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L(3,2,1)-and L(4,3,2,1)-labeling Problems on Circular-ARC Graphs

Sk Amanathulla[*] and Madhumangal Pal[*]

ABSTRACT

For a given graph $G=(V,E)$, the $L(3,2,1)$- and $L(4,3,2,1)$-labeling problems assign the labels to the vertices of G. Let Z^* be the set of non-negative integers. An $L(3,2,1)$- and $L(4,3,2,1)$-labeling of a graph G is a function $f:V\to Z^*$ such that $|f(x)-f(y)|\geq k-d(x,y)$, for $k=4,5$ respectively, where $d(x,y)$ represents the distance (minimum number of edges of a shortest path) between the vertices x and y, and $1\leq d(x,y)\leq k-1$. The $L(3,2,1)$- and $L(4,3,2,1)$-labeling numbers of a graph G, are denoted by $\lambda_{3,2,1}(G)$ and $\lambda_{4,3,2,1}(G)$ and they are the difference between highest and lowest labels used in $L(3,2,1)$- and $L(4,3,2,1)$-labeling respectively. In [4], Calamoneri *et al*. have been studied $L(h,k)$-labeling of co-comparability graphs and circular-arc graphs. Motivated from this paper, we have studied $L(3,2,1)$- and $L(4,3,2,1)$-labeling problems on circular-arc graphs.

In this paper, for circular-arc graph G, it is shown that $\lambda_{3,2,1}(G)\leq 9\Delta-6$ and $\lambda_{4,3,2,1}(G)\leq 16\Delta-12$, where Δ represents the maximum degree of the vertices. These bounds we obtain are the first bounds for the problems on circular-arc graphs. Also two algorithms are designed to label a circular-arc graph by maintaining $L(3,2,1)$-and $L(4,3,2,1)$-labeling conditions. The time complexities of both the algorithms are $O(n\Delta^2)$, where n represent the number of vertices of G.

Keywords: $L(3,2,1)$-labeling, $L(4,3,2,1)$-labeling, frequency assignment, circular-arc graph, network.

Mathematics subject classification: 05C85, 68R10

1. INTRODUCTION

The frequency assignment problem is a problem where the task is to assign a frequency (non-negative integer) to a given group of televisions or radio transmitters so that interfering transmitters are assigned frequency with at least a minimum allowed separation. Frequency assignment problem is motivated from the distance labeling problem of graphs. It is to find a proper assignment of channels to transmitters in a wireless network. The level of interference between any two radio stations correlates with the geographic locations of the stations. Closer stations have a stronger interference and thus there must be a greater difference between their assigned channels.

Hale [14] introduced a graph theory model of channel assignment problem which is known as vertex coloring problem. In, 1988 Roberts proposed a variation of the frequency assignment problem in which

[*] Department of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University, Midnapore — 721102, INDIA, *E-mail: amanat87math@gmail.com, mmpalvu@gmail.com*

'closed' transmitter must receive different frequency and 'very closed' transmitter must receive a frequency at least two apart. Two vertices x and y are said to be 'very closed' and 'closed' if the distance between x and y is 1 and 2 respectively. Griggs and Yeh [13] defined the $L(2,1)$-labeling of a graph $G=(V,E)$ as a function f which assigns every x, y in V a label from the set of positive integers such that $|f(x)-f(y)|\geq 3-d(x,y)$, where $d(x,y)$ represent the distance between the vertices x and y, and $1\leq d(x,y)\leq 2$. The minimum span over all possible labeling functions of $L(h,k)$-labeling is denoted by $\lambda_{h,k}(G)$ and is called $\lambda_{h,k}$ - number of G.

An $L(3,2,1)$-labeling of a graph $G=(V,E)$ is a function f from its vertex set V to the set of non-negative integers such that $|f(x)-f(y)|\geq 3$ if $d(x,y)=1$, $|f(x)-f(y)|\geq 2$ if $d(x,y)=2$ and $|f(x)-f(y)|\geq 1$ if $d(x,y)=3$. The $L(3,2,1)$-labeling number, $\lambda_{3,2,1}(G)$, of G is the smallest non-negative integer k such that G has a $L(3,2,1)$-labeling of span k. Also, an $L(4,3,2,1)$-labeling of a graph $G=(V,E)$ is a function f from its vertex set V to the set of non-negative integers such that $|f(x)-f(y)|\geq 4$ if $d(x,y)=1$, $|f(x)-f(y)|\geq 3$ if $d(x,y)=2$, $|f(x)-f(y)|\geq 2$ if $d(x,y)=3$ and $|f(x)-f(y)|\geq 1$ if $d(x,y)=4$. The $L(4,3,2,1)$-labeling number, $\lambda_{4,3,2,1}(G)$, of G is the smallest non-negative integer k such that G has a $L(4,3,2,1)$-labeling of span k. Frequency assignment problem has been widely studied in the past [2,3,7,8,13,14,15,16,17,18,19,25,28,29,30]. Later Calamoneri studied $L(\delta 1,\delta 2,1)$-labeling of eight grids [6] and also Atta *et al*. studied $L(4,3,2,1)$-labeling for Simple Graphs [1]. We focus our attention on $L(3,2,1)$-labeling and $L(4,3,2,1)$-labeling of circular-arc graphs. Different bounds for $\lambda_{3,2,1}(G)$ and $\lambda_{4,3,2,1}(G)$ were obtained for various type of graphs. The upper bound of $\lambda_{p,1}(G)$ of any graph G is $\Delta^2-(p-1)\Delta-2$ [5], where Δ is the degree of the graph. In [10], Clipperton *et al.* showed that $\lambda_{3,2,1}(G)\leq\Delta^3+\Delta^2+3\Delta$ for any graph. Later Chai *et al.* [9] improved this upper bound and showed that $\lambda_{3,2,1}(G)\leq\Delta^3+2\Delta$ for any graph. In [20], Lui and Shao studied the $L(3,2,1)$-labeling of planer graph and showed that $\lambda_{3,2,1}(G)\leq 15(\Delta^2-\Delta+1)$. In [9], Chia *et al.* also showed that $\lambda_{3,2,1}(G)=2n+5$ if T is a complete n-ary tree of height $h\geq 3$ and for any tree $2\Delta+1\leq\lambda_{3,2,1}(G)\leq 2\Delta+3$. In [21,22,23], Pal *et al*. studied some problems on interval graphs. In [11], Jean studied about $L(d,2,1)$-labeling of simple graph and showed that $\lambda_{d,2,1}(K_n)=d(n-1)+1$ where K_n is complete graph with n vertices and also show that $\lambda_{d,2,1}(K_{m,n})=d+2(m+n)-3$. Kim *et al*. show that $\lambda_{3,2,1}(K_3\square C_n)=15$ when $n\geq 28$ and $n\equiv 0(mod 5)$, where $K_3\square C_n$ is the Cartesian product of complete graphs K_3 and the cycle C_n. Again, $\lambda_{4,3,2,1}(G)\leq\Delta^3+2\Delta^2+6\Delta$ for any graph G [12]. In [26], Paul *et al*. showed that $\lambda_{2,1}(G)\leq\Delta+w$ for interval graph and they also shown that $\lambda_{2,1}(G)\leq\Delta+3w$ for circular-arc graph, where w represents the size of the maximum clique. Also in [31], Sk Amanathulla *et al.* shown that $\lambda_{0,1}(G)\leq\Delta$ and $\lambda_{1,1}(G)\leq 2\Delta$ for circular-arc graphs.

