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# ANTICIPATED BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS WITH CONTINUOUS COEFFICIENTS

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Abstract. In this paper we prove the existence of solutions to 1-dimensional anticipated backward stochastic differential equations with continuous coefficients. We also establish the existence of a minimal solution. Finally we derive a related comparison theorem for these minimal solutions.

## 1. Introduction

In 2009, Peng and Yang [5] defined a new kind of backward stochastic differential equation (BSDE for short), called an anticipated BSDE, as follows:

$$\begin{cases} Y_t = \xi_T + \int_t^T f(s, Y_s, Z_s, Y_{s+\delta(s)}, Z_{s+\zeta(s)})ds - \int_t^T Z_s dW_s, & t \in [0,T]; \\ Y_t = \xi_t, & t \in [T, T+K]; \\ Z_t = \eta_t, & t \in [T, T+K]. \end{cases}$$

In [5] existence, uniqueness and comparison theorems were proved for solutions of these equations with similar Lipschitz coefficients, (i.e., satisfying $(H1)$ in Section 2). In this paper, we prove that if the similar Lipschitz assumption is relaxed, the results of existence and comparison theorem for anticipated BSDEs still hold.

Lepeltier and Martin [2] generalized the existence theorem for solutions of BSDEs from Lipschitz coefficients to continuous coefficients. Based on [2], Liu and Ren [3] proved a related comparison theorem. Consequently, a natural question is: does there exist a solution for anticipated BSDEs with continuous coefficients? Moreover, does the comparison theorem still hold for the case? In this paper we provide positive answers.

To treat this problem, we shall use the comparison theorem proved in [5] for anticipated BSDEs with similar Lipschitz coefficients. There are then no anticipated terms for $Z$ in anticipated BSDEs, that is, the anticipated BSDE has to be the following form:

$$\begin{cases} Y_t = \xi_T + \int_t^T f(s, Y_s, Z_s, Y_{s+\delta(s)})ds - \int_t^T Z_s dW_s, & t \in [0,T]; \\ Y_t = \xi_t, & t \in [T, T+K]. \end{cases}$$

The paper is organized as follows. Section 2 presents some results for BSDEs and anticipated BSDEs. In Section 3 we prove the existence theorem of solutions to anticipated BSDEs with continuous coefficients. We also show there exists a

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minimal solution for this kind of equation. We establish the related comparison theorem for the minimal solutions in Section 4. This paper includes a lot of detailed analysis. It is non-trivial and, we hope, of interest.

## 2. Preliminaries

Let $(\Omega, \mathscr{F}, P, \mathscr{F}_t,\ t \geq 0)$ be a complete stochastic basis such that $\mathscr{F}_0$ contains all $P$-null elements of $\mathscr{F}$ and suppose that the filtration is generated by a $d$-dimensional standard Brownian motion $W = (W_t)_{t\geq 0}$. Given $T > 0$. For all $n \in \mathbb{N}$, denote the Euclidean norm in $\mathbb{R}^n$ by $|\cdot|$. Denote:

$L^2(\mathscr{F}_T;\mathbb{R}^m) =$\{$\mathbb{R}^m$-valued $\mathscr{F}_T$-measurable random variable $\xi$ satisfying that $E[|\xi|^2] < \infty$\};

$L^2_{\mathscr{F}}(0,T;\mathbb{R}^m) =$\{$\mathbb{R}^m$-valued $\mathscr{F}_t$-adapted stochastic process $\varphi_\cdot$ satisfying that $E[\int_0^T |\varphi_t|^2 dt] < \infty$\};

$S^2_{\mathscr{F}}(0,T;\mathbb{R}^m) =$\{continuous process $\varphi_\cdot$ in $L^2_{\mathscr{F}}(0,T;\mathbb{R}^m)$ satisfying that $E[\sup_{0\leq t\leq T} |\varphi_t|^2] < \infty$\}.

If $m = 1$, we denote $L^2(\mathscr{F}_T,\mathbb{R})$ by $L^2(\mathscr{F}_T)$, $L^2_{\mathscr{F}}(0,T;\mathbb{R})$ by $L^2_{\mathscr{F}}(0,T)$ and $S^2_{\mathscr{F}}(0,T;\mathbb{R})$ by $S^2_{\mathscr{F}}(0,T)$.

Consider the anticipated BSDE:

$$\begin{cases} -dY_s = f(s, Y_s, Z_s, Y_{s+\delta(s)}, Z_{s+\zeta(s)})ds - Z_s dW_s, & s \in [0,T]; \\ \quad Y_s = \xi_s, & s \in [T, T+K]; \\ \quad Z_s = \eta_s, & s \in [T, T+K]. \end{cases} \tag{2.1}$$

Here $\delta(\cdot)$ and $\zeta(\cdot)$ are two $\mathbb{R}^+$-valued continuous functions defined on $[0,T]$ such that

**(i)** there exists a constant $K \geq 0$ such that for any $s \in [0,T]$,

$$s + \delta(s) \leq T + K; \qquad s + \zeta(s) \leq T + K.$$

**(ii)** there exists a constant $L \geq 0$ such that for any $s \in [0,T]$ and nonnegative and integrable $g(\cdot)$,

$$\int_s^T g(r+\delta(r))dr \leq L \int_s^{T+K} g(r)dr; \qquad \int_s^T g(r+\zeta(r))dr \leq L \int_s^{T+K} g(r)dr.$$

Assume that for any $s \in [0,T],\ f(s,\omega,y,z,\xi,\eta) : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times L^2(\mathscr{F}_r;\mathbb{R}^m) \times L^2(\mathscr{F}_{r'};\ \mathbb{R}^{m\times d}) \longrightarrow L^2(\mathscr{F}_s,\mathbb{R}^m)$, where $r, r' \in [s, T+K]$, and $f$ satisfies the following conditions:

**(H1)** similar Lipschitz condition: there exists a constant $C > 0$, such that for any $s \in [0,T],\ y, y' \in \mathbb{R}^m,\ z, z' \in \mathbb{R}^{m\times d},\ \xi_\cdot, \xi'_\cdot \in L^2_{\mathscr{F}}(s, T+K;\mathbb{R}^m),\ \eta_\cdot, \eta'_\cdot \in L^2_{\mathscr{F}}(s, T+K;\ \mathbb{R}^{m\times d}),\ r, t \in [s, T+K]$, we have

$$\begin{aligned} &|f(s,y,z,\xi_r,\eta_t) - f(s,y',z',\xi'_r,\eta'_t)| \\ &\leq C(|y-y'| + |z-z'| + E^{\mathscr{F}_s}[|\xi_r - \xi'_r| + |\eta_t - \eta'_t|]). \end{aligned}$$

**(H2)** $E[\int_0^T |f(s,0,0,0,0)|^2 ds] < \infty$.

The following three lemmas give the existence and uniqueness results for adapted solutions of anticipated BSDEs with similar Lipschitz coefficients, the estimate of the solutions and the comparison result for 1-dimensional related anticipated BSDEs, respectively. (See [5]).

**Lemma 2.1.** *Suppose that $f$ satisfies $(H1)$ and $(H2)$, $\delta,\zeta$ satisfy $(i)$ and $(ii)$. Then for arbitrary given terminal conditions $\xi_\cdot \in S^2_{\mathscr{F}}(T,T+K;\mathbb{R}^m)$, $\eta_\cdot \in L^2_{\mathscr{F}}(T,T+K;\mathbb{R}^{m\times d})$, the anticipated BSDE (2.1) has a unique solution, i.e., there exists a unique pair of $\mathscr{F}_t$-adapted processes $(Y_\cdot,Z_\cdot) \in S^2_{\mathscr{F}}(0,T+K;\mathbb{R}^m)\times L^2_{\mathscr{F}}(0,T+K;\mathbb{R}^{m\times d})$ satisfying equation (2.1).*

**Lemma 2.2.** *Assume that $f$ satisfies $(H1)$ and $(H2)$, $\delta$ and $\zeta$ satisfy $(i)$ and $(ii)$. Then there exists a positive constant $C_0$ only depending on $C$ in $(H1)$, $L$ in $(ii)$ and $T$ such that for any $\xi_\cdot \in S^2_{\mathscr{F}}(T,T+K;\mathbb{R}^m)$, $\eta_\cdot \in L^2_{\mathscr{F}}(T,T+K;\mathbb{R}^{m\times d})$, the solution $(Y_\cdot,Z_\cdot)$ to anticipated BSDE (2.1) satisfies*

$$\begin{aligned} &E^{\mathscr{F}_t}[\sup_{t\le s\le T}|Y_s|^2+\int_t^T|Z_s|^2ds] \\ &\le C_0E^{\mathscr{F}_t}[\,|\xi_T|^2+\int_T^{T+K}(|\xi_s|^2+|\eta_s|^2)ds+(\int_t^T|f(s,0,0,0,0)|ds)^2], \end{aligned} \tag{2.2}$$

*for any $t\in[0,T]$.*

**Lemma 2.3.** *Let $(Y^{(1)}_\cdot,Z^{(1)}_\cdot)$ and $(Y^{(2)}_\cdot,Z^{(2)}_\cdot)$ be respectively the solutions of the following two 1-dimensional anticipated BSDEs:*

$$\begin{cases} Y_t^{(j)}=\xi_T^{(j)}+\int_t^T f_j(s,Y_s^{(j)},Z_s^{(j)},Y^{(j)}_{s+\delta(s)})ds-\int_t^T Z_s^{(j)}dW_s, & 0\le t\le T;\\ Y_t^{(j)}=\xi_t^{(j)}, & T\le t\le T+K, \end{cases}$$

*where $j=1,2$. Assume that $\xi^{(1)}_\cdot,\xi^{(2)}_\cdot\in S^2_{\mathscr{F}}(T,T+K)$, $\delta$ satisfies $(i),(ii)$ and $f_1,f_2$ satisfy $(H1),(H2)$, furthermore, for any $t\in[0,T]$, $y\in\mathbb{R}$, $z\in\mathbb{R}^d$, $f_2(t,y,z,\cdot)$ is increasing, that is, $f_2(t,y,z,\theta_r)\ge f_2(t,y,z,\theta'_r)$, if $\theta_r\ge\theta'_r$, $\theta_\cdot,\theta'_\cdot\in L^2_{\mathscr{F}}(t,T+K)$, $r\in[t,T+K]$. If $\xi_s^{(1)}\ge\xi_s^{(2)}$, $s\in[T,T+K]$, and $f_1(t,y,z,\theta_r)\ge f_2(t,y,z,\theta_r)$, $t\in[0,T]$, $y\in\mathbb{R}$, $z\in\mathbb{R}^d$, $\theta_\cdot\in L^2_{\mathscr{F}}(t,T+K)$, $r\in[t,T+K]$, then*

$$Y_t^{(1)}\ge Y_t^{(2)},\qquad a.e.,\ a.s.$$

For completeness we quote the following four lemmas from Peng [4]. Lemma 2.4 gives two estimates for the solution to a simple BSDE. Lemma 2.5 is an existence and uniqueness theorem for BSDEs. Both Lemma 2.6 and Lemma 2.7 are comparison theorems for solutions of BSDEs. Lemma 2.6 can also be found in El Karoui, Peng and Quenez [1]. Lemma 2.7 can be easily obtained from Lemma 2.6.

