $8Q_1^{(m)} + 1$ are perfect squares.

We prove that it is also true for $\alpha = m + 1$.

Consider $8Q_1^{(m+1)} + 1 = 8.1 + 1 = 9 = 3^2$.

(From table 1(d) $Q_1^{(\alpha)} = 1$, for all α).

$\therefore$ By the Principle of Mathematical Induction $8Q_n^{(\alpha)} + 1$ is a perfect square if n=0, 1. which completes the proof of the theorem.

Theorem 4 : $P_n^{(\alpha)}$ is a generalized triangular number if $n = 0, 1$.

Proof : We prove the theorem by proving $8P_n^{(\alpha)} + 1$ is a perfect square.

i.e., we prove $8P_n^{(\alpha)} + 1$ is a perfect square if $n = 0, 1$ for all integers $\alpha > 0$.

To show this we use the Principle of Mathematical Induction on α.

From Lemma 5, $8P_n^{(1)} + 1$ is a perfect square if $n = 1$.

When n = 0, $8P_n^{(1)} + 1 = 1$ which is a perfect square (from table 1(b)).

For $\alpha = 1$ the result is true.

Assume that it is true for $\alpha = m$.

Observing that $8P_0^{(m)} + 1$ and $8P_1^{(m)} + 1$ are perfect squares

We prove that it is also true for $\alpha = m + 1$.

Consider $8P_0^{(m+1)} + 1 = 1 = 1^2$.

(From table 1(b), $P_0^{(\alpha)} = 0$ for all α).

$\therefore$ By the Principle of Mathematical Induction $8P_n^{(\alpha)} + 1$ is perfect square if n = 0, 1.

Hence $P_n^{(\alpha)}$ is a generalized triangular number if n = 0, 1.

which completes the proof of the theorem.

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