In [4], Calamoneri *et al*. have been studied a lot of result about $L(h,k)$-labeling of co-comparability graphs, interval graphs and circular-arc graphs. They have shown that $\lambda_{h,k}(G)\leq max(h,2k)2\Delta+k$ for co-

comparability graphs, $\lambda_{h,k}(G) \leq max(h, 2k)\Delta$ for interval graphs. Also they have proved $\lambda_{h,k}(G) \leq max(h, 2k)\Delta + hw$ for circular-arc graphs. Motivated from this paper we have studied $L(3,2,1)$-and $L(4,3,2,1)$-labeling of circular-arc graphs.

In this paper, for circular-arc graphs G, it is shown that the upper bounds of $L(3,2,1)$-and $L(4,3,2,1)$-labeling are $9\Delta-6$ and $16\Delta-12$ respectively, where Δ represents the maximum degree of the vertices. Also two algorithms are designed to label a circular-arc graph by maintaining $L(3,2,1)$-and $L(4,3,2,1)$-labeling conditions. The time complexities of both the algorithms are $O(n\Delta^2)$, where n represent the number of vertices of *G*.

The remaining part of the paper is organized as follows. Some notations and definitions are presented in Section 2. In Section 3, some lemmas related to our work and an algorithm to $L(3,2,1)$-label a circular-arc graph are presented. Section 4 is devoted to $L(4,3,2,1)$-labeling problem of circular-arc graph. In Section 5, a conclusion is made.

2. PRELIMINARIES AND NOTATIONS

The graphs used in this work are simple, finite without self loop or multiple edges. A graph $G = (V, E)$ is called an intersection graph for a finite family *F* of a non-empty set if there is a one-to-one correspondence between *F* and *V* such that two sets in *F* have non-empty intersection if and only if there corresponding vertices in *V* are adjacent to each other. We call *F* an intersection model of *G*. For an intersection model *F*, we use $G(F)$ to denote the intersection graph for *F*. Depending on the nature of the set *F* one gets different intersection graphs. For a survey on intersection graph see [24].

The class of circular-arc graph is a very important subclass of intersection graph. A graph is a circular-arc graph if there exists a family *A* of arcs around a circle and a one-to-one correspondence between vertices of *G* and arcs *A*, such that two distinct vertices are adjacent in *G* if and only if there corresponding arcs intersect in *A*. Such a family of arcs is called an arc representation for *G*. Also, it is observed that an arc A_k of *A* and a vertex v_k of *V* are one and same thing. A circular-arc graph and its corresponding circular-arc representation are shown in Figure 1.

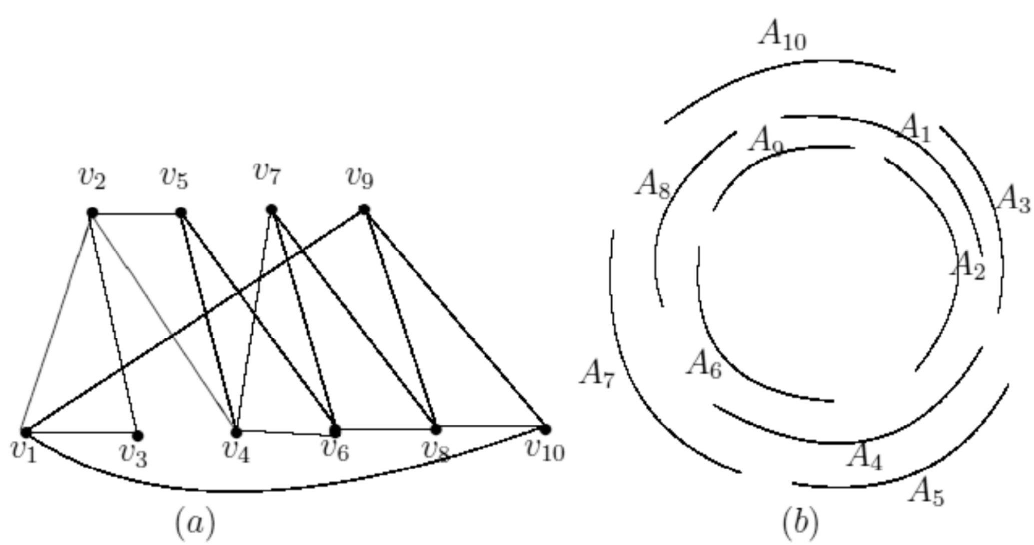

Figure 1: A circular-arc graph and its corresponding circular-arc representation

A graph G is a proper circular-arc (PCA) graph if there exists an arc representation for G such that no arc is properly included in another. The circular-arc graphs used in this work may or may not be proper. It is assumed that all the arcs must cover the circle, otherwise the circular-arc graph is nothing but an interval graph. The degree of the vertex v_k corresponding to the arc A_k is denoted by $d(v_k)$ and is defined by the maximum number of arcs which are adjacent to A_k. The maximum degree or the degree of a circular-arc graph G, denoted by $\Delta(G)$ or by Δ, is the maximum degree of all vertices corresponding to the arcs of G.

Let $A=\{A_1, A_2, A_3, \dots, A_n\}$ be a set of arcs around a circle. While going in a clockwise direction, the point at which we first encounter an arc is called the starting point of the arc. Similarly, the point at which we leave an arc is called the finishing point of the arc. A set $C \subseteq V$ is called a clique if every pair of vertices of C has an edge. The number of vertices of the clique represents its size. A clique is called maximal if there is no clique of G which properly contains C as a subset. A clique with r vertices is called r-clique. A clique is called maximum if there is no clique of G of larger cardinality. The size of the maximum clique is denoted by $w(G)$ or by w.