**Lemma 2.4.** *For a fixed $\xi\in L^2(\mathscr{F}_T)$ and $g_0(\cdot)$ which is an $\mathscr{F}_t$-adapted process satisfying $E[(\int_0^T|g_0(t)|dt)^2]<\infty$, there exists a unique pair of processes $(y_\cdot,z_\cdot)\in L^2_{\mathscr{F}}(0,T;\mathbb{R}^{1+d})$ satisfying the following BSDE:*

$$y_t=\xi+\int_t^T g_0(s)ds-\int_t^T z_sdW_s,\qquad t\in[0,T].$$

*If $g_0(\cdot) \in L^2_{\mathscr{F}}(0,T)$, then $(y_\cdot, z_\cdot) \in S^2_{\mathscr{F}}(0,T) \times L^2_{\mathscr{F}}(0,T;\mathbb{R}^d)$. We have the following basic estimate:*

$$\begin{aligned}&|y_t|^2 + E^{\mathscr{F}_t}[\int_t^T (\frac{\beta}{2}|y_s|^2 + |z_s|^2)e^{\beta(s-t)}ds] \\ &\leq E^{\mathscr{F}_t}[|\xi|^2 e^{\beta(T-t)}] + \frac{2}{\beta}E^{\mathscr{F}_t}[\int_t^T |g_0(s)|^2 e^{\beta(s-t)}ds],\end{aligned} \tag{2.3}$$

*where $\beta > 0$ is an arbitrary constant.*

**Lemma 2.5.** *Assume that $g = g(\omega, t, y, z) : \Omega \times [0,T] \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \longrightarrow \mathbb{R}^m$ satisfies the following conditions:*
***(a)*** *$g(\cdot, y, z)$ is an $\mathbb{R}^m$-valued and $\mathscr{F}_t$-adapted process satisfying Lipschitz condition in $(y,z)$, i.e., there exists $\rho > 0$ such that for any $y, y' \in \mathbb{R}^m, z, z' \in \mathbb{R}^{m\times d}$,*

$$|g(t,y,z) - g(t,y',z')| \leq \rho(|y-y'| + |z-z'|).$$

***(b)*** *$g(\cdot, 0, 0) \in L^2_{\mathscr{F}}(0,T;\mathbb{R}^m)$.*
*Then for any given terminal condition $\xi \in L^2(\mathscr{F}_T;\mathbb{R}^m)$, BSDE*

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \qquad 0 \leq t \leq T \tag{2.4}$$

*has a unique solution, i.e., there exists a unique pair of $\mathscr{F}_t$-adapted processes $(Y_\cdot, Z_\cdot) \in S^2_{\mathscr{F}}(0,T;\mathbb{R}^m) \times L^2_{\mathscr{F}}(0,T;\mathbb{R}^{m\times d})$ satisfying equation (2.4).*

**Lemma 2.6.** *Assume $g_j(\omega, t, y, z) : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}$ satisfies (a) and (b), $\xi^{(j)} \in L^2(\mathscr{F}_T)$, $j = 1, 2$. Let $(Y^{(1)}_\cdot, Z^{(1)}_\cdot)$ and $(Y^{(2)}_\cdot, Z^{(2)}_\cdot)$ be respectively the solutions of BSDEs as follows:*

$$Y^{(j)}_t = \xi^{(j)} + \int_t^T g_j(s, Y^{(j)}_s, Z^{(j)}_s)ds - \int_t^T Z^{(j)}_s dW_s, \quad 0 \leq t \leq T,$$

*where $j = 1, 2$. If $\xi^{(1)} \geq \xi^{(2)}$ and $g_1(t, Y^{(1)}_t, \ Z^{(1)}_t) \geq g_2(t, Y^{(1)}_t, Z^{(1)}_t)$, a.e., a.s., then*

$$Y^{(1)}_t \geq Y^{(2)}_t, \qquad a.e., \ a.s.$$

**Lemma 2.7.** *We make the same assumption as in Lemma 2.6. If $\xi^{(1)} \geq \xi^{(2)}$, $g_1(t,y,z) \geq g_2(t,y,z)$, $t \in [0,T]$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$, then*

$$Y^{(1)}_t \geq Y^{(2)}_t, \qquad a.e., \ a.s.$$

Lemma 2.8 and Lemma 2.9 can also be found in Lepeltier and Martin [2]. Lemma 2.8 is one of the basic lemmas required to prove both Lemma 2.9 in [2], and Theorem 3.2 in Section 3. Lemma 2.9 is the existence theorem for BSDEs with continuous coefficients.

**Lemma 2.8.** *Assume $f : \mathbb{R}^m \longrightarrow \mathbb{R}$ is a continuous function with linear growth, that is, there exists a constant $K < \infty$ such that for any $x \in \mathbb{R}^m$, $|f(x)| \leq K(1 + |x|)$. Then the sequence of functions*

$$f_n(x) = \inf_{y \in \mathbb{Q}^m} \{f(y) + n|x-y|\} \tag{2.5}$$

*is well defined for any $n \in \mathbb{N}, n \geq K$ and it satisfies:*
***(I)*** *linear growth: for any $x \in \mathbb{R}^m$, $|f_n(x)| \leq K(1+|x|)$;*

***(II)*** *monotonicity in* $n$ : *for any* $x \in \mathbb{R}^m$, $f_n(x) \nearrow$ ;
***(III)*** *Lipschitz continuous condition: for any* $x, y \in \mathbb{R}^m$, $|f_n(x) - f_n(y)| \leq n|x-y|$;
***(IV)*** *strong convergence: if* $x_n \to x,\ n \to \infty$, *then* $f_n(x_n) \to f(x),\ n \to \infty$.

**Lemma 2.9.** *Let* $\mathscr{P}$ *is the predictable* $\sigma$*-field and*

$$H^2(\mathbb{R}^p) = \{X. : [0,T] \times \Omega \longrightarrow \mathbb{R}^p;\ X. \in \mathscr{P} \text{ and } \|X.\|^2 = E[\int_0^T |X_s|^2 ds] < \infty\}.$$

*Assume* $g : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}$ *is* $\mathscr{P} \times \mathscr{B}(\mathbb{R}^{1+d})$ *measurable function, which satisfies*

***(H3)*** *linear growth: there exists* $K' < \infty$ *such that for any* $t \in [0,T]$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$, $|g(t,y,z)| \leq K'(1 + |y| + |z|)$.

***(H4)*** *for fixed* $t, \omega$, $f(t, \omega, \cdot, \cdot)$ *is continuous.*

*If* $\xi \in L^2(\mathscr{F}_T)$, *then the BSDE*

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \qquad t \in [0,T] \tag{2.6}$$

*has an adapted solution* $(Y., Z.) \in H^2(\mathbb{R}^{1+d})$, *where* $Y.$ *is a continuous process and* $Z.$ *is predictable. Also, there is a minimal solution* $(\hat{Y}., \hat{Z}.)$ *of equation (2.6), in the sense that for any other solution* $(Y., Z.)$ *of equation (2.6), we have* $\hat{Y}_t \leq Y_t$, *a.e., a.s.*

Lemma 2.10 is the comparison theorem for the minimal solutions of BSDEs with continuous coefficients (see Liu and Ren [2]).

**Lemma 2.10.** *Let* $(\hat{Y}.^{(i)}, \hat{Z}.^{(i)}),\ i = 1, 2$ *be the minimal solutions to the following equations, respectively,*

$$Y_t^{(i)} = \xi^{(j)} + \int_t^T g_i(s, Y_s^{(i)}, Z_s^{(i)})ds - \int_t^T Z_s^{(i)} dW_s, \qquad t \in [0,T],$$

*where for* $i = 1, 2$, $\xi^{(j)} \in L^2(\mathscr{F}_T)$, *for any* $y \in \mathbb{R}$, $z \in \mathbb{R}^d$, $g_i(\cdot, y, z) \in H^2(\mathbb{R})$, *moreover,* $g_i$ *satisfies* $(H3)$ *and* $(H4)$. *If* $g_1(t,y,z) \geq g_2(t,y,z)$, $t \in [0,T]$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$ *and* $\xi^{(1)} \geq \xi^{(2)}$, *a.e., then*

$$\hat{Y}_t^{(1)} \geq \hat{Y}_t^{(2)}, \qquad a.e.,\ a.s.$$

*Remark* 2.11. The results of Lemma 2.9 and Lemma 2.10 will hold for adapted processes if we change the conditions 'predictable' into 'adapted' in the above two lemmas.

## 3. Existence Theorem of Multiple Solutions to Anticipated BSDEs With Continuous Coefficients

From now on, we only consider 1-dimensional solutions $Y.$ of anticipated BSDEs. We introduce a new definition:

**Definition 3.1.** Let $s \leq t$ be two fixed times. The functional $\varphi : L^2(\mathscr{F}_t) \longrightarrow L^2(\mathscr{F}_s)$ is *continuous* in $L^2(\mathscr{F}_t)$ if for any $\xi_n, \eta \in L^2(\mathscr{F}_t)$ satisfying $\xi_n \to \eta$ in $L^2(\mathscr{F}_t)$, then $\varphi(\xi_n) \to \varphi(\eta)$ in $L^2(\mathscr{F}_s)$ holds.

