EXPLICIT FORMULAE FOR P_n AND Q_n

NARESH PATEL AND PUNIT SHRIVASTAVA

ABSTRACT: In this paper, we have derived some explicit formulae for Pell numbers P_n and Q_n Pell-Lucas numbersin terms of ceiling and floor functions. We have also derived their predecessor and successor formulae to get predecessor or successor of any given Pell or Pell-Lucas number.

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1. INTRODUCTION

Define the sequences $\{U_n\}$ and $\{V_n\}$ for all integers n by

$$\begin{cases} U_n = pU_{n-1} + U_{n-2},\ U_0 = 0,\ U_1 = 1, \\ V_n = pV_{n-1} + V_{n-2},\ V_0 = 2,\ V_1 = p. \end{cases}$$

For $p = 1$, we write $\{U_n\} = \{F_n\}$ and $\{V_n\} = \{L_n\}$, which are the Fibonacci [1] and Lucas numbers respectively. Their Binet forms, obtained by using standard techniques for solving linear recurrences, are

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + b^n,$$

where α and β are the roots of $x^2 - x - 1 = 0$.

For $p = 2$, we write

$$\begin{cases} P_n = 2P_{n-1} + P_{n-2},\ P_0 = 0,\ P_1 = 1, \\ Q_n = 2Q_{n-1} + Q_{n-2},\ Q_0 = 2,\ Q_1 = 2. \end{cases}$$

Here $\{P_n\}$ and $\{Q_n\}$ are the Pell and Pell-Lucas Sequences respectively. Their Binet forms are given by [6]

$$P_n = \frac{\gamma^n - \delta^n}{\gamma - \delta} \quad \text{and} \quad Q_n = \gamma^n + \delta^n,$$

where γ and δ are the roots of $x^2 - 2x - 1 = 0$ i.e., $\gamma = 1+\sqrt{2}$ and $\delta = 1-\sqrt{2}$.

2. Explicit Formulae

Here we derive explicit formula for P_n and Q_n. Since $|\delta| < 1$, so when n is large $\delta^n \to 0$ and hence using Binet Formula we can write $P_n \approx \frac{\gamma^n}{2\sqrt{2}}$. So we compute the value of $\frac{\gamma^n}{2\sqrt{2}}$ for the first ten values of n and look for a pattern:

$$\frac{\gamma}{2\sqrt{2}} \approx 0.85355339, \qquad \frac{\gamma^2}{2\sqrt{2}} \approx 2.060660172,$$

$$\frac{\gamma^3}{2\sqrt{2}} \approx 4.974873734, \qquad \frac{\gamma^4}{2\sqrt{2}} \approx 12.01040764,$$

$$\frac{\gamma^5}{2\sqrt{2}} \approx 28.995689091, \qquad \frac{\gamma^6}{2\sqrt{2}} \approx 70.00178567,$$

$$\frac{\gamma^7}{2\sqrt{2}} \approx 168.9992604, \qquad \frac{\gamma^8}{2\sqrt{2}} \approx 408.0003064,$$

$$\frac{\gamma^9}{2\sqrt{2}} \approx 984.9998731, \qquad \frac{\gamma^{10}}{2\sqrt{2}} \approx 2378.000053.$$

Now add $\frac{1}{2}$ to each and see the pattern which emerges:

$$\frac{\gamma}{2\sqrt{2}} + \frac{1}{2} \approx 1.35355339, \qquad \frac{\gamma^2}{2\sqrt{2}} + \frac{1}{2} \approx 2.560660172,$$

$$\frac{\gamma^3}{2\sqrt{2}} + \frac{1}{2} \approx 5.474873734, \qquad \frac{\gamma^4}{2\sqrt{2}} + \frac{1}{2} \approx 12.51040764,$$

$$\frac{\gamma^5}{2\sqrt{2}} + \frac{1}{2} \approx 29.495689091, \qquad \frac{\gamma^6}{2\sqrt{2}} + \frac{1}{2} \approx 70.50178567,$$

$$\frac{\gamma^7}{2\sqrt{2}} + \frac{1}{2} \approx 169.4992604, \qquad \frac{\gamma^8}{2\sqrt{2}} + \frac{1}{2} \approx 408.5003064,$$

$$\frac{\gamma^9}{2\sqrt{2}} + \frac{1}{2} \approx 985.4998731, \qquad \frac{\gamma^{10}}{2\sqrt{2}} + \frac{1}{2} \approx 2378.500053.$$