Notations: Let G be a circular-arc graph with arc set A, we define the following objects:

(i) $L(A_k)$: the set of labels which are used before labeling the arc $A_k, A_k \in A$.

(ii) $L_i(A_k)$: the set of labels which are used to label the vertices at distance i ($i=1,2,3,4$) from the arc A_k, before labeling the arc A_k, $A_k \in A$.

(iii) $L_{ivl}(A_k)$: the set of all valid labels to label the arc A_k satisfying the condition of distance 'one', 'one and two', 'one, two and three' of $L(3,2,1)$-labeling from the arc A_k, before labeling A_k, for ($i=1,2,3$) respectively.

(iv) $L'_{ivl}(A_k)$: the set of all valid labels to label the arc A_k satisfying the condition of distance 'one', 'one and two', 'one, two and three', 'one, two, three and four' of $L(4,3,2,1)$-labeling from the arc A_k, before labeling A_k, for ($i=1,2,3,4$) respectively.

(v) f_j : the label of the arc A_j, $A_j \in A$.

(vi) L: the label set, i.e. the set of labels used to label the circular-arc graph G completely.

Definition 1. For a circular-arc graph G, for each arc $A_j \in A$ the set S_{A_j} is defined as

(a) all arcs of S_{A_j} are adjacent to A_j,

(b) no two arcs of S_{A_j} are adjacent, and

(c) each S_{A_j} is maximal.

3. L(3,2,1)-LABELING OF CIRCULAR-ARC GRAPHS

In this section, we present some lemmas related to the proposed algorithm. Also, an algorithm is designed to solve $L(3,2,1)$-labeling problem on circular-arc graph. The time complexity of the algorithm is also calculated.

Lemma 1. For a circular-arc graph $G, |L_i(A_k)| \leq 2\Delta - 2$, $i = 2, 3, 4$ for any arc $A_k \in A$.

Proof. Case 1: Let $i = 2$ and let G be a circular-arc graph and A_k be any arc of G. Also let $|L_2(A_k)| = m$. This implies that m distinct labels are used to label the arcs which are at distance two from the arc A_k, before labeling the arc A_k.

Since Δ is the degree of the graph G so, A_k is adjacent to at most Δ arcs of G. Since G is a circular-arc graph, among the arcs those are adjacent to A_k, there must exists at most two arcs (the arcs of maximum length) in opposite direction of the arc A_k, each of which are adjacent to at most $\Delta - 1$ arcs (except A_k) of G, obviously these arcs are at distance two from A_k. In figure 2, all the two distances vertices of A_k are adjacent to either A_{k_1} or A_{k_2}. Except A_k, A_{k_1} is adjacent at most $\Delta - 1$ arcs. Similarly, except A_k, A_{k_2} is adjacent at most $\Delta - 1$ arcs. Hence, $m \leq 2(\Delta - 1)$, i.e. $|L_2(A_k)| \leq 2\Delta - 2$.

Case 2: Let $i = 3$ and let G be a circular-arc graph and A_k be any arc of G and let $|L_3(A_k)| = r$. This implies that r distinct labels are used to label the arcs which are at distance three from the arc A_k, before labeling the arc A_k.

Since Δ is the degree of the graph G so, A_k is adjacent to at most Δ arcs of G. Since G is a circular-arc graph, among the arcs those are adjacent to A_k, there must exists at most two arcs (the arcs of maximum length) in opposite direction of the arc A_k, each of which are adjacent to at most Δ arcs of G. In figure 2, A_{k_1} and A_{k_2} are those arcs each of which are adjacent to at most Δ arcs of G. Among the arcs those are adjacent to A_{k_1} and of distance two from A_k, there exists at most one arc (the arcs of maximum length) which is adjacent to at most Δ–1 arcs (expect A_{p_1}) obviously these arcs are at distance three from A_k. Again among the arcs which are adjacent to A_{k_2} and of distance two from A_k, there exists at most one arc (the arcs of maximum length) which is adjacent to at most Δ–1 arcs (except A_{p_2}), obviously these arcs are at distance three from A_k. In figure 2, all the three distances arcs are adjacent to either A_{p_1} or A_{p_2}. Except A_{k_1}, A_{p_1} is adjacent at most Δ – 1 arcs. Similarly, except A_{k_2}, A_{p_2} is adjacent at most $\Delta - 1$ arcs. Hence, $r \leq 2(\Delta - 1)$, i.e. $|L_3(A_k)| \leq 2\Delta - 2$.

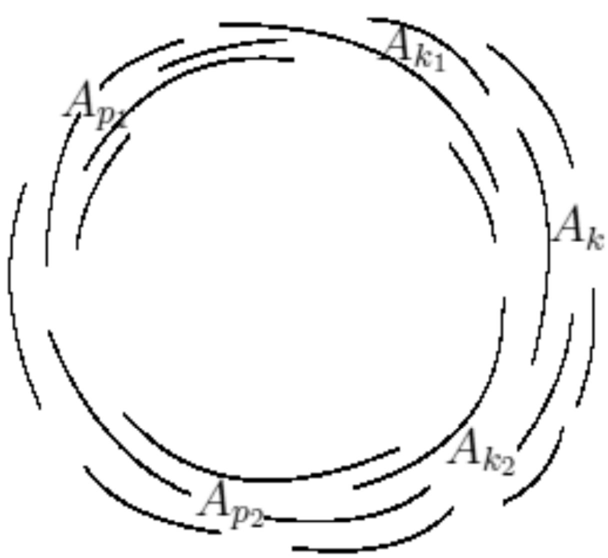

Figure 2: A circular-arc graph

The proof of the other case is similar.

Lemma 2. For a circular-arc graph G, $L_i(A_k) \subseteq L(A_k)$, for any arc A_k of G and $i = 1,2,3,4$.

Proof. Any label used to label a circular-arc graph G belong to $L(A_k)$. So any label $l \in L_i(A_k)$ implies $l \in L(A_k)$, for $i = 1,2,3,4$. Hence $L_i(A_k) \subseteq L(A_k)$, for any arc A_k of G and $i = 1,2,3,4$.

Lemma 3. $L_{kvl}(A_j)$ is the non empty largest set satisfying the condition of distance 1,2,,...,k for $k = 1,2,3$, of $L(3,2,1)$-labeling, where $l \le p$ for all $l \in L_{kvl}(A_j)$ and $p = max\{L(A_j)\} + 3$, for any $A_j \in A$ and $k = 1,2,3$.