Consider the anticipated BSDE:

$$\begin{cases} Y_t = \xi_T + \int_t^T f(s, Y_s, Z_s, Y_{s+\delta(s)})ds - \int_t^T Z_s dW_s, & t \in [0,T]; \\ Y_t = \xi_t, & t \in [T, T+K]. \end{cases} \tag{3.1}$$

Here $\delta(\cdot)$ is an $\mathbb{R}^+$-valued continuous function defined on $[0,T]$ satisfying $(i)$ and $(ii)$. Assume that for any $s \in [0,T]$, $f(s,\omega,y,z,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^d \times L^2(\mathscr{F}_r) \longrightarrow L^2(\mathscr{F}_s)$, where $r \in [s, T+K]$, and $f$ satisfies the following conditions:

**(H5)** linear growth: there exists a constant $\hat{C} > 0$, such that for any $s \in [0,T]$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$, $\theta_\cdot \in L^2_{\mathscr{F}}(s, T+K)$, $r \in [s, T+K]$, we have

$$|f(s,y,z,\theta_r)| \leq \hat{C}(1 + |y| + |z| + E^{\mathscr{F}_s}[|\theta_r|]).$$

**(H6)** for fixed $s \in [0,T]$, $f(s,\cdot,\cdot,\cdot)$ is continuous, and for any $t \in [0,T]$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$, $f(t,y,z,\cdot)$ is increasing, moreover, for any $\xi \in L^2(\mathscr{F}_r)$, $r \in [t, T+K]$, $f(t,y,z,E^{\mathscr{F}_t}[\xi]) = f(t,y,z,\xi)$ holds.

**Assumption 1:** $\mathscr{F}$ contains all subsets of $\Omega$.

The following result is the existence theorem for a solution to an anticipated BSDE with continuous coefficients.

**Theorem 3.2.** *Suppose Assumption 1 holds, $f$ satisfies $(H5)$ and $(H6)$, and $\delta$ satisfies $(i)$ and $(ii)$. Then for an arbitrary given terminal condition $\xi_\cdot \in S^2_{\mathscr{F}}(T, T+K)$ with $\xi_T \in L^2(\mathscr{F}_T)$, there exists a pair of adapted processes $(Y_\cdot, Z_\cdot) \in S^2_{\mathscr{F}}(0, T+K) \times L^2_{\mathscr{F}}(0,T;\, \mathbb{R}^d)$ satisfying equation (3.1). Also, there is a minimal solution $\hat{Y}_\cdot$ of equation (3.1), in the sense that for any other solution $Y_\cdot$ of equation (3.1), we have $\hat{Y}_t \leq Y_t$, a.e., a.s.*

Before proving Theorem 3.2, we give some lemmas. Lemma 3.3 shows a limit of a sequence of solutions for anticipated BSDEs with similar Lipschitz and monotonic coefficients is still a solution of an anticipated BSDE. Similarly to Lemma 2.8, Lemma 3.5 shows that a continuous functional can be a limit of a sequence of similar Lipschitz functionals. Lemma 3.6 shows that the sequence of functionals defined in Lemma 3.5 inherits the monotony of the variable from the continuous functional which is the limit of the above sequence.

**Lemma 3.3.** *Consider the following anticipated BSDEs:*

$$\begin{cases} Y_t^{(n)} = \xi_T^{(n)} + \int_t^T f_n(s, Y_s^{(n)}, Z_s^{(n)}, Y_{s+\delta(s)}^{(n)})ds - \int_t^T Z_s^{(n)} dW_s, & t \in [0,T]; \\ Y_t^{(n)} = \xi_t^{(n)}, & t \in [T, T+K], \end{cases}$$

*where $n \in \mathbb{N}$. Assume $\delta$ satisfies $(i)$ and $(ii)$, and for any $n \in \mathbb{N}$, $\xi_\cdot^{(n)} \in S^2_{\mathscr{F}}(T, T+K)$ with $\xi_T^{(1)} \in L^2(\mathscr{F}_T)$, $f_n$ satisfies $(H1)$ and*
***(H2)′*** *for any $n \in \mathbb{N}$, $t \in [0,T]$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$, $f_n(t,y,z,\cdot)$ is increasing, and there exists a constant $\mu > 0$ such that*

$$E[(\int_0^T |f_n(s,0,0,0)|ds)^2] \leq \mu, \quad \textit{for any } n \in \mathbb{N}.$$

*If for any* $t \in [0,T]$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$, $\theta_\cdot \in L^2_{\mathscr{F}}(t, T+K)$, $r \in [t, T+K]$, $f_n(t,y,z,\theta_r) \nearrow f(t,y,z,\theta_r)$, $n \to \infty$, *and for any* $s \in [T, T+K]$, $\xi_s^{(n)} \nearrow \xi_s$, $n \to \infty$, *moreover,* $\xi_\cdot \in S^2_{\mathscr{F}}(T, T+K)$ *with* $\xi_T \in L^2(\mathscr{F}_T)$, *then the anticipated BSDE*

$$\begin{cases} Y_t = \xi_T + \int_t^T f(s, Y_s, Z_s, Y_{s+\delta(s)})ds - \int_t^T Z_s dW_s, & t \in [0,T]; \\ Y_t = \xi_t, & t \in [T, T+K] \end{cases} \tag{3.2}$$

*has a solution* $(Y_\cdot, Z_\cdot) \in S^2_{\mathscr{F}}(0, T+K) \times L^2_{\mathscr{F}}(0,T;\mathbb{R}^d)$ *and*

$$Y_t = \sup_{n \in \mathbb{N}} Y_t^{(n)}, \qquad a.e.,\ a.s.$$

*Proof.* Since for any $s \in [T, T+K]$, $\xi_s^{(n)} \nearrow \xi_s$, $n \to \infty$, we have for any $s \in [T, T+K]$, $\xi_s - \xi_s^{(n)} \searrow 0$, $n \to \infty$. Because $\xi_\cdot, \xi_\cdot^{(1)}, \xi_\cdot^{(2)}, \ldots \in L^2_{\mathscr{F}}(T, T+K)$, by Levi′s lemma we know $\xi_s^{(n)} \to \xi_s$ in $L^2_{\mathscr{F}}(T, T+K)$. Hence, $\{\xi_\cdot, \xi_\cdot^{(1)}, \xi_\cdot^{(2)}, \ldots\}$ is bounded in $L^2_{\mathscr{F}}(T, T+K)$. Denote its bounded by $A$. By Lemma 2.1 we know for any $n \in \mathbb{N}$, the anticipated BSDE

$$\begin{cases} Y_t^{(n)} = \xi_T^{(n)} + \int_t^T f_n(s, Y_s^{(n)}, Z_s^{(n)}, Y_{s+\delta(s)}^{(n)})ds - \int_t^T Z_s^{(n)} dW_s, & t \in [0,T]; \\ Y_t^{(n)} = \xi_t^{(n)}, & t \in [T, T+K] \end{cases}$$

has a unique solution $(Y_\cdot^{(n)}, Z_\cdot^{(n)})$. From Lemma 2.2 there exists a positive constant $C_0$ only depending on $C$ in $(H1)$, $L$ in $(ii)$ and $T$ such that for any $n \in \mathbb{N}$, we have

$$\begin{aligned} &E[\sup_{0 \le t \le T} |Y_t^{(n)}|^2 + \int_0^T |Z_t^{(n)}|^2 dt] \\ &\le C_0 E[|\xi_T^{(n)}|^2 + \int_T^{T+K} |\xi_t^{(n)}|^2 ds + (\int_0^T |f_n(t,0,0,0)|dt)^2]. \end{aligned}$$

By $(H2)'$, we know

$$\begin{aligned} &E[\int_0^T (|Y_t^{(n)}|^2 + |Z_t^{(n)}|^2)dt] \\ &\le (T+1)C_0 E[|\xi_T^{(n)}|^2 + A + \mu] \le (T+1)C_0 E[|\xi_T^{(1)}|^2 + |\xi_T|^2 + A + \mu]. \end{aligned}$$

Because $\xi_T^{(1)}, \xi_T \in L^2(\mathscr{F}_T)$, we deduce that $\{(Y_\cdot^{(n)}, Z_\cdot^{(n)});\ n \in \mathbb{N}\}$ is bounded in $L^2_{\mathscr{F}}(0,T;\mathbb{R}^{1+d})$. Denote its bounded by $B$. By Lemma 2.3, $\{Y_\cdot^{(n)}\}$ is increasing in $n$, then for any $\omega \in \Omega$, set

$$\tilde{Y}_t(\omega) = \begin{cases} \sup_{n \in \mathbb{N}} Y_t^{(n)}(\omega), & t \in [0,T]; \\ \xi_t(\omega) & t \in [T, T+K]. \end{cases}$$

Since for any $t \in [0,T]$,

$$\mathbb{I}_{\{\omega:\ \tilde{Y}_t(\omega) \ge 0\}} |Y_t^{(n)}(\omega)| \nearrow \mathbb{I}_{\{\omega:\ \tilde{Y}_t(\omega) \ge 0\}} |\tilde{Y}_t(\omega)|, \qquad n \to \infty,$$

and

$$\mathbb{I}_{\{\omega:\ \tilde{Y}_t(\omega) < 0\}} |Y_t^{(n)}(\omega)| \searrow \mathbb{I}_{\{\omega:\ \tilde{Y}_t(\omega) < 0\}} |\tilde{Y}_t(\omega)|, \qquad n \to \infty,$$

by Levi′s lemma,

$$E[\int_0^T |Y_t^{(n)}|^2 dt] \to E[\int_0^T |\tilde{Y}_t|^2 dt], \qquad n \to \infty.$$

So $E[\int_0^T |\tilde{Y}_t|^2 dt] \le B$, moreover, $|\tilde{Y}_\cdot| \in L^2_{\mathscr{F}}(0,T)$. Therefore,

$$Q((\omega,t) \in \Omega \times [0,T];\ \tilde{Y}_t(\omega) < \infty) = 1,$$

where $Q$ is a probability on $\Omega \times [0,T]$ with $Q|_\Omega = P$. Thus, also by Levi′s lemma we deduce $E[\int_0^T |\tilde{Y}_t - Y_t^{(n)}|^2 dt] \to 0,\ n \to \infty$. That is, $Y_\cdot^{(n)} \to \tilde{Y}_\cdot$ in $L^2_{\mathscr{F}}(0,T)$. Hence $Y^{(n)}$ converges uniformly to $\tilde{Y}$. As for any $n \in \mathbb{N}$, $Y_\cdot^{(n)}$ is continuous in $[0,T]$, $\tilde{Y}_\cdot$ is also continuous in $[0,T]$. Because $\xi_T^{(n)} \nearrow \xi_T,\ n \to \infty$, and $\xi_T^{(1)}, \xi_T \in L^2(\mathscr{F}_T)$, by Levi′s lemma we see $\xi_T^{(n)} \to \xi_T$ in $L^2(\mathscr{F}_T)$. For any $n,m \in \mathbb{N}$, applying Itô′s formula to $|Y_s^{(n)} - Y_s^{(m)}|^2$ on $[0,T]$,

$$\begin{aligned}
&E[|Y_0^{(n)} - Y_0^{(m)}|^2 + \textstyle\int_0^T |Z_s^{(n)} - Z_s^{(m)}|^2 ds] \\
&= E[|\xi_T^{(n)} - \xi_T^{(m)}|^2 \\
&\quad +2\textstyle\int_0^T (Y_s^{(n)} - Y_s^{(m)})(f_n(s, Y_s^{(n)}, Z_s^{(n)}, Y_{s+\delta(s)}^{(n)}) - f_m(s, Y_s^{(m)}, Z_s^{(m)}, Y_{s+\delta(s)}^{(m)}))ds].
\end{aligned}$$