Thus, we get

$$\left\lfloor \frac{\gamma}{2\sqrt{2}}+\frac{1}{2} \right\rfloor \approx \lfloor 1.35355339 \rfloor = 1, \qquad \left\lfloor \frac{\gamma^2}{2\sqrt{2}}+\frac{1}{2} \right\rfloor \approx \lfloor 2.560660172 \rfloor = 2,$$

$$\left\lfloor \frac{\gamma^3}{2\sqrt{2}}+\frac{1}{2} \right\rfloor \approx \lfloor 5.474873734 \rfloor = 5, \qquad \left\lfloor \frac{\gamma^4}{2\sqrt{2}}+\frac{1}{2} \right\rfloor \approx \lfloor 12.51040764 \rfloor = 12,$$

$$\left\lfloor \frac{\gamma^5}{2\sqrt{2}}+\frac{1}{2} \right\rfloor \approx \lfloor 29.495689091 \rfloor = 29, \qquad \left\lfloor \frac{\gamma^6}{2\sqrt{2}}+\frac{1}{2} \right\rfloor \approx \lfloor 70.50178567 \rfloor = 70,$$

$$\left\lfloor \frac{\gamma^7}{2\sqrt{2}}+\frac{1}{2} \right\rfloor \approx \lfloor 169.4992604 \rfloor = 169, \qquad \left\lfloor \frac{\gamma^8}{2\sqrt{2}}+\frac{1}{2} \right\rfloor \approx \lfloor 408.5003064 \rfloor = 408,$$

$$\left\lfloor \frac{\gamma^9}{2\sqrt{2}}+\frac{1}{2} \right\rfloor \approx \lceil 985.4998731 \rceil = 985, \qquad \left\lfloor \frac{\gamma^{10}}{2\sqrt{2}}+\frac{1}{2} \right\rfloor \approx \lfloor 2378.500053 \rfloor = 2378.$$

Hence we found that $\left\lfloor \frac{\gamma^n}{2\sqrt{2}}+\frac{1}{2} \right\rfloor = P_n$. Theorem 1 confirms this result. To establish it, we need the following Lemma 1 [5].

Lemma 1: $0 < \dfrac{\delta^n}{2\sqrt{2}}+\dfrac{1}{2} < 1.$

Proof: Since $\delta < 0, |\delta| = -\delta$. Also, since $0 < |\delta| < 1, 0 < \delta|^n < 1$, ,

So $0 < |\delta|^n < \dfrac{2\sqrt{2}}{2} \Rightarrow 0 < \dfrac{|\delta|^n}{2\sqrt{2}} < \dfrac{1}{2}.$

Case I: Let n be even. Then $|\delta|^n = \delta^n$, so $0 < \frac{|\delta|^n}{2\sqrt{2}} < \frac{1}{2}$ and hence

$$\frac{1}{2} < \frac{|\delta|^n}{2\sqrt{2}}+\frac{1}{2} < 1.$$

Case II: Let n be odd. Then $|\delta|^n = -\delta^n$, so $0 < \frac{-\delta^n}{2\sqrt{2}} < \frac{1}{2}$ and hence

$$0 > \frac{\delta^n}{2\sqrt{2}} > \frac{-1}{2} \Rightarrow 0 < \frac{|\delta|^n}{2\sqrt{2}}+\frac{1}{2} < \frac{1}{2}.$$

Thus in both cases,

$$0 < \frac{|\delta|^n}{2\sqrt{2}} + \frac{1}{2} < 1 .$$

This establishes the lemma.

Theorem 1: $P_n = \left\lfloor \frac{\gamma^n}{2\sqrt{2}} + \frac{1}{2} \right\rfloor.$

Proof: Using Binet formula, we can write

$$P_n = \frac{\gamma^n - \delta^n}{\gamma - \delta} = \left(\frac{\gamma^n}{2\sqrt{2}} + \frac{1}{2} \right) - \left(\frac{\delta^n}{2\sqrt{2}} + \frac{1}{2} \right) \tag{1}$$

$$\therefore \left(\frac{\gamma^n}{2\sqrt{2}} + \frac{1}{2} \right) = P_n + \left(\frac{\delta^n}{2\sqrt{2}} + \frac{1}{2} \right) < P_n + 1 \qquad \text{(Using Lemma 1)}$$

Since $\left(\frac{\delta^n}{2\sqrt{2}} + \frac{1}{2} \right) > 0,$ it follows from (1) that $P_n < \left(\frac{\gamma^n}{2\sqrt{2}} + \frac{1}{2} \right).$