Proof. Since $L_i(A_j) \subseteq L(A_j)$ for $i = 1,2,3$ (by Lemma 2) and $p = max\{L(A_j)\} + 3$, so $|p - l_i| \ge 3$ for any $l_i \in L_i(A_j), i = 1,2,3$. Therefore, $p \in L_{kvl}(A_j)$ for $k = 1,2,3$. Hence $L_{kvl}(A_j)$ is non empty set for $k = 1,2,3$.

Again, let B be any set of labels satisfying the condition of distance 1,2,...,k for $k = 1,2,3$, of $L(3,2,1)$-labeling, where $l \le p$ for all $l \in B$. Also, let $b \in B$. Then $|b - l_i| \ge 4 - i$ for any $l_i \in L_i(A_j)$ and for $i = k-2, k-1, k$, where $i > 0$. Thus, $b \in L_{kvl}(A_j)$, for $k = 1,2,3$. So $b \in B$ implies $b \in L_{kvl}(A_j)$, for $k = 1,2,3$. Therefore, $B \subseteq L_{kvl}(A_j)$, for $k = 1,2,3$. Since B is arbitrary, so $L_{kvl}(A_j)$ is the largest set of labels satisfying the condition of distance 1,2,...,k for $k = 1,2,3$, of $L(3,2,1)$-labeling, where $l \le p$ for all $l \in L_{kvl}(A_j)$ for $k = 1,2,3$.

Now, we discuss about the bounds of $\lambda_{3,2,1}(G)$ for a circular-arc graphs.

Theorem 1. For any circular-arc graph G, $\lambda_{3,2,1}(G) \ge 2k + 1$ where $k = \max_{A_j \in A} |S_{A_j}|$, $j = 1,2,3,\ldots,n$.

Proof. Let G be a circular-arc graph and $A = \{A_1, A_2, A_3, \ldots, A_n\}$. Let $A_\alpha \in A$ such that $|S_{A_\alpha}| = \max_{A_j \in A} |S_{A_j}| = k$, then clearly $S_{A_\alpha} \cup \{A_\alpha\}$ forms a subgraph of G. Thus when we label this subgraph by $L(3,2,1)$-labeling, then any member of S_{A_α} and A_α takes labels so that each differs the other by at least 3 and all other members get labels so that each label differs from the other by at least 2. Thus exactly, $2k+1$ labels (namely $0,3,5,7,\ldots,2k+1$) are needed to label the subgraph $S_{A_\alpha} \cup \{A_\alpha\}$. Hence, $\lambda_{3,2,1}(G) \ge 2k + 1$.

Theorem 2. For any circular-arc graph G, $\lambda_{3,2,1}(G) \le 9\Delta - 6$, where Δ is the degree of the graph G.

Proof. Let the total number of arcs of the circular-arc graph G be n and the set of arcs $A = \{A_1, A_2, A_3, \ldots, A_n\}$. Let $L(A_k) = \{0,1,2,\ldots,9\Delta - 6\}$, where $A_k \in A$. Then $|L(A_k)| = 9\Delta - 5$. Now $\lambda_{3,2,1}(G) \le 9\Delta - 6$, if we can prove that the label in the set $L(A_k)$ is sufficient to label all the arcs of G. Suppose, we are going to label the arc A_k by $L(3,2,1)$-labeling. We know that $|L_1(A_k)| \le \Delta$. So in the extreme unfavorable cases at least $(9\Delta - 5) - 3\Delta = 6\Delta - 5$ labels of the set $L(A_k)$ are available satisfying the condition of distance one of $L(3,2,1)$-labeling. Also, since $|L_2(A_k)| \le 2\Delta - 2$, (by Lemma 1). So in the extreme unfavorable cases at least $(6\Delta - 5) - 2(2\Delta - 2) = 2\Delta - 1$ labels of the set $L(A_k)$ are available satisfying the condition of distance one and two of $L(3,2,1)$-labeling. Again, since $|L_3(A_k)| \le 2\Delta - 2$, (by Lemma 1), so in the most unfavorable cases at least one (viz: $(2\Delta-1)-(2\Delta-2)=1$) label of the set $L(A_k)$ is available satisfying $L(3,2,1)$-labeling condition. Since A_k is arbitrary, so we can label any arc of the circular-arc graph satisfying $L(3,2,1)$-labeling condition by using the labels from the set $L(A_k)$.

If we take $L(A_k)$ so that $L(A_k) \subseteq \{0,1,2,\ldots,9\Delta-6\}$ and we are going to label the arc A_k by $L(3,2,1)$-labeling, then by similar arguments, it follows that the set $L(A_k)$ may or may not contain a label satisfying $L(3,2,1)$-labeling condition. Hence, $\lambda_{3,2,1}(G) \le 9\Delta - 6$.

3.1. Algorithm for L(3,2,1)-labeling

In this subsection, an algorithm to $L(3,2,1)$-label a circular-arc graph is designed. The main idea of the algorithm is discuss below:

First we find out the set of labels which satisfies the condition of distance one of $L(3,2,1)$-labeling. Among these labels we find the set of labels which also satisfies the condition of distance two of $L(3,2,1)$-labeling. Among these labels we find the set of labels which satisfies the condition of distance three of $L(3,2,1)$-labeling. Then we take the least element of that set, which obviously satisfies $L(3,2,1)$-labeling condition.

Algorithm L321

Input: A set of ordered arcs A of a circular-arc graph.

//assume that the arcs are ordered with respect to clockwise direction i.e. $A = \{A_1, A_2, A_3, \ldots, A_n\}$ //

Output: f_j, the $L(3,2,1)$-label of A_j, $j = 1,2,3,\ldots,n$.

Initialization: $f_1 = 0$;

$L(A_2) = \{0\}$;

for each $j=2$ to $n-1$ compute $L_1(A_j)$, $L_2(A_j)$ and $L_3(A_j)$

for $i = 0$ to r, //where $r = max\{L(A_j)\}+3$ //

for $k=1$ to $|L_1(A_j)|$

if $|i-l_k| \ge 3$, then $L_{1vl}(A_j) = \{i\}$ //where $l_k \in L_1(A_j)$ //

end for;

end for;