Using the Hölder inequality and Schwarz inequality, we have

$$\begin{aligned}
&E[\textstyle\int_0^T |Z_s^{(n)} - Z_s^{(m)}|^2 ds] \\
&\le 2E[(\textstyle\int_0^T |f_n(s, Y_s^{(n)}, Z_s^{(n)}, Y_{s+\delta(s)}^{(n)}) - f_m(s, Y_s^{(m)}, Z_s^{(m)}, Y_{s+\delta(s)}^{(m)})|^2 ds)^{\frac{1}{2}} \\
&\qquad \cdot(\textstyle\int_0^T |Y_s^{(n)} - Y_s^{(m)}|^2 ds)^{\frac{1}{2}}] + E[|\xi_T^{(n)} - \xi_T^{(m)}|^2] \\
&\le 2\{E[\textstyle\int_0^T |f_n(s, Y_s^{(n)}, Z_s^{(n)}, Y_{s+\delta(s)}^{(n)}) - f_m(s, Y_s^{(m)}, Z_s^{(m)}, Y_{s+\delta(s)}^{(m)})|^2 ds]\}^{\frac{1}{2}} \\
&\quad \cdot\{E[\textstyle\int_0^T |Y_s^{(n)} - Y_s^{(m)}|^2 ds]\}^{\frac{1}{2}} + E[|\xi_T^{(n)} - \xi_T^{(m)}|^2] \\
&\le 2\sqrt{2}\{E[\textstyle\int_0^T (|f_n(s, Y_s^{(n)}, Z_s^{(n)}, Y_{s+\delta(s)}^{(n)})|^2 + |f_m(s, Y_s^{(m)}, Z_s^{(m)}, Y_{s+\delta(s)}^{(m)})|^2) ds]\}^{\frac{1}{2}} \\
&\quad \cdot\{E[\textstyle\int_0^T |Y_s^{(n)} - Y_s^{(m)}|^2 ds]\}^{\frac{1}{2}} + E[|\xi_T^{(n)} - \xi_T^{(m)}|^2] \\
&\le 4\{E[\textstyle\int_0^T (|f_n(s, Y_s^{(n)}, Z_s^{(n)}, Y_{s+\delta(s)}^{(n)}) - f_n(s,0,0,0)|^2 + |f_n(s,0,0,0)|^2 \\
&\qquad +|f_m(s, Y_s^{(m)}, Z_s^{(m)}, Y_{s+\delta(s)}^{(m)}) - f_m(s,0,0,0)|^2 + |f_m(s,0,0,0)|^2) ds]\}^{\frac{1}{2}} \\
&\quad \cdot\{E[\textstyle\int_0^T |Y_s^{(n)} - Y_s^{(m)}|^2 ds]\}^{\frac{1}{2}} + E[|\xi_T^{(n)} - \xi_T^{(m)}|^2] \\
&\le E[|\xi_T^{(n)} - \xi_T^{(m)}|^2] + 4\{E[\textstyle\int_0^T |Y_s^{(n)} - Y_s^{(m)}|^2 ds]\}^{\frac{1}{2}} \\
&\quad \cdot\{E[\textstyle\int_0^T 3C^2(|Y_s^{(n)}|^2 + |Z_s^{(n)}|^2 + |Y_{s+\delta(s)}^{(n)}|^2 + |Y_s^{(m)}|^2 + |Z_s^{(m)}|^2 + |Y_{s+\delta(s)}^{(m)}|^2) ds] \\
&\qquad +2\mu\}^{\frac{1}{2}} \\
&\le E[|\xi_T^{(n)} - \xi_T^{(m)}|^2] \\
&\quad +4(2\mu + 6BC^2 + 6LAC^2 + 6LBC^2)^{\frac{1}{2}}\{E[\textstyle\int_0^T |Y_s^{(n)} - Y_s^{(m)}|^2 ds]\}^{\frac{1}{2}}.
\end{aligned}$$

Thus $(Z_\cdot^{(n)})$ is a Cauchy sequence in $L^2_{\mathscr{F}}(0,T;\mathbb{R}^d)$. We denote the limit by $\tilde{Z}_\cdot \in L^2_{\mathscr{F}}(0,T;\mathbb{R}^d)$. Since $f_n \nearrow f,\ n \to \infty$,

$$|f_n(s, \tilde{Y}_s, \tilde{Z}_s, \tilde{Y}_{s+\delta(s)}) - f(s, \tilde{Y}_s, \tilde{Z}_s, \tilde{Y}_{s+\delta(s)})| \searrow 0, \qquad n \to \infty.$$

Hence by Dominated convergence theorem, for any $t \in [0,T]$,

$$E[\int_t^T |f_n(s, \tilde{Y}_s, \tilde{Z}_s, \tilde{Y}_{s+\delta(s)}) - f(s, \tilde{Y}_s, \tilde{Z}_s, \tilde{Y}_{s+\delta(s)})|^2 ds] \to 0, \qquad n \to \infty.$$

Therefore

$$
\begin{aligned}
&E[\int_t^T |f_n(s,Y_s^{(n)},Z_s^{(n)},Y_{s+\delta(s)}^{(n)})-f(s,\tilde{Y}_s,\tilde{Z}_s,\tilde{Y}_{s+\delta(s)})|^2ds]\\
&\leq 2E[\int_t^T |f_n(s,Y_s^{(n)},Z_s^{(n)},Y_{s+\delta(s)}^{(n)})-f_n(s,\tilde{Y}_s,\tilde{Z}_s,\tilde{Y}_{s+\delta(s)})|^2ds]\\
&\ +2E[\int_t^T |f_n(s,\tilde{Y}_s,\tilde{Z}_s,\tilde{Y}_{s+\delta(s)})-f(s,\tilde{Y}_s,\tilde{Z}_s,\tilde{Y}_{s+\delta(s)})|^2ds]\\
&\leq 6C^2E[\int_t^T(|Y_s^{(n)}-\tilde{Y}_s|^2+|Z_s^{(n)}-\tilde{Z}_s|^2+|Y_{s+\delta(s)}^{(n)}-\tilde{Y}_{s+\delta(s)}|^2)ds]\\
&\ +2E[\int_t^T |f_n(s,\tilde{Y}_s,\tilde{Z}_s,\tilde{Y}_{s+\delta(s)})-f(s,\tilde{Y}_s,\tilde{Z}_s,\tilde{Y}_{s+\delta(s)})|^2ds]\to 0,\quad n\to\infty.
\end{aligned}
$$

Taking limits of the following anticipated BSDE

$$
\begin{cases}
Y_t^{(n)}=\xi_T^{(n)}+\int_t^T f_n(s,Y_s^{(n)},Z_s^{(n)},Y_{s+\delta(s)}^{(n)})ds-\int_t^T Z_s^{(n)}dW_s, & t\in[0,T];\\
Y_t^{(n)}=\xi_t^{(n)}, & t\in[T,T+K],
\end{cases}
$$

we obtain

$$
\begin{cases}
\tilde{Y}_t=\xi_T+\int_t^T f(s,\tilde{Y}_s,\tilde{Z}_s,\tilde{Y}_{s+\delta(s)})ds-\int_t^T \tilde{Z}_sdW_s, & t\in[0,T];\\
\tilde{Y}_t=\xi_t, & t\in[T,T+K].
\end{cases}
$$

That is, $(\tilde{Y}_\cdot,\tilde{Z}_\cdot)\in L^2_{\mathscr{F}}(0,T+K)\times L^2_{\mathscr{F}}(0,T;\mathbb{R}^d)$ is the solution to anticipated BSDE (3.2). By the Burkholder-Davis-Gundy inequality, we have

$$
\begin{aligned}
&E[\sup_{t\in[0,T]}|Y_t^{(n)}-\tilde{Y}_t|^2]\\
&\leq 3E[|\xi_T^{(n)}-\xi_T|^2]+3E[\sup_{t\in[0,T]}|\int_t^T(Z_s^{(n)}-\tilde{Z}_s)dW_s|^2]\\
&\ +3E[\sup_{t\in[0,T]}|\int_t^T(f_n(s,Y_s^{(n)},Z_s^{(n)},Y_{s+\delta(s)}^{(n)})-f(s,\tilde{Y}_s,\tilde{Z}_s,\tilde{Y}_{s+\delta(s)}))ds|^2]\\
&\leq 3E[|\xi_T^{(n)}-\xi_T|^2]+6E[\sup_{t\in[0,T]}|\int_0^t(Z_s^{(n)}-\tilde{Z}_s)dW_s|^2]\\
&\ +3TE[\sup_{t\in[0,T]}\int_t^T|f_n(s,Y_s^{(n)},Z_s^{(n)},Y_{s+\delta(s)}^{(n)})-f(s,\tilde{Y}_s,\tilde{Z}_s,\tilde{Y}_{s+\delta(s)})|^2ds]\\
&\leq 3E[|\xi_T^{(n)}-\xi_T|^2]+24E[\int_0^T|Z_s^{(n)}-\tilde{Z}_s|^2ds]\\
&\ +3TE[\int_0^T|f_n(s,Y_s^{(n)},Z_s^{(n)},Y_{s+\delta(s)}^{(n)})-f(s,\tilde{Y}_s,\tilde{Z}_s,\tilde{Y}_{s+\delta(s)})|^2ds].
\end{aligned}
$$

So

$$
E[\sup_{t\in[0,T]}|Y_t^{(n)}-\tilde{Y}_t|^2]\to 0,\qquad n\to\infty.
$$

Hence $Y_\cdot^{(n)}\to\tilde{Y}_\cdot$ in $S^2_{\mathscr{F}}(0,T)$. Because $S^2_{\mathscr{F}}(0,T)$ is a Banach space, we know $\{\tilde{Y}_t\}_{t\in[0,T]}\in S^2_{\mathscr{F}}(0,T)$. Noting $\xi_\cdot\in S^2_{\mathscr{F}}(T,T+K)$, we obtain $\{\tilde{Y}_t\}_{t\in[0,T+K]}\in S^2_{\mathscr{F}}(0,T+K)$. □

*Remark* 3.4. We can see from the above lemma that $(H1)$ is not a necessary condition for the existence of a solution to an anticipated BSDE because $f$ may not satisfy $(H1)$.