Thus,

$$P_n < \left(\frac{\gamma^n}{2\sqrt{2}} + \frac{1}{2} \right) < P_{n+1} .$$

Consequently,

$$P_n = \left\lfloor \frac{\gamma^n}{2\sqrt{2}} + \frac{1}{2} \right\rfloor. \tag{2}$$

For example,

$$\frac{\gamma^{20}}{2\sqrt{2}} + \frac{1}{2} \approx 15994428.5, \quad \text{so} \quad \left\lfloor \frac{\gamma^{20}}{2\sqrt{2}} + \frac{1}{2} \right\rfloor \approx \lfloor 15994428.5 \rfloor = 15994428 = P_{20}$$

Corollary 1: $P_n = \left\lceil \frac{\gamma^n}{2\sqrt{2}} - \frac{1}{2} \right\rceil.$

Proof: Since $\lfloor x \rfloor = \lceil x \rceil - 1$ for non-integral real number *x*, it follows that

$$P_n = \left\lceil \frac{\gamma^n}{2\sqrt{2}} + \frac{1}{2} \right\rceil - 1 \qquad \text{(Using 2)}$$

But $\lceil x + n \rceil = \lceil x \rceil + n$ for integer *n*

$$\therefore\ P_n = \left\lceil \frac{\gamma^n}{2\sqrt{2}} + \frac{1}{2} - 1 \right\rceil = \left\lceil \frac{\gamma^n}{2\sqrt{2}} - \frac{1}{2} \right\rceil.$$

For example,

$$.\frac{\gamma^{20}}{2\sqrt{2}} - \frac{1}{2} \approx 2377.500053 \qquad \therefore \left\lceil \frac{\gamma^{20}}{2\sqrt{2}} - \frac{1}{2} \right\rceil \approx \lceil 2377.500053 \rceil = 2378 = P_{10}.$$

Similarly,

$$\frac{\gamma^{15}}{2\sqrt{2}} - \frac{1}{2} \approx 195024.5 \qquad \therefore \left\lceil \frac{\gamma^{15}}{2\sqrt{2}} - \frac{1}{2} \right\rceil \approx \lceil 195024.5 \rceil = 195025 = P_{15}.$$

Here is yet another interesting observation:

$$\left\lceil \frac{\gamma}{2\sqrt{2}} \right\rceil = P_1, \quad \left\lceil \frac{\gamma^3}{2\sqrt{2}} \right\rceil = P_3, \quad \left\lceil \frac{\gamma^5}{2\sqrt{2}} \right\rceil = P_5, \quad \left\lceil \frac{\gamma^7}{2\sqrt{2}} \right\rceil = P_7, \quad \left\lceil \frac{\gamma^9}{2\sqrt{2}} \right\rceil = P_9.$$

$$\& \quad \left\lfloor \frac{\gamma^2}{2\sqrt{2}} \right\rfloor = P_2, \quad \left\lfloor \frac{\gamma^4}{2\sqrt{2}} \right\rfloor = P_4, \quad \left\lfloor \frac{\gamma^6}{2\sqrt{2}} \right\rfloor = P_6, \quad \left\lfloor \frac{\gamma^8}{2\sqrt{2}} \right\rfloor = P_8, \quad \left\lfloor \frac{\gamma^{10}}{2\sqrt{2}} \right\rfloor = P_{10}.$$

Thus, we have

$$\left\lfloor \frac{\gamma^{2n}}{2\sqrt{2}} \right\rfloor = P_{2n} \quad \text{and} \quad \left\lceil \frac{\gamma^{2n+1}}{2\sqrt{2}} \right\rceil = P_{2n+1}.$$

The following corollary confirms these two observations:

Corollary 2: $\left\lfloor \frac{\gamma^{2n}}{2\sqrt{2}} \right\rfloor = P_{2n}$ and $\left\lceil \frac{\gamma^{2n+1}}{2\sqrt{2}} \right\rceil = P_{2n+1}$

Proof: Let n be even. Then, using Lemma 1, we have

$$\frac{1}{2} < \frac{\delta^n}{2\sqrt{2}} < 1, \quad \text{so} \quad -\frac{1}{2} > -\frac{\delta^n}{2\sqrt{2}} > -1.$$

Then,

$$\frac{\gamma^n}{2\sqrt{2}} - \frac{1}{2} > \frac{\gamma^n}{2\sqrt{2}} - \frac{\delta^n}{2\sqrt{2}} > \frac{\gamma^n}{2\sqrt{2}} - 1,$$