for $k=1$ to 2

for $m=1$ to $|L_{kvl}(A_j)|$

for $n=1$ to $|L_{k+1}(A_j)|$

if $|l_m - p_n| \ge 3-k$, then $L_{k+1vl}(A_j) = \{l_m\}$

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            //where $l_m \in L_{kvl}(A_j)$ and $p_n \in L_{k+1}(A_j)$ //
            end for;
        end for;
    end for;
    $f_j = \min\{L_{3vl}(A_j)\}$;
    $L(A_{j+1}) = L(A_j) \cup \{f_j\}$;
    end for;
    for $i = 0$ to $s$, where $s = max\{L(A_n)\} + 3$
        for $k = 1$ to $|L_1(A_n)|$
            if $|i - l_k| \geq 3$, then $L_{1vl}(A_n) = \{i\}$ //where $l_k \in L_1(A_n)$ //
        end for;
    end for;
    for k = 1 to 2
        for $m = 1$ to $|L_{kvl}(A_n)|$
            for $q = 1$ to $|L_{k+1}(A_n)|$
                if $|l_m - p_q| \geq 3 - k$, then $L_{k+1vl}(A_n) = \{l_m\}$
                    //where $l_m \in L_{k+1vl}(A_n)$ and $p_q \in L_{k+1}(A_n)$ //
                end for;
            end for;
        end for;
    $f_n = \min\{L_{3vl}(A_n)\}$;
    $L = L(A_n) \cup \{f_n\}$;
    end L321.
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Theorem 3. The Algorithm L321 correctly labels the vertices of a circular-arc graph using L321 L(3,2,1)-labeling condition.

Proof. Let $A = \{A_1, A_2, A_3, \ldots, A_n\}$, also let $f_1 = 0$, $L(A_2) = \{0\}$. If the graph has only one vertex then $L(A_2)$ is sufficient to label the whole graph and obviously, $\lambda_{3,2,1}(G) = 0$.

If the graph has more than one vertex then the set $L(A_2)$ is insufficient to label the whole graph G, because in this case more than one label is required and $L(A_2)$ contains only one label. Suppose, we are going to label the arc $A_j \in A$. $L_{kvl}(A_j)$ is the non empty largest set satisfying the condition of distance 1,2,...,k for $k = 1,2,3,$ of $L(3,2,1)$-labeling, where $l \leq p$ for all $l \in L_{kvl}(A_j)$ and $p = max\{L(A_j)\} + 3$, for any $A_j \in A$ and $k = 1,2,3$ (by Lemma 3). Also no label $l \notin L_{3vl}(A_j)$ and $l \leq p$ satisfying the condition of

$L(3,2,1)$-labeling of graph. So the labels on the set are the only valid labels for A_j, which is less than or equal to p and satisfying $L(3,2,1)$-labeling condition.

Our aim is to label the arc A_j by using as few labels as possible, satisfying $L(3,2,1)$-labeling condition. So $f_j = q$, where $q = min\{L_{3vl}(A_j)\}$. Now q is the least label for A_j, because no label less than q satisfies $L(3,2,1)$-labeling condition. Since A_j is arbitrary so this algorithm spent minimum number of labels to label any arc of a circular-arc graph satisfying $L(3,2,1)$-labeling condition and $\lambda_{3,2,1}(G) = max\{L(A_n) \cup \{f_n\}\}$.

Theorem 4. A circular-arc graph can be $L(3,2,1)$-labeled using $O(n\Delta^2)$ time, where n, and Δ represent number of vertices and the degree of the graph G.

Proof. Let L be the label set and |L|be its cardinality. According to the algorithm L321, $|L_i(A_k)| \leq |L|$ for i=1,2,3, for any $A_k \in A$, and also $r \leq 9\Delta - 3$, where $r = max\{L(A_j)\} + 3$. So we can compute $L_{1vl}(A_j)$ using at most $|L|(9\Delta - 3)$ time, i.e. using at most $O(\Delta |L|)$ time. Also, $|L_{kvl}(A_j)| \leq 9\Delta - 6$ for $k = 1,2$, so for each $k = 1,2$, $L_{k+1vl}(A_j)$ can be computed using at most $|L|(9\Delta - 6)$ time, i.e. using at most $O(\Delta |L|)$ time. This process is repeated for $n-1$ times. So the total time complexity for the algorithm L321 is $O((n-1)\Delta |L|) = O(n\Delta |L|)$. Since, $|L| \leq 9\Delta - 5$, therefore the running time for the algorithm L321 is $O(n\Delta^2)$.

Illustration of the algorithm L321

Let us consider a circular-arc graph of Fig. 3 to illustrate the algorithm L321.

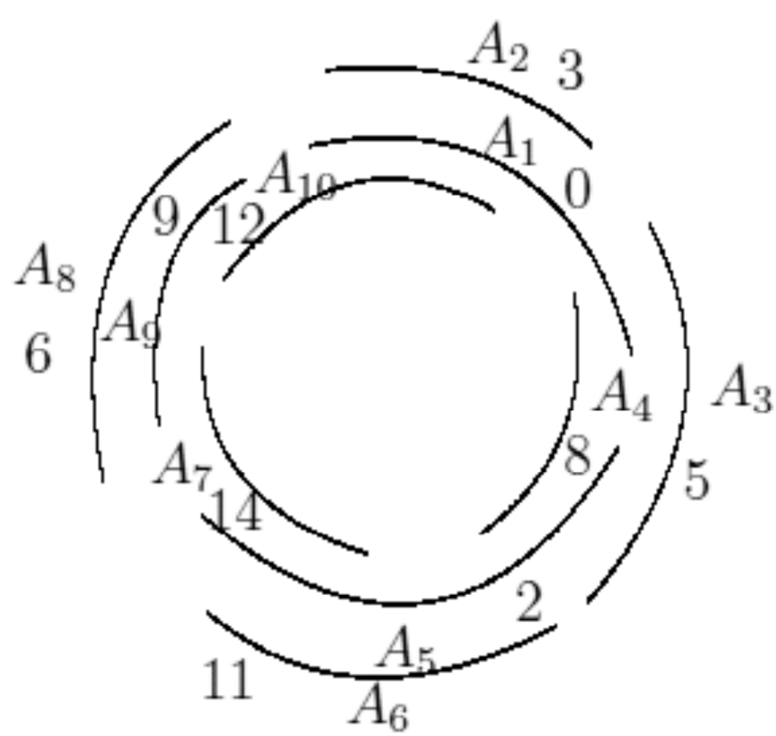

Figure 3: Illustration of Algorithm *L*321

For this graph, $A=\{A_1,A_2,A_3,\ldots,A_{10}\}$ and $\Delta=4$.

$f_j=$ the label of the arc A_j, for $j=1,2,3,\ldots,10$.

$f_1=0$, $L(A_2)=\{0\}$.

Iteration 1: For $j=2$.

$L_1(A_2)=\{0\}$, $L_2(A_2)=\phi$, $L_3(A_2)=\phi$.

$L_{1vl}(A_2)=\{3\}$, $L_{2vl}(A_2)=\{3\}$, $L_{3vl}(A_2)=\{3\}$.