**Lemma 3.5.** *Let $t,s\in[0,T]$ be two fixed times with $t\geq s$. Assume $f:L^2(\mathscr{F}_t)\longrightarrow L^2(\mathscr{F}_s)$ is continuous in $L^2(\mathscr{F}_t)$, and there exists a constant $\tilde{C}<\infty$ such that for* *any* $\eta \in L^2(\mathscr{F}_t)$, $|f(\eta)| \leq \tilde{C}(1+E^{\mathscr{F}_s}[|\eta|])$. *If for any* $\xi \in L^2(\mathscr{F}_t)$, $f(E^{\mathscr{F}_s}[\xi]) = f(\xi)$ *holds, then the sequence of functions*

$$f_n(\eta) = E^{\mathscr{F}_s}[\inf_{\xi \in L^2(\mathscr{F}_t)} \{f(\xi) + nE^{\mathscr{F}_s}[|\eta - \xi|]\}] \tag{3.3}$$

*is well defined for* $n \geq \tilde{C}$ *and also* $f_n$ *satisfies*
***(a)*** *for any* $\eta \in L^2(\mathscr{F}_t)$, $|f_n(\eta)| \leq \tilde{C}(1 + E^{\mathscr{F}_s}[|\eta|])$;
***(b)*** *for any* $\eta \in L^2(\mathscr{F}_t)$, $f_n(\eta) \nearrow$;
***(c)*** *for any* $\eta, \xi \in L^2(\mathscr{F}_t)$, $|f_n(\eta) - f_n(\xi)| \leq nE^{\mathscr{F}_s}[|\eta - \xi|]$;
***(d)*** *for any* $\eta \in L^2(\mathscr{F}_t)$, $f_n(\eta) \to f(\eta)$, *a.e.*

*Proof.* It is obvious that $f_n$ is well defined when $n \in \mathbb{N}, n \geq \tilde{C}$ and that $f_n \leq f$. Since $\mathscr{F}$ contains all subsets of $\Omega$, we conclude every function defined on $\Omega$ and valued in $\mathbb{R}$ is $\mathscr{F}$-measurable, in particularly, $\inf_{\xi \in L^2(\mathscr{F}_t)} \{f(\xi) + nE^{\mathscr{F}_s}[|\eta - \xi|]\}$ is an $\mathscr{F}$-measurable random variable. Thus $f_n(\eta)$ is $\mathscr{F}_s$-measurable. $(b)$ holds from the definition of $f_n$ directly.
$(a)$ For any $\eta \in L^2(\mathscr{F}_t)$, we have $f_n(\eta) \leq f(\eta) \leq \tilde{C}(1 + E^{\mathscr{F}_s}[|\eta|])$ and

$$f_n(\eta) \geq E^{\mathscr{F}_s}[\inf_{\xi \in L^2(\mathscr{F}_t)} \{-\tilde{C} - \tilde{C}E^{\mathscr{F}_s}[|\xi|] + nE^{\mathscr{F}_s}[|\eta - \xi|]\}] \geq -\tilde{C}(1 + E^{\mathscr{F}_s}[|\eta|]).$$

That is, $(a)$ holds.
$(c)$ for any $\eta, \xi \in L^2(\mathscr{F}_t)$, for any $\varepsilon > 0$, there exists a $\xi_\varepsilon \in L^2(\mathscr{F}_t)$ such that

$$\begin{aligned} f_n(\eta) &\geq f(\xi_\varepsilon) + nE^{\mathscr{F}_s}[|\eta - \xi_\varepsilon|] - \varepsilon \\ &= f(\xi_\varepsilon) + nE^{\mathscr{F}_s}[|\xi - \xi_\varepsilon|] + nE^{\mathscr{F}_s}[|\eta - \xi_\varepsilon|] - nE^{\mathscr{F}_s}[|\xi - \xi_\varepsilon|] - \varepsilon \\ &\geq f(\xi_\varepsilon) + nE^{\mathscr{F}_s}[|\xi - \xi_\varepsilon|] - nE^{\mathscr{F}_s}[|\eta - \xi|] - \varepsilon \\ &\geq f_n(\xi) - nE^{\mathscr{F}_s}[|\eta - \xi|] - \varepsilon. \end{aligned}$$

Thus, interchanging the roles of $\eta$ and $\xi$, and noting $\varepsilon > 0$ is an arbitrary constant we obtain $|f_n(\eta) - f_n(\xi)| \leq nE^{\mathscr{F}_s}[|\eta - \xi|]$.
$(d)$ For any $\eta \in L^2(\mathscr{F}_t)$, there exists a $\xi_n \in L^2(\mathscr{F}_t)$ such that for any $n \in \mathbb{N}, n > \tilde{C}$,

$$f(\eta) \geq f_n(\eta) \geq f(\xi_n) + nE^{\mathscr{F}_s}[|\eta - \xi_n|] - \frac{1}{n}.$$

Hence $f(\xi_n) + nE^{\mathscr{F}_s}[|\eta - \xi_n|] \leq f(\eta) + \dfrac{1}{n}$. Since $f$ has linear growth, we have

$$\begin{aligned} &f(\xi_n) + nE^{\mathscr{F}_s}[|\eta - \xi_n|] \\ &\geq -\tilde{C}(1 + E^{\mathscr{F}_s}[|\xi_n|]) + nE^{\mathscr{F}_s}[|\eta - \xi_n|] \\ &\geq -\tilde{C}(1 + E^{\mathscr{F}_s}[|\xi_n|]) + nE^{\mathscr{F}_s}[|\xi_n| - |\eta|] \\ &\geq -\tilde{C} + (n - \tilde{C})E^{\mathscr{F}_s}[|\xi_n|] - nE^{\mathscr{F}_s}[|\eta|]. \end{aligned}$$

So when $n \in \mathbb{N}, n > \tilde{C}$, we derive,

$$E^{\mathscr{F}_s}[|\xi_n|] \leq \frac{1}{n - \tilde{C}} f(\eta) + \frac{n}{n - \tilde{C}} E^{\mathscr{F}_s}[|\eta|] + \frac{n\tilde{C} + 1}{n(n - \tilde{C})}.$$

As for any $n \in \mathbb{N}, n > \tilde{C}$,

$$\begin{aligned}
&E[|\frac{1}{n-\tilde{C}} f(\eta) + \frac{n}{n-\tilde{C}} E^{\mathscr{F}_s}[|\eta|] + \frac{n\tilde{C}+1}{n(n-\tilde{C})}|^2] \\
&\leq 3E[\frac{1}{(n-\tilde{C})^2}|f(\eta)|^2 + \frac{1}{(1-\frac{\tilde{C}}{n})^2}(E^{\mathscr{F}_s}[|\eta|])^2 + (\frac{\tilde{C}+\frac{1}{n}}{n-\tilde{C}})^2] \\
&\leq 3E[|f(\eta)|^2 + (1+\tilde{C})^2 E^{\mathscr{F}_s}[|\eta|^2] + (1+\tilde{C})^2] \\
&\leq 3E[|f(\eta)|^2 + (1+\tilde{C})^2|\eta|^2 + (1+\tilde{C})^2] < \infty,
\end{aligned}$$

we know $\{E^{\mathscr{F}_s}[|\xi_n|];\ n \in \mathbb{N}, n > \tilde{C}\}$ is bounded in $L^2(\mathscr{F}_s)$, hence also in $L^2(\mathscr{F}_t)$. Because $f$ has linear growth we obtain $\{f(\xi_n);\ n \in \mathbb{N}, n > \tilde{C}\}$ is bounded in $L^2(\mathscr{F}_s)$. Therefore

$$\overline{\lim_{n\to\infty}} E[(nE^{\mathscr{F}_s}[|\eta - \xi_n|])^2] \leq \overline{\lim_{n\to\infty}} E[(f(\eta) - f(\xi_n) + \frac{1}{n})^2] < \infty.$$

Thus $\lim_{n\to\infty} E[(E^{\mathscr{F}_s}[|\eta - \xi_n|])^2] = 0$. So $\lim_{n\to\infty} E[|E^{\mathscr{F}_s}[\eta - \xi_n]|^2] = 0$. That is, $E^{\mathscr{F}_s}[\xi_n] \to E^{\mathscr{F}_s}[\eta]$ in $L^2(\mathscr{F}_t)$. Since $f$ is continuous in $L^2(\mathscr{F}_t)$, we have $f(E^{\mathscr{F}_s}[\xi_n]) \to f(E^{\mathscr{F}_s}[\eta])$ in $L^2(\mathscr{F}_s)$. Note that for any $\zeta \in L^2(\mathscr{F}_t)$, $f(E^{\mathscr{F}_s}[\zeta]) = f(\zeta)$ holds, we deduce $f(\xi_n) \to f(\eta)$ in $L^2(\mathscr{F}_s)$. Therefore, there exists a subsequence $\{\xi_{n_l};\ l \in \mathbb{N}\} \subseteq \{\xi_n;\ n \in \mathbb{N}\}$ such that $\lim_{l\to\infty} f(\xi_{n_l}) = f(\eta)$, a.e. Since for any $n \in \mathbb{N}, n > \tilde{C}$, $f(\eta) \geq f_n(\eta) \geq f(\xi_n) - \frac{1}{n}$ holds, we derive $\lim_{l\to\infty} f_{n_l}(\eta) = f(\eta)$, a.e. On the other hand, since $f_n \nearrow$ and $f_n \leq f$, for any $\zeta \in L^2(\mathscr{F}_t)$, we can define a function $f'(\zeta) = \lim_{n\to\infty} f_n(\zeta)$. Because $\{f_{n_l};\ l \in \mathbb{N}\}$ is a subsequence of $\{f_n;\ n \in \mathbb{N}\}$, we know for above $\zeta$, $\lim_{n\to\infty} f_{n_l}(\zeta) = f'(\zeta)$. Thus $f'(\eta) = f(\eta)$, a.e., i.e., $f_n(\eta) \to f(\eta)$, a.e. □

**Lemma 3.6.** *We make the same assumptions as in Lemma 3.5. Suppose $f$ is increasing in $\eta$. Then for any $n \in \mathbb{N}$, $n \geq \tilde{C}$, $f_n$ defined in Lemma 3.5 are increasing in $\eta$.*