Or

$$\frac{\gamma^n}{2\sqrt{2}} - 1 < P_n < \frac{\gamma^n}{2\sqrt{2}} - \frac{1}{2}.$$

But, $\lfloor x \rfloor \leq x$ and $\lfloor x \rfloor = \lfloor x \rfloor + n$

$$\therefore \quad \left\lfloor \frac{\gamma^n}{2\sqrt{2}} \right\rfloor - 1 < P_n < \frac{\gamma^n}{2\sqrt{2}} - \frac{1}{2}$$

Or

$$\left\lfloor \frac{\gamma^n}{2\sqrt{2}} \right\rfloor - 1 < P_n < \frac{\gamma^n}{2\sqrt{2}}.$$

Thus,

$$P_n = \left\lfloor \frac{\gamma^n}{2\sqrt{2}} \right\rfloor.$$

Similarly, we can establish the case when n is odd and similar proof can be given to the second part also.

Theorem 2: $Q_n = \left\lfloor \gamma^n + \frac{1}{2} \right\rfloor.$

For example, $\gamma^{13} + \frac{1}{2} \approx 94642.50001$

$$\therefore \quad \left\lfloor \gamma^n + \frac{1}{2} \right\rfloor = 94642 = Q_{13}.$$

Corollary 3: (i) $Q_n = \lceil \gamma^n - \frac{1}{2} \rceil$

(ii) $Q_{2n} = \lceil (\gamma)^{2n} \rceil$ and $Q_{2n+1} = \lfloor \gamma^{2n+1} \rfloor$.

For example, $\gamma^9 - \frac{1}{2} \approx 2785.500359$

$$\therefore \left\lfloor \gamma^9 - \frac{1}{2} \right\rfloor = 2786 = Q_9, \quad \lceil \gamma^{12} \rceil = \lceil 39201.99997 \rceil = 39202 = Q_{12},$$

and
$$\lfloor \gamma^{17} \rfloor = \lfloor 3215042 \rfloor = 3215042 = Q_{17}.$$

In every explicit formula, we needed to know the value of n in order to compute P_n. But knowing a Pell number, we can easily compute its successor. The next theorem provides such a formula, but first we need to lay some groundwork in the form of a lemma, similar to Lemma 1.

Lemma 2: If $n \geq 2$, then $0 < \frac{1}{2} - \delta^n < 1$.

Proof: Since $|\delta| < 0.414, |\delta|^2 < \frac{1}{2}$, so $|\delta|^n < \frac{1}{2}$ when $n \geq 2$.

Since $|\delta|^n = |\delta^n|$, this yields $-\frac{1}{2} < \delta^n < \frac{1}{2}$. Then $-1 < \delta^n - \frac{1}{2} < 0$;

That is, $0 < \frac{1}{2} - \delta^n < 1$.

Theorem 3: $P_{n+1} = \lfloor \gamma P_n + \frac{1}{2} \rfloor, n \geq 2$.

Proof: By Binet formula

$$P_n = \frac{\gamma^n - \delta^n}{\gamma - \delta}$$

$$\Rightarrow \quad \gamma P_n = \frac{\gamma^{n+1} - \gamma\delta^n}{2\sqrt{2}}$$

$$= \frac{\gamma^{n+1} - \gamma\delta(\delta^{n-1}) + \delta^{n+1} - \delta^{n+1}}{2\sqrt{2}}$$

$$= \frac{(\gamma^{n+1} - \delta^{n+1}) + \{(-\gamma\delta)\delta^{n-1} + \delta^{n+1}\}}{2\sqrt{2}} \qquad (\because \gamma\delta = -1)$$

$$= P_{n+1} + \frac{\delta^{n-1}(-\gamma\delta+\delta^2)}{2\sqrt{2}}$$

$$= P_{n+1} + \frac{\delta^{n-1}\{(-\delta)\ \ (\gamma-\delta)\}}{2\sqrt{2}}$$

$$= P_{n+1} + \frac{\delta^{n-1}\{-2\sqrt{2}(-\delta)\ \}}{2\sqrt{2}} \qquad (\because\ \gamma-\delta = 2\sqrt{2})$$

$$= P_{n+1} - \delta^n$$

$$\therefore\quad \gamma P_n + \frac{1}{2} = P_{n+1} + \left(\frac{1}{2} - \delta^n\right) \tag{3}$$

Since $(\frac{1}{2} - \delta^n) > 0,$ this implies that $P_{n+1} < (\gamma P_n + \frac{1}{2})$.