Therefore, $f_2=\min\{L_{3vl}(A_2)\}=3$ and $L(A_3)=L(A_2)\cup\{f_2\}=\{0\}\cup\{3\}=\{0,3\}$.

Iteration 2: For $j=3$.

$L_1(A_3)=\{0\}$, $L_2(A_3)=\{3\}$, $L_3(A_3)=\phi$.

$L_{1vl}(A_3)=\{3,4,5,6\}$, $L_{2vl}(A_3)=\{5,6\}$, $L_{3vl}(A_3)=\{5,6\}$.

So $f_3=\min\{L_{3vl}(A_3)\}=5$ and $L(A_4)=L(A_3)\cup\{f_3\}=\{0,3,\}\cup\{5\}=\{0,3,5\}$.

Iteration 3: For $j=4$.

$L_1(A_4)=\{0,5\}$, $L_2(A_4)=\{3\}$, $L_3(A_4)=\phi$.

$L_{1vl}(A_4)=\{8\}$, $L_{2vl}(A_4)=\{8\}$, $L_{3vl}(A_4)=\{8\}$.

Therefore, $f_4=\min\{L_{3vl}(A_4)\}=8$ and $L(A_5)=L(A_4)\cup\{f_4\}=\{0,3,5\}\cup\{8\}=\{0,3,5,8\}$.

Iteration 4: For $j=5$.

$L_1(A_5)=\{5,8\}$, $L_2(A_5)=\{0\}$, $L_3(A_5)=\{3\}$.

$L_{1vl}(A_5)=\{0,1,2,11\}$, $L_{2vl}(A_5)=\{2,11\}$, $L_{3vl}(A_5)=\{2,11\}$.

Therefore, $f_5=\min\{L_{3vl}(A_5)\}=2$ and $L(A_6)=L(A_5)\cup\{f_5\}=\{0,3,5,8\}\cup\{2\}=\{0,2,3,5,8\}$.

In this way $f_6=11$, $f_7=14$ $f_8=6$, $f_9=9$, and finally, $f_{10}=12$. The vertices and the label of the corresponding vertices its are given below:

Vertices	*A1*	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
L (3, 2, 1)-labels	0	3	5	8	2	11	14	6	9	12

4. L(4,3,2,1)-LABELING OF CIRCULAR-ARC GRAPH

By extending the idea of $L(3,2,1)$-labeling, we design an algorithm for $L(4,3,2,1)$-labeling of circular-arc graph. In this section, we present some lemmas related to our work, bounds of $L(4,3,2,1)$-labeling, the algorithm L4321 and time complexity of the proposed algorithm L4321

Lemma 4. $L'_{kvl}(A_j)$ is the non empty largest set satisfying the condition of distance 1,2,..., k for $k=1,2,3,4$ of $L(4,3,2,1)$-labeling, where $l \le p$ for all $l \in L'_{kvl}(A_j)$ and $p = max\{L(A_j)\}+4$, for any $A_j \in A$ and $k=1,2,3,4$.

Proof. Since $L_i(A_j) \subseteq L(A_j)$ for $i=1,2,3,4$ (by Lemma 3) and $p = max\{L(A_j)\}+4$, so $|p-l_i| \ge 4$ for any $l_i \in L_i(A_j), i=1,2,3,4$. Therefore, $p \in L'_{kvl}(A_j)$ for $k=1,2,3,4$. Hence, $L'_{kvl}(A_j)$ is non empty set for $k=1,2,3,4$.

Again, let B be any set of labels satisfying the condition of distance 1,2,...,k for $k=1,2,3,4$ of $L(4,3,2,1)$-labeling , where $l \le p$ for all $l \in B$. Also, let $b \in B$. Then $|b-l_i| \ge 5-i$ for any $l_i \in L_i(A_j)$ and for $i=k-3, k-2, k-1, k$, where $i>0$. Thus, $b \in L'_{kvl}(A_j)$, for $k=1,2,3,4$. So $b \in B$ implies $b \in L'_{kvl}(A_j)$, for $k=1,2,3,4$. Therefore, $B \subseteq L'_{kvl}(A_j)$, for $k=1,2,3,4$. Since, B is arbitrary, so $L'_{kvl}(A_j)$ is the largest set of labels satisfying the condition of distance 1,2,...,k for $k=1,2,3,4$ of $L(4,3,2,1)$-labeling, where $l \le p$ for all $l \in L'_{kvl}(A_j)$, for $k=1,2,3,4$.

Hence the lemma.

Now, we discuss about the upper bound of $\lambda_{4,3,2,1}(G)$ of a circular-arc graphs.

Theorem 5. For any circular-arc graph G, $\lambda_{4,3,2,1}(G) \ge 3k+1$ where $k = \max_{A_j \in A} |S_{A_j}|$, $j=1,2,3,\ldots,n$.

Proof. Let G be a circular-arc graph and $A=\{A_1, A_2, A_3, \ldots, A_n\}$. Let $A_\alpha \in A$ such that $|S_{A_\alpha}| = \max_{A_j \in A} |S_{A_j}| = k$, then clearly $S_{A_\alpha} \cup \{A_\alpha\}$ forms a subgraph of G. Thus, when we label this subgraph by $L(4,3,2,1)$-labeling, then any one member of S_{A_α} and A_α takes labels so that each differs the other by at least 4 and all other members get labels so that each label differs from the other by at least 3. Thus exactly, $3k+1$ labels (namely $0,4,7,9,\ldots,3k+1$) are needed to label the subgraph $S_{A_\alpha} \cup \{A_\alpha\}$. Hence, $\lambda_{3,2,1}(G) \ge 3k+1$.

Theorem 6. For any circular-arc graph G, $\lambda_{4,3,2,1}(G) \le 16\Delta - 12$, where Δ is the degree of the graph G.

Proof. Let G be a circular-arc graph having n arcs and the set of arcs be $A=\{A_1, A_2, A_3, \ldots, A_n\}$.

Also, let $L(A_k) = \{0,1,2,\ldots,16\Delta-12\}$, where $A_k \in A$. Then $|L(A_k)| = 16\Delta - 11$.