*Proof.* Suppose $\eta$ and $\eta'$ are two arbitrary elements in $L^2(\mathscr{F}_t)$ satisfying $\eta \leq \eta'$. For any $\xi \in L^2(\mathscr{F}_t)$, set $\xi' = \mathbb{I}_{\{\xi\geq\eta\}}(2\eta - \xi) + \mathbb{I}_{\{\xi<\eta\}}\xi$. Then $\xi' \in L^2(\mathscr{F}_t)$, $\xi' - \xi = \mathbb{I}_{\{\xi\geq\eta\}}(2\eta - 2\xi) \leq 0$, $\xi' - \eta = \mathbb{I}_{\{\xi\geq\eta\}}(\eta - \xi) + \mathbb{I}_{\{\xi<\eta\}}(\xi - \eta) \leq 0$, $f(\xi') \leq f(\xi)$ and $E^{\mathscr{F}_s}[|\xi - \eta|] = E^{\mathscr{F}_s}[\eta - \xi']$. So $f(\xi) + nE^{\mathscr{F}_s}[|\eta - \xi|] \geq f(\xi') + nE^{\mathscr{F}_s}[\eta - \xi']$. Thus by equation (3.3) we have

$$\begin{aligned}
f_n(\eta) &= E^{\mathscr{F}_s}[\inf_{\xi\in L^2(\mathscr{F}_t)}\{f(\xi) + nE^{\mathscr{F}_s}[|\eta - \xi|]\}] \\
&= E^{\mathscr{F}_s}[\inf_{\xi\in L^2(\mathscr{F}_t),\xi\leq\eta}\{f(\xi) + nE^{\mathscr{F}_s}[\eta - \xi]\}].
\end{aligned}$$

Similarly $f_n(\eta') = E^{\mathscr{F}_s}[\inf_{\xi\in L^2(\mathscr{F}_t),\xi\leq\eta'}\{f(\xi) + nE^{\mathscr{F}_s}[\eta' - \xi]\}]$. For any $\xi \in L^2(\mathscr{F}_t)$ satisfying $\xi \leq \eta'$, set $\zeta = \mathbb{I}_{\{\eta\leq\xi\leq\eta'\}}(\eta + \xi - \eta') + \mathbb{I}_{\{\xi<\eta\}}\xi$. Then we obtain $\zeta \in L^2(\mathscr{F}_t)$, $\zeta - \xi = \mathbb{I}_{\{\eta\leq\xi\leq\eta'\}}(\eta - \eta') \leq 0$, $\zeta - \eta = \mathbb{I}_{\{\eta\leq\xi\leq\eta'\}}(\xi - \eta') + \mathbb{I}_{\{\xi<\eta\}}(\xi - \eta) \leq 0$, $f(\zeta) \leq f(\xi)$, and $E^{\mathscr{F}_s}[\eta - \zeta] = E^{\mathscr{F}_s}[\mathbb{I}_{\{\eta\leq\xi\leq\eta'\}}(\eta' - \xi) + \mathbb{I}_{\{\xi<\eta\}}(\eta - \xi)] \leq E^{\mathscr{F}_s}[\eta' - \xi]$. Therefore

$$f(\xi) + nE^{\mathscr{F}_s}[\eta' - \xi] \geq f(\zeta) + nE^{\mathscr{F}_s}[\eta - \zeta].$$

Hence

$$\begin{aligned} f_n(\eta') &= E^{\mathscr{F}_s}[\inf_{\xi\in L^2(\mathscr{F}_t),\xi\leq\eta'}\{f(\xi)+nE^{\mathscr{F}_s}[\eta'-\xi]\}] \\ &\geq E^{\mathscr{F}_s}[\inf_{\xi\in L^2(\mathscr{F}_t),\xi\leq\eta',\zeta=\mathbb{I}_{\{\eta\leq\xi\leq\eta'\}}(\eta+\xi-\eta')+\mathbb{I}_{\{\xi<\eta\}}\xi}\{f(\zeta)+nE^{\mathscr{F}_s}[\eta-\zeta]\}] \\ &\geq E^{\mathscr{F}_s}[\inf_{\zeta\in L^2(\mathscr{F}_t),\zeta\leq\eta}\{f(\zeta)+nE^{\mathscr{F}_s}[\eta-\zeta]\}] = f_n(\eta). \end{aligned}$$

□

*Proof of Theorem 3.2.* Denote, for any fixed $t\in[0,T]$, $\eta\in L^2(\mathscr{F}_r)$, $r\in[t,T+K]$, the sequence associated with $f(t,\cdot,\cdot,\eta)$ in Lemma 2.8 by $\{g_n(t,\cdot,\cdot,\eta);\ n\in\mathbb{N},\ n\geq\hat{C}\}$, where $\hat{C}$ is given in $(H5)$, that is, for any $y,z\in\mathbb{Q}^{1+d}$,

$$g_n(t,y,z,\eta)=\inf_{u,v\in\mathbb{Q}^{1+d}}\{f(t,u,v,\eta)+n|y-u|+n|z-v|\}.$$

Also, denote for any $n\in\mathbb{N}$, $n\geq\hat{C}$, for fixed $t\in[0,T]$, $y\in\mathbb{R}$, $z\in\mathbb{R}^d$, the sequence associated with $g_n(t,y,z,\cdot)$ in Lemma 3.5 by $\{g_{nm}(t,y,z,\cdot);\ m\in\mathbb{N},\ m\geq\hat{C}\}$, that is, for any $\eta\in L^2(\mathscr{F}_r)$, $r\in[t,T+K]$,

$$g_{nm}(t,y,z,\eta)=E^{\mathscr{F}_t}[\inf_{\xi\in L^2(\mathscr{F}_r)}\{g_n(t,y,z,\xi)+mE^{\mathscr{F}_t}[|\eta-\xi|]\}].$$

For any $n\in\mathbb{N}$, $n\geq\hat{C}$, define $f_n(t,y,z,\eta)=g_{nn}(t,y,z,\eta)$, $t\in[0,T]$, $y\in\mathbb{R}$, $z\in\mathbb{R}^d$, $\eta\in L^2(\mathscr{F}_r)$, $r\in[t,T+K]$. Then by Lemma 2.8 and Lemma 3.5 for $n\in\mathbb{N}$, $n\geq\hat{C}$, $f_n(t,y,z,\eta)$ is $\mathscr{F}_t$-measurable and it satisfies:

(1) for any $t\in[0,T]$, $y\in\mathbb{R}$, $z\in\mathbb{R}^d$, $\eta\in L^2(\mathscr{F}_r)$, $r\in[t,T+K]$,

$$|f_n(t,y,z,\eta)|\leq\hat{C}(1+|y|+|z|+E^{\mathscr{F}_t}[|\eta|]);$$

(2) for any $t\in[0,T]$, $y\in\mathbb{R}$, $z\in\mathbb{R}^d$, $\eta\in L^2(\mathscr{F}_r)$, $r\in[t,T+K]$, $f_n(t,y,z,\eta)\nearrow$;

(3) for any $t\in[0,T]$, $y,y'\in\mathbb{R}$, $z,z'\in\mathbb{R}^d$, $\eta,\eta'\in L^2(\mathscr{F}_r)$, $r\in[t,T+K]$,

$$|f_n(t,y,z,\eta)-f_n(t,y',z',\eta')|\leq n(|y-y'|+|z-z'|+E^{\mathscr{F}_t}[|\eta-\eta'|]);$$

(4) for any $t\in[0,T]$, $y\in\mathbb{R}$, $z\in\mathbb{R}^d$, $\eta\in L^2(\mathscr{F}_r)$, $r\in[t,T+K]$, $f_n(t,y,z,\eta)\to f(t,y,z,\eta)$, a.e.

We prove the above four statements first. In fact, it is obvious that $f_n$ is well defined when $n\in\mathbb{N}$, $n\geq\hat{C}$ and that $f_n\leq g_n\leq f$.

Proof of (2): For any $n,m\in\mathbb{N}$, $n\geq m\geq\hat{C}$, we have $f_n=g_{nn}\geq g_{nm}$ by Lemma 3.5 and $g_n\geq g_m$ by Lemma 2.8, hence $g_{nm}\geq g_{mm}=f_m$. Then $f_n\geq f_m$.

Proof of (1): For any $t\in[0,T]$, $y\in\mathbb{R}$, $z\in\mathbb{R}^d$, $\eta\in L^2(\mathscr{F}_r)$, $r\in[t,T+K]$, we have

$$f_n(t,y,z,\eta)\leq g_n(t,y,z,\eta)\leq f(t,y,z,\eta)\leq\hat{C}(1+|y|+|z|+E^{\mathscr{F}_t}[|\eta|]).$$

On the other hand, for any $\xi\in L^2(\mathscr{F}_r)$, $r\in[t,T+K]$,

$$\begin{aligned} &g_n(t,y,z,\xi)=\inf_{u,v\in\mathbb{Q}^{1+d}}\{f(t,u,v,\xi)+n|y-u|+n|z-v|\} \\ &\geq\inf_{u,v\in\mathbb{Q}^{1+d}}\{-\hat{C}(1+|u|+|v|+E^{\mathscr{F}_t}[|\xi|])+\hat{C}|y-u|+\hat{C}|z-v|\} \\ &\geq-\hat{C}(1+|y|+|z|+E^{\mathscr{F}_t}[|\xi|]). \end{aligned}$$

Similarly, we obtain

$$\begin{aligned}f_n(t,y,z,\eta) &= E^{\mathscr{F}_t}[\inf_{\xi\in L^2(\mathscr{F}_r)}\{g_n(t,y,z,\xi)+nE^{\mathscr{F}_t}[|\eta-\xi|]\}]\\ &\geq E^{\mathscr{F}_t}[\inf_{\xi\in L^2(\mathscr{F}_r)}\{-\hat{C}(1+|y|+|z|+E^{\mathscr{F}_t}[|\xi|])+\hat{C}E^{\mathscr{F}_t}[|\eta-\xi|]\}]\\ &\geq -\hat{C}(1+|y|+|z|+E^{\mathscr{F}_t}[|\eta|]).\end{aligned}$$

Proof of (3): for any $t\in[0,T],\ y,y'\in\mathbb{R},\ z,z'\in\mathbb{R}^d,\ \eta,\eta'\in L^2(\mathscr{F}_r)$, $r\in[t,T+K]$,

$$\begin{aligned}&|f_n(t,y,z,\eta)-f_n(t,y',z',\eta')| = |g_{nn}(t,y,z,\eta)-g_{nn}(t,y',z',\eta')|\\ &\leq |g_{nn}(t,y,z,\eta)-g_{nn}(t,y,z,\eta')|+|g_{nn}(t,y,z,\eta')-g_{nn}(t,y',z',\eta')|.\end{aligned}$$

By Lemma 3.5 $(c)$ we derive

$$|f_n(t,y,z,\eta)-f_n(t,y',z',\eta')|\leq nE^{\mathscr{F}_t}[|\eta-\eta'|]+|g_{nn}(t,y,z,\eta')-g_{nn}(t,y',z',\eta')|.$$