Besides, since $(\frac{1}{2} - \delta^n) < 1,$ using (3), we can write $\gamma P_n + \frac{1}{2} < P_{n+1} + 1$.

Thus, $P_{n+1} < (\gamma P_n + \frac{1}{2}) < P_{n+1} + 1$, so $P_{n+1} = \lfloor \gamma P_n + \frac{1}{2} \rfloor$..

For example, let $P_n = 985$. Its successor is given by $\lfloor 985\gamma + \frac{1}{2} \rfloor = \lfloor 2378.500359 \rfloor = 2378$ as expected. Substituting for γ in the formula for P_n yields the following result:

Corollary 4: $P_{n+1} = \left\lfloor \frac{2P_n + 2\sqrt{2}P_n + 1}{2} \right\rfloor,\ n \geq 2.$

We can use the recursive formula theorem 3 or corollary 4 to compute the ratio $\frac{P_{n+1}}{P_n}$ as $n \to \infty$, as the following corollary demonstrates. Its proof employs the following fact; $\lfloor x \rfloor = k$ if then $x = k + \theta$, where $0 \leq \theta < 1$.

Corollary 5: $\lim_{n\to\infty} \frac{P_{n+1}}{P_n} = \gamma.$

Proof: By Theorem 3, $P_{n+1} = \gamma P_n + \frac{1}{2} + \theta$ where $0 \leq \theta < 1$.

$$\frac{P_{n+1}}{P_n} = \gamma + \frac{1}{2P_n} + \frac{\theta}{P_n}$$

$$\therefore\quad \lim_{n\to\infty} \frac{P_{n+1}}{P_n} = \gamma.$$

Since $\lfloor x \rfloor = \lceil x \rceil - 1$, for any non-integral real number *x*. We can express these two formulas in terms of the ceiling function, as the next corollary states:

Corollary 6: (i) $P_{n+1} = \lceil \gamma P_n - \frac{1}{2} \rceil,\ n \geq 2.$

(ii) $P_{n+1} = \left\lfloor \frac{2P_n + 2\sqrt{2}P_n - 1}{2} \right\rfloor,\ n \geq 2.$

For example, the successor of the Pell number 2378 is given by $\lceil \gamma 2378 - \frac{1}{2} \rceil = \lceil 5740.499851 \rceil = 5741$.

Theorem 4: $P_{n+1} = \lfloor \gamma Q_n + \frac{1}{2} \rfloor,\ n \geq 2.$

For example, the successor of the Pell-Lucas number 478 is given by $\lceil \gamma 478 + \frac{1}{2} \rceil = \lceil 1154.494083 \rceil = 1154$. Notice that $Q_7 = 478$ and $Q_8 = 1154$.

Corollary 7: $\lim_{n \to \infty} \frac{Q_{n+1}}{Q_n} = \gamma.$

Corollary 6: (i) $Q_{n+1} = \left\lfloor \frac{2Q_n + 2\sqrt{2} + 1}{2} \right\rfloor,\ n \geq 2.$

(ii) $Q_{n+1} = \lceil \gamma Q_n - \frac{1}{2} \rceil,\ n \geq 2.$

(iii) $Q_{n+1} = \left\lceil \frac{2Q_n - 2\sqrt{2}Q_n - 1}{2} \right\rceil,\ n \geq 2.$

For example, the successor of the Pell-Lucas number 2786 is given by $\lceil 2786\gamma - \frac{1}{2} \rceil = \lceil 6725.498985 \rceil = 6726$. Notice that $Q_9 = 2786$ and $Q_{10} = 6726$.

There is yet another recursive formula that expresses each Pell number in terms of its predecessor and one that expresses each Pell-Lucas number in terms of its predecessor. We find both in the following theorem:

Theorem 5: (i) $P_{n+1} = \frac{2P_n + \sqrt{8P_n^2 + 4(-1)^n}}{2}$

(ii) $Q_{n+1} = \frac{2Q_n + \sqrt{8\,[Q_n^2 - 4(-1)^n]}}{2}$

This theorem can easily be proved using following three identities [2]:

$$2P_{n+1} = 2P_n + Q_n$$

$$2Q_{n+1} = 8P_n + 2Q_n$$

$$Q_n^2 - 8P_n^2 = 4(-1)^n.$$

There is still another formula that expresses a Pell number in terms of its predecessor.

Theorem 6: $P_{n+1} = \left\lfloor \frac{2P_n + 1 + \sqrt{8P_n^2 - 4P_n + 1}}{2} \right\rfloor,\ n \geq 2$.