Now $\lambda_{4,3,2,1}(G) \le 16\Delta-12$, if we can prove that the label in the set $L(A_k)$ is sufficient to label all the arcs of G. Suppose, we are going to label the arc A_k by $L(4,3,2,1)$-labeling. We know that $|L_1(A_k)| \le \Delta$. So in the extreme unfavorable cases at least $(16\Delta-11)-4\Delta = 12\Delta-11$ labels of the set $L(A_k)$ are available satisfying the condition of distance one of $L(4,3,2,1)$-labeling. Also, since $|L_2(A_k)| \le 2\Delta-2$, (by Lemma 1). So in the worst case at least $(12\Delta-11)-3(2\Delta-2) = 6\Delta-5$ labels of the set $L(A_k)$ are available satisfying the condition of distance one and two of $L(4,3,2,1)$-labeling. Again since $|L_3(A_k)| \leq 2\Delta - 2$, (by Lemma 1), so in the most unfavorable cases at least $(6\Delta - 5) - 2(2\Delta - 2) = 2\Delta - 1$ labels of the set $L(A_k)$ are available satisfying the condition of distance one, two and three of $L(4,3,2,1)$-labeling. Finally, since $|L_4(A_k)| \leq 2\Delta - 2$, (by Lemma 1), so in the most unfavorable cases at least one (viz: $(2\Delta - 1) - (2\Delta - 2) = 1$) label of the set $L(A_k)$ is available satisfying $L(4,3,2,1)$-labeling condition. Since A_k is arbitrary, so we can label any arc of the circular-arc graph satisfying $L(4,3,2,1)$-labeling condition by using the labels of the set $L(A_k)$.

If we take $L(A_k)$ so that $L(A_k) \subseteq \{0,1,2,\ldots,16\Delta - 12\}$ and we are going to label the arc A_k by $L(4,3,2,1)$-labeling, then by similar arguments, it follows that the set $L(A_k)$ may or may not contain a label satisfying $L(4.3,2,1)$-labeling condition. Hence, $\lambda_{4,3,2,1}(G) \leq 16\Delta - 12$.

4.1. Algorithm for L(4, 3, 2, 1)-labeling

In this subsection we present an algorithm to solve $L(4,3,2,1)$-labeling of circular-arc graph.

Algorithm L4321

Input: A set of ordered arcs A of a circular-arc graph.

//assume that the arcs are ordered with respect to clockwise direction namely $A_1, A_2, A_3, \ldots, A_n$ where $A = \{A_1, A_2, A_3, \ldots, A_n\}$ //

Output: f_j, the $L(4,3,2,1)$-label of A_j, $j = 1,2,3,\ldots,n$.

Initialization: $f_1 = 0$;

$L(A_2) = \{0\}$;

for each $j = 2$ to n–1 compute $L_p(A_j)$ for $p = 1,2,3,4$

for $i = 0$ to r, where $r = max\{L(A_j)\} + 4$

for $k = 1$ to $|L_1(A_j)|$

if $|i - l_k| \geq 4$, then $L'_{1vl}(A_j) = \{i\}$ //where $l_k \in L_1(A_j)$//

end for;

end for;

for $k = 1$ to 3

for $m = 1$ to $|L'_{kvl}(A_j)|$

for $n = 1$ to $|L_{k+1}(A_j)|$

if $|l_m - p_n| \geq 4 - k$, then $L'_{k+1vl}(A_j) = \{l_m\}$

//where $l_m \in L'_{kvl}(A_j)$ and $p_n \in L_{k+1}(A_j)$//

end for;

end for;
end for;
$f_j = \min\{L'_{4vl}(A_j)\}$;
$L(A_{j+1}) = L(A_j) \cup \{f_j\}$;
end for;
for $i = 0$ to s, where $s = max\{L(A_n)\} + 4$
for $k = 1$ to $|L_1(A_n)|$
if $|i - l_k| \geq 4$, then $L'_{1vl}(A_n) = \{i\}$ //where $l_k \in L_1(A_n)$ //
end for;
end for;
for $k = 1$ to 3
for $m = 1$ to $|L'_{kvl}(A_n)|$
for $q = 1$ to $|L_{k+1}(A_n)|$
if $|l_m - p_q| \geq 4 - k$, then $L'_{k+1vl}(A_n) = \{l_m\}$
//where $l_m \in L'_{kvl}(A_n)$ and $p_n \in L_{k+1}(A_n)$ //
end for;
end for;
end for;
$f_n = \min\{L'_{4vl}(A_n)\}$;
$L = L(A_n) \cup \{f_n\}$;
end L4321.

Theorem 7. The Algorithm L4321 correctly labels the vertices of a circular-arc graph using L(4,3,2,1)-labeling condition.

Proof. Let $A = \{A_1, A_2, A_3, \ldots, A_n\}$, also let $f_1 = 0$, $L(A_2) = \{0\}$. If the graph has only one vertex then $L(A_2)$ is sufficient to label the whole graph and obviously, $\lambda_{4,3,2,1}(G) = 0$.

If the graph has more than one vertex then the set $L(A_2)$ is insufficient to label the whole graph *G*. Suppose, we are going to label the arc $A_j \in A$. $L'_{kvl}(A_j)$ is the non empty largest set satisfying the condition of distance $1, 2, \ldots\ldots, k$ for $k = 1, 2, 3, 4$ of $L(4,3,2,1)$-labeling, where $l \leq p$ for all $l \in L'_{kvl}(A_j)$ and $p = max\{L(A_j)\} + 4$, for any $A_j \in A$ and $k = 1, 2, 3, 4$ (by Lemma 4). So, the labels in the set $L'_{4vl}(A_j)$ are the only valid labels for A_j, which is less than or equal to p and satisfying $L(4,3,2,1)$-labeling condition.

Our aim is to label the arc A_j by using least possible label by $L(4,3,2,1)$-labeling. So $f_j = q$, where $q = min\{L'_{4vl}(A_j)\}$. Now q is the least label for A_j, because no label less than q satisfies $L(4, 3, 2, 1)$-labeling condition. Since A_j is arbitrary, so this algorithm spent minimum number of labels to label any arc of a circular-arc graph by $L(4,3,2,1)$-labeling and $\lambda_{4,3,2,1}(G) = max\{L(A_n) \cup \{f_n\}\}$.

Hence the theorem.

Theorem 8. A circular-arc graph can be $L(4,3,2,1)$-labeled using $O(n\Delta^2)$ time, where n and Δ represent number of vertices and the degree of the graph G respectively.

Proof. Let L be the label set and |L| be the cardinality of L. According to the algorithm L4321, $|L_i(A_k)| \leq |L|$ for i = 1, 2, 3, 4 and for any $A_k \in A$, and also $r \leq 16\Delta - 8$, where $r = max\{L(A_j)\} + 4$. So $L'_{1vl}(A_j)$ is computed using at most $|L|(16\Delta - 8)$ time, i.e. using at most $O(\Delta |L|)$ time. Also $|L'_{kvl}(A_j)| \leq 16\Delta - 11$ for $k = 1,2,3$, so $L'_{k+1vl}(A_j)$ can be computed using at most $|L|(16\Delta - 11)$ time, i.e. using at most $O(\Delta |L|)$ time for each $k = 1,2,3$. This process is repeated for $n-1$ times. So, the total time complexity for the algorithm L4321 is $O((n-1)\Delta |L|) = O(n\Delta |L|)$. Since, $|L| \leq 16\Delta - 11$, therefore the running time for the algorithm L4321 is $O(n\Delta^2)$.