For any $y'\in\mathbb{Q}$, $z'\in\mathbb{Q}^d$, for any $\varepsilon>0$, there exists $\xi_\varepsilon\in L^2(\mathscr{F}_r)$ such that

$$g_{nn}(t,y',z',\eta')\geq g_n(t,y',z',\xi_\varepsilon)+nE^{\mathscr{F}_t}[|\eta'-\xi_\varepsilon|]-\varepsilon.$$

So

$$\begin{aligned}&g_{nn}(t,y,z,\eta')-g_{nn}(t,y',z',\eta')\\ &\leq g_n(t,y,z,\xi_\varepsilon)+nE^{\mathscr{F}_t}[|\eta'-\xi_\varepsilon|]-g_n(t,y',z',\xi_\varepsilon)-nE^{\mathscr{F}_t}[|\eta'-\xi_\varepsilon|]+\varepsilon\\ &\leq n(|y-y'|+|z-z'|)+\varepsilon.\end{aligned}$$

Noting $\varepsilon$ is arbitrary, $g_{nn}(t,y,z,\eta')-g_{nn}(t,y',z',\eta')\leq n(|y-y'|+|z-z'|)$ holds. Similarly we know $g_{nn}(t,y',z',\eta')-g_{nn}(t,y,z,\eta')\leq n(|y-y'|+|z-z'|)$, hence $|g_{nn}(t,y,z,\eta')-g_{nn}(t,y',z',\eta')|\leq n(|y-y'|+|z-z'|)$. Therefore,

$$|f_n(t,y,z,\eta)-f_n(t,y',z',\eta')|\leq n(|y-y'|+|z-z'|+E^{\mathscr{F}_t}[|\eta-\eta'|]).$$

Proof of (4): For any $n\in\mathbb{N}$, $n\geq\hat{C}$, $t\in[0,T]$, $y\in\mathbb{R}$, $z\in\mathbb{R}^d$, $\eta\in L^2(\mathscr{F}_r)$, $r\in[t,T+K]$, denote the set $\{\omega\in\Omega;\lim_{m\to\infty}g_{nm}(t,y,z,\eta)=g_n(t,y,z,\eta)\}$ by $A_n$. Then by Lemma 3.5 $(d)$ we know $P(A_n)=1$ and $P(A_n^c)=0$. By Lemma 2.8 $(d)$ we have for any $\omega\in\Omega$, $\lim_{n\to\infty}g_n(t,y,z,\eta)(\omega)=f(t,y,z,\eta)(\omega)$. Denote $A:=\bigcap_{n\in\mathbb{N},n\geq\hat{C}}A_n$. So

$$P(A)=1-P(A^c)=1-P(\bigcup_{n\in\mathbb{N},n\geq\hat{C}}A_n^c)\geq 1-\sum_{n=\hat{C}}^{\infty}P(A^c)=1.$$

Thus, for any $\omega\in A$, for any $\varepsilon>0$, there exists an $N\in\mathbb{N}$ such that for any $n>N\vee\hat{C}$, the following inequality holds:

$$0<f(t,y,z,\eta)(\omega)-g_n(t,y,z,\eta)(\omega)<\frac{\varepsilon}{2}.$$

For above $\omega$ and $\varepsilon$, for any $n\in\mathbb{N},n\geq\hat{C}$, there exists an $M\in\mathbb{N}$ such that for any $m>M\vee\hat{C}$, the following inequality holds:

$$0<g_n(t,y,z,\eta)(\omega)-g_{nm}(t,y,z,\eta)(\omega)<\frac{\varepsilon}{2}.$$

Then for any $n>N\vee M\vee\hat{C}$, we derive

$$0<f(t,y,z,\eta)(\omega)-g_{nn}(t,y,z,\eta)(\omega)<\varepsilon.$$

Hence $P(\omega \in \Omega; \lim_{n\to\infty} g_{nn}(t,y,z,\eta)(\omega) = f(t,y,z,\eta)(\omega)) = P(A) = 1$, i.e., $g_{nn}(t,y,z,\eta) \to f(t,y,z,\eta))$, a.e.

Let us return again to the proof of Theorem 3.2. By Lemma 3.6 for any $n \in \mathbb{N}$, $n \geq \hat{C}$, $t \in [0,T]$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$, $f_n(t,y,z,\cdot)$ is increasing. Thus for any $n \in \mathbb{N}$, $n \geq \hat{C}$, $f_n$ satisfies $(H1)$ and $(H2)'$. Hence for any $n \in \mathbb{N}$, $n \geq \hat{C}$, we deduce that the BSDE

$$\begin{cases} Y_t^{(n)} = \xi_T + \int_t^T f_n(s, Y_s^{(n)}, Z_s^{(n)}, Y_{s+\delta(s)}^{(n)})ds - \int_t^T Z_s^{(n)} dW_s, & t \in [0,T]; \\ Y_t^{(n)} = \xi_t, & t \in [T+K] \end{cases}$$

has a unique adapted solution $(Y_\cdot^{(n)}, Z_\cdot^{(n)})$ in $S^2_{\mathscr{F}}(0,T+K) \times L^2_{\mathscr{F}}(0,T;\mathbb{R}^d)$. By Lemma 3.3, equation (3.1) has a solution $(Y_\cdot, Z_\cdot) \in S^2_{\mathscr{F}}(0,T+K) \times L^2_{\mathscr{F}}(0,T;\mathbb{R}^d)$ and

$$Y_t = \sup_{n\in\mathbb{N}, n\geq\hat{C}} Y_t^{(n)}, \qquad \text{a.e., a.s.}$$

We now prove the existence of a minimal solution. Suppose $(Y'_\cdot, Z'_\cdot)$ is an another solution of equation (3.1). For any $n \geq \hat{C}$, $n \in \mathbb{N}$, we shall compare $Y'_\cdot$ and $Y_\cdot^{(n)}$: Set

$$\begin{cases} \bar{Y}_t^{(1)} = \xi_T + \int_t^T f_n(s, \bar{Y}_s^{(1)}, \bar{Z}_s^{(1)}, Y'_{s+\delta(s)})ds - \int_t^T \bar{Z}_s^{(1)} dW_s, & t \in [0,T]; \\ \bar{Y}_t^{(1)} = \xi_t, & t \in [T, T+K]. \end{cases}$$

By Lemma 2.5, we deduce there exists a unique pair of $\mathscr{F}_t$-adapted processes $(\bar{Y}_\cdot^{(1)}, \bar{Z}_\cdot^{(1)}) \in S^2_{\mathscr{F}}(0,T) \times L^2_{\mathscr{F}}(0,T;\mathbb{R}^d)$ satisfying the above BSDE. Because for any $s \in [0,T]$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$, $f(s,y,z,Y'_{s+\delta(s)}) \geq f_n(s,y,z,Y'_{s+\delta(s)})$, by Lemma 2.10 we obtain $Y'_t \geq \bar{Y}_t^{(1)}$, a.e., a.s. Set

$$\begin{cases} \bar{Y}_t^{(2)} = \xi_T + \int_t^T f_n(s, \bar{Y}_s^{(2)}, \bar{Z}_s^{(2)}, \bar{Y}_{s+\delta(s)}^{(1)})ds - \int_t^T \bar{Z}_s^{(2)} dW_s, & t \in [0,T]; \\ Y_t^{(4)} = \xi_t, & t \in [T, T+K]. \end{cases}$$

Since for any $t \in [0,T]$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$, $f_n(t,y,z,\cdot)$ is increasing and $Y'_t \geq \bar{Y}_t^{(1)}$, a.e., a.s., by Lemma 2.7 we know $\bar{Y}_t^{(1)} \geq \bar{Y}_t^{(2)}$, a.e., a.s. For $m = 3, 4, \cdots$, we consider the following classical BSDE:

$$\begin{cases} \bar{Y}_t^{(m)} = \xi_T + \int_t^T f_n(s, \bar{Y}_s^{(m)}, \bar{Z}_s^{(m)}, \bar{Y}_{s+\delta(s)}^{(m-1)})ds - \int_t^T \bar{Z}_s^{(m)} dW_s, & t \in [0,T]; \\ \bar{Y}_t^{(m)} = \xi_t, & t \in [T, T+K]. \end{cases}$$

Similarly we have $\bar{Y}_t^{(2)} \geq \bar{Y}_t^{(3)} \geq \cdots \geq \bar{Y}_t^{(m)} \geq \cdots$, a.e., a.s. Set $\beta = 18\hat{C}^2 L + 18\hat{C}^2 + 3$, where $\hat{C}$ and $L$ are two constants given in $(H5)$ and $(ii)$, respectively. For any $p > 0$, $l \in \mathbb{N}$, we introduce a norm in the Banach space $L^2_{\mathscr{F}}(0,p;\mathbb{R}^l)$ :

$$\| \nu(\cdot) \|_\beta = (E[\int_0^p |\nu_s|^2 e^{\beta s} ds])^{\frac{1}{2}}.$$

Clearly it is equivalent to the original norm of $L^2_{\mathscr{F}}(0,p;\mathbb{R}^l)$. For $m \geq 2$, by the basic estimate (2.3) we have

$$\begin{aligned}
&E[\int_0^T (\frac{\beta}{2}|\bar{Y}_s^{(m)} - \bar{Y}_s^{(m-1)}|^2 + |\bar{Z}_s^{(m)} - \bar{Z}_s^{(m-1)}|^2)e^{\beta s}ds] \\
&\leq \frac{2}{\beta}E[\int_0^T |f_n(s,\bar{Y}_s^{(m)},\bar{Z}_s^{(m)},\bar{Y}_{s+\delta(s)}^{(m-1)}) - f_n(s,\bar{Y}_s^{(m-1)},\bar{Z}_s^{(m-1)},\bar{Y}_{s+\delta(s)}^{(m-2)})|^2 e^{\beta s}ds] \\
&\leq \frac{6\hat{C}^2}{\beta}E[\int_0^T (|\bar{Y}_s^{(m)} - \bar{Y}_s^{(m-1)}|^2 + |\bar{Z}_s^{(m)} - \bar{Z}_s^{(m-1)}|^2 + |\bar{Y}_{s+\delta(s)}^{(m-1)} - \bar{Y}_{s+\delta(s)}^{(m-2)}|^2) \\
&\qquad \cdot e^{\beta s}ds] \\
&\leq \frac{6\hat{C}^2}{\beta}E[\int_0^T (|\bar{Y}_s^{(m)} - \bar{Y}_s^{(m-1)}|^2 + |\bar{Z}_s^{(m)} - \bar{Z}_s^{(m-1)}|^2)e^{\beta s}ds] \\
&\quad + \frac{6\hat{C}^2 L}{\beta}E[\int_0^{T+K} |\bar{Y}_s^{(m-1)} - \bar{Y}_s^{(m-2)}|^2 e^{\beta s}ds] \\
&= \frac{6\hat{C}^2}{\beta}E[\int_0^T (|\bar{Y}_s^{(m)} - \bar{Y}_s^{(m-1)}|^2 + |\bar{Z}_s^{(m)} - \bar{Z}_s^{(m-1)}|^2)e^{\beta s}ds] \\
&\quad + \frac{6\hat{C}^2 L}{\beta}E[\int_0^T |\bar{Y}_s^{(m-1)} - \bar{Y}_s^{(m-2)}|^2 e^{\beta s}ds].
\end{aligned}$$