Proof: Since $$Q_n - 2P_n = 2(P_{n-1} + P_n) - 2P_n = 2P_{n-1} \qquad (4)$$

Also, $Q_n^2 - 8P_n^2 = 4(-1)^n$, where $n \geq 1$. When $n \geq 2$, $4(-1)^n \leq 4P_{n-1}$.

Therefore, when $n \geq 2$, we have

$$Q_n^2 - 8P_n^2 \leq 4P_{n-1}$$

Or $$Q_n^2 - 8P_n^2 \leq 2(Q_n - 2P_n) \qquad \text{Using (4)}$$

Or $$(Q_n - 1)^2 \leq 8P_n^2 - 4P_n + 1 \qquad (5)$$

But, $Q_n = 2P_{n+1} - 2P_n$

$\therefore$ Using (5), we can write

$$(2P_{n+1} - 2P_n - 1)^2 \leq 8P_n^2 - 4P_n + 1.$$

Thus, $$(2P_{n+1} - 2P_n - 1) \leq \sqrt{8P_n^2 - 4P_n + 1}$$

Or $$P_{n+1} \leq \frac{2P_n + 1 + \sqrt{8P_n^2 - 4P_n + 1}}{2} \qquad (6)$$

Also, $Q_n + 2P_n = 2P_{n+1}$. So when $n \geq 2$,

$$4(-1)^n < 4P_{n-1}$$

Or $$-4(-1)^n > -4P_{n-1}$$

$\therefore$ $$Q_n^2 - 8P_n^2 > -2(Q_n + 2P_n)$$

Or $$Q_n^2 + 2Q_n > 8P_n^2 - 4P_n$$

Or $$(Q_n + 1)^2 > 8P_n^2 - 4P_n + 1$$

$\therefore$ $$(2P_{n+1} - 2P_n + 1)^2 > 8P_n^2 - 4P_n + 1$$

Thus, $$(2P_{n+1} - 2P_n + 1) > \sqrt{8P_n^2 - 4P_n + 1}$$

$$P_{n+1} > \frac{2P_n - 1 + \sqrt{8P_n^2 - 4P_n + 1}}{2} \tag{7}$$

$$P_{n+1} > \left\lfloor \frac{2P_n - 1 + \sqrt{8P_n^2 - 4P_n + 1}}{2} \right\rfloor \tag{8}$$

From equation (6) and (7), we have

$$\left\lfloor \frac{2P_n - 1 + \sqrt{8P_n^2 - 4P_n + 1}}{2} \right\rfloor < P_{n+1} \le \frac{2P_n + 1 + \sqrt{8P_n^2 - 4P_n + 1}}{2}.$$

Since P_{n+1} is an integer, it follows that

$$P_{n+1} = \left\lfloor \frac{2P_n + 1 + \sqrt{8P_n^2 - 4P_n + 1}}{2} \right\rfloor, \; n \ge 2.$$

For example, the successor of the Pell number 985 is given by

$$\left\lfloor \frac{2(985) + 1 + \sqrt{8(985)^2 - 4(985) + 1}}{2} \right\rfloor = \lfloor 2378.14685 \rfloor = 2378.$$

Similar to Pell numbers, there is a formula for Pell-Lucas numbers also given as follows:

Theorem 7: $Q_{n+1} = \left\lfloor \frac{2Q_n + 1 + \sqrt{8Q_n^2 - 4Q_n + 1}}{2} \right\rfloor, \; n \ge 4.$

For example, the successor of the Pell-Lucas number 1154 is given by

$$\left\lfloor \frac{2(1154) + 1 + \sqrt{8(1154)^2 - 4(1154) + 1}}{2} \right\rfloor = \lfloor 2786.148936 \rfloor = 2786.$$

We can also compute the predecessor of a given Pell number, as the following theorem states:

Theorem 8: $P_n = \left\lfloor \frac{1}{\gamma}\left(P_{n+1} + \frac{1}{2}\right) \right\rfloor,\ n \geq 2$.