Illustration of the algorithm L4321

To illustrate the algorithm we consider a circular-arc graph of Fig. 4.

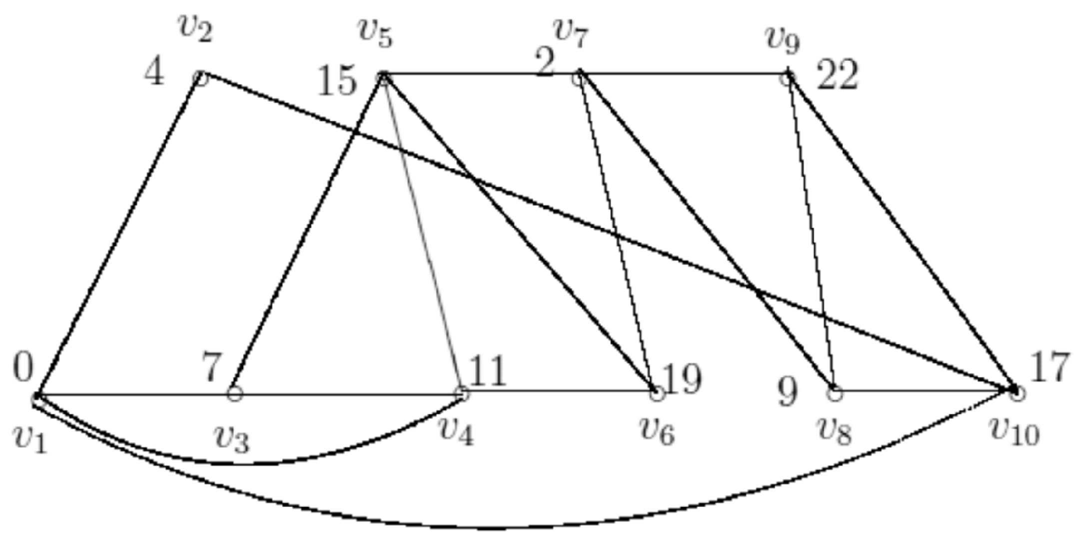

Figure 4: Illustration of Algorithm L4321

For this graph, $V = \{v_1, v_2, v_3, \ldots, v_{10}\}$ and $\Delta = 4$.

$f_j =$ The label of the vertex v_j, for $j = 1,2,3,\ldots,10$.

$f_1 = 0$, $L(v_2) = \{0\}$.

Iteration 1: For $j = 2$.

$L_1(v_2) = \{0\}$, $L_2(v_2) = \phi$, $L_3(v_2) = \phi$, $L_4(v_2) = \phi$.

$L'_{1vl}(v_2) = \{4\}$, $L'_{2vl}(v_2) = \{4\}$, $L'_{3vl}(v_2) = \{4\}$, $L'_{4vl}(v_2) = \{4\}$.

Therefore, $f_2 = \min\{L'_{4vl}(v_2)\} = 4$ and $L(v_3) = L(v_2) \cup \{f_2\} = \{0\} \cup \{4\} = \{0,4\}$.

Iteration 2: For $j = 3$.

$L_1(v_3) = \{0\}$, $L_2(v_3) = \{4\}$, $L_3(v_3) = \phi$, $L_4(v_3) = \phi$

$L'_{1vl}(v_3) = \{4,5,6,7,8\}$, $L'_{2vl}(v_3) = \{7,8\}$, $L'_{3vl}(v_3) = \{7,8\}$, $L'_{4vl}(v_3) = \{7,8\}$.

Therefore, $f_3 = \min\{L'_{4vl}(v_3)\} = 7$ and $L(v_4) = L_0(v_3) \cup \{f_3\} = \{0,4\} \cup \{7\} = \{0,4,7\}$.

Iteration 3: For $j = 4$.

$L_1(v_4) = \{0,7\}$, $L_2(v_4) = \{4\}$, $L_3(v_4) = \phi$, $L_4(v_4) = \phi$

$L'_{1vl}(v_4) = \{11\}$, $L'_{2vl}(v_4) = \{11\}$, $L'_{3vl}(v_4) = \{11\}$, $L'_{4vl}(v_4) = \{11\}$.

Therefore, $f_4 = \min\{L'_{4vl}(v_4)\} = 11$ and $L(v_5) = L(v_4) \cup \{f_4\} = \{0,4,7\} \cup \{11\} = \{0,4,7,11\}$.

Iteration 4: For $j = 5$.

$L_1(v_5) = \{7,11\}$, $L_2(v_5) = \{0\}$, $L_3(v_5) = \{4\}$, $L_4(v_4) = \phi$.

$L'_{1vl}(v_5) = \{0,1,2,3,15\}$, $L'_{2vl}(v_5) = \{3,15\}$, $L'_{3vl}(v_5) = \{15\}$, $L'_{4vl}(v_5) = \{15\}$.

Therefore, $f_5 = \min\{L'_{4vl}(v_5)\} = 15$ and $L(v_6) = L(v_5) \cup \{f_5\} = \{0,4,7,11\} \cup \{15\} = \{0,4,7,11,15\}$.

In this way $f_6 = 19$, $f_7 = 2$ $f_8 = 9$, $f_9 = 22$, and finally, $f_{10} = 17$.

The vertices and the label of the corresponding vertices are shown below:

Vertices	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}
L (4, 3, 2, 1)-labels	0	4	7	11	15	19	2	9	22	17

5. CONCLUSION

In this paper, we determine the upper bounds for $\lambda_{3,2,1}$ and $\lambda_{4,3,2,1}$ for a circular-arc graph *G*, and have shown that $\lambda_{3,2,1}(G) \leq 9\Delta - 6$ and $\lambda_{4,3,2,1}(G) \leq 16\Delta - 12$. These are the first bounds for the problems on circular-arc graphs. Also, two algorithms are designed to $L(3,2,1)$-label and $L(4,3,2,1)$-label for circular-arc graphs. The running time for both the algorithm is $O(n\Delta^2)$.

Since the upper bounds are not tight, so there is a chance for new upper bounds for the problems. Also the time complexities of the proposed algorithms may be reduced.

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