Noting $\beta = 18\hat{C}^2 L + 18\hat{C}^2 + 3$, we deduce

$$\begin{aligned}
&\frac{2}{3}E[\int_0^T (|\bar{Y}_s^{(m)} - \bar{Y}_s^{(m-1)}|^2 + |\bar{Z}_s^{(m)} - \bar{Z}_s^{(m-1)}|^2)e^{\beta s}ds] \\
&\leq E[\int_0^T (\frac{\beta}{2}|\bar{Y}_s^{(m)} - \bar{Y}_s^{(m-1)}|^2 + |\bar{Z}_s^{(m)} - \bar{Z}_s^{(m-1)}|^2)e^{\beta s}ds] \\
&\leq \frac{1}{3}E[\int_0^T |\bar{Y}_s^{(m-1)} - \bar{Y}_s^{(m-2)}|^2 e^{\beta s}ds] \\
&\leq \frac{1}{3}E[\int_0^T (|\bar{Y}_s^{(m-1)} - \bar{Y}_s^{(m-2)}|^2 + |\bar{Z}_s^{(m-1)} - \bar{Z}_s^{(m-2)}|^2)e^{\beta s}ds].
\end{aligned}$$

Hence

$$\begin{aligned}
&E[\int_0^{T+K} |\bar{Y}_s^{(m)} - \bar{Y}_s^{(m-1)}|^2 e^{\beta s}ds + \int_0^T |\bar{Z}_s^{(m)} - \bar{Z}_s^{(m-1)}|^2 e^{\beta s}ds] \\
&= E[\int_0^T (|\bar{Y}_s^{(m)} - \bar{Y}_s^{(m-1)}|^2 + |\bar{Z}_s^{(m)} - \bar{Z}_s^{(m-1)}|^2)e^{\beta s}ds] \\
&\leq (\frac{1}{2})^{m-2}E[\int_0^T (|\bar{Y}_s^{(2)} - \bar{Y}_s^{(1)}|^2 + |\bar{Z}_s^{(2)} - \bar{Z}_s^{(1)}|^2)e^{\beta s}ds] \\
&= (\frac{1}{2})^{m-2}E[\int_0^{T+K} |\bar{Y}_s^{(2)} - \bar{Y}_s^{(1)}|^2 e^{\beta s}ds + \int_0^T |\bar{Z}_s^{(2)} - \bar{Z}_s^{(1)}|^2)e^{\beta s}ds].
\end{aligned}$$

It follows that $(\bar{Y}_\cdot^{(m)})_{m\in\mathbb{N}}$ and $(\bar{Z}_\cdot^{(m)})_{m\in\mathbb{N}}$ are respectively Cauchy sequences in $L^2_{\mathscr{F}}(0,\ T+K)$ and in $L^2_{\mathscr{F}}(0,T;\mathbb{R}^d)$. Denote their limits by $\bar{Y}_\cdot$ and $\bar{Z}_\cdot$, respectively. Because $L^2_{\mathscr{F}}(0,T+K)$ and $L^2_{\mathscr{F}}(0,T;\mathbb{R}^d)$ are both Banach spaces, we obtain $(\bar{Y}_\cdot,\bar{Z}_\cdot) \in L^2_{\mathscr{F}}(0,T+K) \times L^2_{\mathscr{F}}(0,T;\mathbb{R}^d)$. Note for any $t \in [0,T]$,

$$\begin{aligned}
&E[\int_t^T |f_n(s,\bar{Y}_s^{(m)},\bar{Z}_s^{(m)},\bar{Y}_{s+\delta(s)}^{(m-1)}) - f_n(s,\bar{Y}_s,\bar{Z}_s,\bar{Y}_{s+\delta(s)})|^2 e^{\beta s}ds] \\
&\leq 3C^2 E[\int_t^T (|\bar{Y}_s^{(m)} - \bar{Y}_s|^2 + |\bar{Z}_s^{(m)} - \bar{Z}_s|^2 + L|\bar{Y}_s^{(m-1)} - \bar{Y}_s|^2)e^{\beta s}ds] \to 0,
\end{aligned}$$

if $n \to \infty$. Therefore $(\bar{Y}_\cdot, \bar{Z}_\cdot)$ satisfies the following anticipated BSDE

$$\begin{cases} Y_t = \xi_T + \int_t^T f_n(s, Y_s, Z_s, Y_{s+\delta(s)})ds - \int_t^T Z_s dW_s, & 0 \le t \le T; \\ Y_t = \xi_t, & T \le t \le T+K. \end{cases}$$

By Lemma 2.5 we know the above equation has a unique solution. Noting $(Y_\cdot^{(n)}, Z_\cdot^{(n)})$ also satisfies this equation, we conclude $\bar{Y}_t = Y_t^{(n)}$, a.e., a.s. Since $Y'_t \ge \bar{Y}_t^{(1)} \ge \bar{Y}_t^{(2)} \ge \cdots \ge \bar{Y}_t$, we derive for any $n \in \mathbb{N}, n \ge \hat{C}$, $Y'_t \ge Y_t^{(n)}$, a.e., a.s. Because $Y_t = \sup_{n\in\mathbb{N}, n\ge\hat{C}} Y_t^{(n)}$, a.e., a.s., we have $Y'_t \ge Y_t$, a.e., a.s. That is, $Y_t = \sup_{n\in\mathbb{N}, n\ge\hat{C}} Y_t^{(n)}$ is just the minimal solution of the anticipated BSDE (3.1). □

## 4. Comparison Theorem for the Minimal Solutions of Anticipated BSDEs With Continuous Coefficients

**Theorem 4.1.** *[Comparison Theorem] Let $\hat{Y}_\cdot^{(1)}$ and $\hat{Y}_\cdot^{(2)}$ be respectively the minimal solutions of the following two anticipated BSDEs:*

$$\begin{cases} Y_t^{(j)} = \xi_T^{(j)} + \int_t^T f^{(j)}(s, Y_s^{(j)}, Z_s^{(j)}, Y_{s+\delta(s)}^{(j)})ds - \int_t^T Z_s^{(j)} dW_s, & t \in [0,T]; \\ Y_t^{(j)} = \xi_t^{(j)}, & t \in [T, T+K], \end{cases}$$

*where $j = 1, 2$. Suppose $\xi_\cdot^{(1)}, \xi_\cdot^{(2)} \in S^2_{\mathscr{F}}(T, T+K)$ with $\xi_T^{(1)}, \xi_T^{(2)} \in L^2(\mathscr{F}_T)$, $f^{(1)}, f^{(2)}$ satisfies $(H5), (H6)$ and $\delta$ satisfy $(i), (ii)$. If $\xi_s^{(1)} \ge \xi_s^{(2)}$, $s \in [T, T+K]$, and $f^{(1)}(t, y, z, \theta_r) \ge f^{(2)}(t, y, z, \theta_r)$, $t \in [0,T]$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$, $\theta_\cdot \in L^2_{\mathscr{F}}(t, T+K)$, $r \in [t, T+K]$, then*

$$\hat{Y}_t^{(1)} \ge \hat{Y}_t^{(2)}, \quad a.e., \ a.s.$$

*Proof.* Denote, for fixed $t$, the sequence associated with $f^{(1)}$ and $f^{(2)}$ in the proof of Theorem 3.2 by $\{f_n^{(1)}, \ n \ge \tilde{C}\}$ and $\{f_n^{(2)}, \ n \ge \tilde{C}\}$, respectively, where $\tilde{C} = \hat{C}^{(1)} \vee \hat{C}^{(2)}$ with $\hat{C}^{(1)}, \hat{C}^{(2)}$ given in $(H.5)$. Then by Lemma 3.6 for any $n \ge \tilde{C}$, $t \in [0,T]$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$, $f_n^{(1)}(t, y, z, \cdot)$ and $f_n^{(2)}(t, y, z, \cdot)$ are both increasing. Thus for any $n \ge \tilde{C}$, $f_n^{(1)}$ and $f_n^{(2)}$ satisfy $(H1)$ and $(H2)'$. Since $f^{(1)}(t, y, z, \theta_r) \ge f^{(2)}(t, y, z, \theta_r)$, $t \in [0,T]$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$, $\theta_\cdot \in L^2_{\mathscr{F}}(t, T+K)$, $r \in [t, T+K]$, also from the proof of Theorem 3.2, we derive $f_n^{(1)}(t, y, z, \theta_r) \ge f_n^{(2)}(t, y, z, \theta_r)$, $t \in [0,T]$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$, $\theta_\cdot \in L^2_{\mathscr{F}}(t, T+K)$, $r \in [t, T+K]$. Hence for $n \in \mathbb{N}, n \ge \hat{C}$, we deduce that each of the following BSDEs has a unique adapted solution $(Y_t^{(n,i)}, Z_t^{(n,i)})$ in $S^2_{\mathscr{F}}(0, T+K) \times L^2_{\mathscr{F}}(0,T)$ :

$$\begin{cases} Y_t^{(n,i)} = \xi_T^{(i)} + \int_t^T f_n^{(i)}(s, Y_s^{(n,i)}, Z_s^{(n,i)}, Y_{s+\delta(s)}^{(n,i)})ds \\ \qquad\qquad - \int_t^T Z_s^{(n,i)} dW_s, & t \in [0,T]; \\ Y_t^{(n,i)} = \xi_t^{(i)}, & t \in [T+K], \end{cases}$$

where $i = 1, 2$. Then by lemma 2.3, we obtain for $n \in \mathbb{N}, n \ge \hat{C}$, $Y_t^{(n,2)} \le Y_t^{(n,1)}$, a.e., a.s. Again from the proof of Theorem 3.2, we have

$$\hat{Y}_t^{(2)} = \sup_{n\in\mathbb{N}, n\geq\hat{C}} Y_t^{(n,2)} \leq \sup_{n\in\mathbb{N}, n\geq\hat{C}} Y_t^{(n,1)} = \hat{Y}_t^{(1)}, \quad \text{a.e., a.s.}$$

□

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