Proof: Since $x - 1 < \lfloor x \rfloor \leq x$, using Theorem 3 we can write

$$\gamma P_n - \frac{1}{2} < P_{n+1} \leq \gamma P_n + \frac{1}{2}$$

Or
$$P_n - \frac{1}{2\gamma} < \frac{P_{n+1}}{\gamma} \leq P_n + \frac{1}{2\gamma}$$

Then,
$$P_n < \frac{1}{\gamma}\left(P_{n+1} + \frac{1}{2}\right) \quad \text{and} \quad P_n \geq \frac{1}{\gamma}\left(P_{n+1} - \frac{1}{2}\right)$$

Or
$$\frac{1}{\gamma}\left(P_{n+1} - \frac{1}{2}\right) < P_n \leq \frac{1}{\gamma}\left(P_{n+1} + \frac{1}{2}\right)$$

Since $\frac{1}{\gamma}(P_{n+1} + \frac{1}{2}) - \frac{1}{\gamma}(P_{n+1} - \frac{1}{2}) = \frac{1}{\gamma} \approx 0.4142$ and P_n is an integer, it follows that

$$P_n = \left\lfloor \frac{1}{\gamma}\left(P_{n+1} + \frac{1}{2}\right) \right\rfloor,\ n \geq 2.$$

For example, the predecessor of the Pell number 13860 is given by $\left\lfloor \frac{1}{\gamma}(13860 + \frac{1}{2}) \right\rfloor = \lfloor 5741.207081 \rfloor = 5741$. Notice that $P_{12} = 13860$ and $P_{11} = 5741$.

Theorem 9: $Q_n = \left\lfloor \frac{1}{\gamma}(Q_{n+1} + \frac{1}{2}) \right\rfloor,\ n \geq 2$.

For example, the predecessor of the Pell-Lucas number 39202 is given by $\left\lfloor \frac{1}{\gamma}(39202 + \frac{1}{2}) \right\rfloor = \lfloor 16238.20718 \rfloor = 16238$. Notice that $Q_{12} = 39202$ and $Q_{11} = 16238$.

Theorem 10: $\left\lfloor \gamma^k P_n + \frac{1}{2} \right\rfloor = P_{n+k},\ n \geq k \geq 1$.

Proof: Since the theorem is true for $k = 1$. Assume that $n \geq k \geq 2$. Using Binet formula,

$$\gamma^k P_n = \frac{\gamma^{n+k} - \gamma^k \delta^n}{2\sqrt{2}}$$

$$= \frac{\gamma^{n+k} - \gamma^k \delta^n + \delta^{n+k} - \delta^{n+k}}{2\sqrt{2}}$$

$$= \frac{\gamma^{n+k} - \delta^{n+k}}{2\sqrt{2}} - \frac{\gamma^k \delta^n - \delta^{n+k}}{2\sqrt{2}}$$

$$= P_{n+k} - \delta^n P_k$$

$$\therefore \quad \gamma^k P_n + \frac{1}{2} = P_{n+k} + \left(\frac{1}{2} - \delta^n\right). \tag{9}$$

Now we shall prove that $0 < (\frac{1}{2} - \delta^n P_k) < 1$. When $n = k$, $|\delta^n P_k|$ has its largest value. Notice that $|\delta^n| \to 0$ as $n \to \infty$.

Also,

$$\left|\delta^k P_k\right| = \left|\delta^k \left(\frac{\gamma^k - \delta^k}{2\sqrt{2}}\right)\right| = \left|\frac{(-1)^k - \delta^{2k}}{2\sqrt{2}}\right|.$$

Case I: Let k be even. Then

$$\left|\delta^k P_k\right| = \left|\frac{1 - \delta^{2k}}{2\sqrt{2}}\right|$$

$$\Rightarrow \qquad \lim_{k\to\infty}\left|\delta^k P_k\right| = \left|\frac{1-0}{2\sqrt{2}}\right| = \frac{1}{2\sqrt{2}} < \frac{1}{2}.$$

Since $|\delta^n| = |\delta^k|$, it follows that $0 < |\delta^n P_k| < \frac{1}{2}$.

Case II: Let k be odd. Then

$$\left|\delta^k P_k\right| = \left|\frac{-1 - \delta^{2k}}{2\sqrt{2}}\right| = \left|\frac{1 + \delta^{2k}}{2\sqrt{2}}\right|$$

When $k = 3$, $\delta^{2k} \approx 0.005050633883$. So

$$\left|\delta^k P_k\right| = \left|\frac{1.005050633883}{2\sqrt{2}}\right| = 0.355339 < \frac{1}{2}.$$

As k increases, δ^{2k} gets smaller and smaller. So $|\delta^k P_k| < \frac{1}{2}$ for $k > 3$ also. Thus $0 < |\delta^n P_k| < \frac{1}{2}$, since $|\delta^n| < |\delta^k|$.

Consequently,

$$0 < \left|\delta^n P_k\right| < \frac{1}{2} \quad \forall n \geq k \geq 2;$$

Or
$$-\frac{1}{2} < \delta^n P_k < \frac{1}{2}$$

Or
$$0 < \frac{1}{2} - \delta^n P_k < 1$$

Using (9), we have

$$P_{n+k} < \gamma^k P_n + \frac{1}{2} < P_{n+1} + 1$$

Thus,

$$\left\lfloor \gamma^k P_n + \frac{1}{2} \right\rfloor = P_{n+k}, \; n \geq k \geq 1.$$

For example,

$$\left\lfloor \gamma^7 P_8 + \frac{1}{2} \right\rfloor = \left\lfloor \gamma^7 (408) + \frac{1}{2} \right\rfloor = \lfloor 195025.3536 \rfloor = 195025 = P_{15} = P_{8+7}.$$

Notice that

$$\left\lfloor \gamma^8 P_7 + \frac{1}{2} \right\rfloor = 195026 \neq P_{15}.$$

Corollary 9: $\left\lceil \gamma^k P_n - \frac{1}{2} \right\rceil = P_{n+k}$ where $n \geq k \geq 1$.

For example,

$$\left\lceil \gamma^9 P_{11} - \frac{1}{2} \right\rceil = \left\lceil \gamma^9 (5741) - \frac{1}{2} \right\rceil = \lceil 15994427.56 \rceil = 15994427 = P_{20} = P_{11+9}.$$

Theorem 11: $\left\lfloor \gamma^k Q_n + \frac{1}{2} \right\rfloor = Q_{n+k}, \; n \geq 4, \; k \geq 1.$

Proof: Since. $\gamma Q_n - Q_{n+1} = \gamma(\gamma^n + \delta^n) - (\gamma^{n+1} + \delta^{n+1}) = \delta^n(\gamma - \delta) = \sqrt{8}\delta^n$ When $n \geq 4$.

$$\left|\sqrt{8}\delta^n\right| \leq \sqrt{8}\delta^4 = \sqrt{8}(1-\sqrt{2})^4 \approx 0.0658 < \frac{1}{2}$$

$$\therefore \quad \left|\gamma Q_n - Q_{n+1}\right| < \frac{1}{2}.$$

Or $0 < \gamma Q_n - Q_{n+1} + \frac{1}{2} < 1$, so $\left\lfloor \gamma Q_n + \frac{1}{2} \right\rfloor = Q_{n+1}$. Thus the theorem is true for $k = 1$. Now assume $n \geq k + 2$ where $k \geq 2$.

Notice that $\gamma^{-2} + \gamma^{-6} = \delta^2 + \delta^6 \approx 0.176620633$.

Since $k \geq 2$, this implies $\gamma^{-2} + \gamma^{-2k-2} < \frac{1}{2}$;

Or $$\gamma^{-k-2}(\gamma^k + \gamma^{-k}) < \frac{1}{2}.$$

Since $n \geq k + 2$, this implies $\gamma^{-n}(\gamma^k + \gamma^{-k}) < \frac{1}{2}$;

$$\therefore \quad \left|\delta^n(\gamma^k - \delta^k)\right| < \frac{1}{2}.$$

That is, $\left|\gamma^k Q_n - Q_{n+k}\right| < \frac{1}{2}$.

This implies that $\left\lfloor \gamma^k Q_n + \frac{1}{2} \right\rfloor = Q_{n+k}$.

For example,

$$\left\lfloor \gamma^3 P_{11} + \frac{1}{2} \right\rfloor = \left\lfloor \gamma^3(16238) + \frac{1}{2} \right\rfloor = \lfloor 228486.4991 \rfloor = 228486 = Q_{14} = Q_{11+3}.$$

Corollary 10: $\left\lceil \gamma^k Q_n - \frac{1}{2} \right\rceil = Q_{n+k}$ where $n \geq 4$, $k \geq 1$.

For example,

$$\left\lfloor \gamma^4 Q_{10} - \frac{1}{2} \right\rfloor = \left\lfloor \gamma^4(6726) - \frac{1}{2} \right\rfloor = \lfloor 228485.505 \rfloor = 228486 = Q_{14} = Q_{10+4}.$$

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Naresh Patel
Assistant Professor (Mathematics)
Government College Jobat
Distt. – Alirajpur (M.P.) India.
E-mail: n_patel_1978@yahoo.co.in

Punit Shrivastava
Lecturer (Mathematics),
Dhar Polytechnic College, Dhar (M.P.) India.
E-mail: puneetsri2001@yahoo.co.in