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ACTION FUNCTIONALS FOR STOCHASTIC DIFFERENTIAL EQUATIONS WITH LÉVY NOISE

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ABSTRACT. This article is about stochastic dynamical systems with small non-Gaussian Lévy noise. We review the recent works on the large deviation techniques that deal with the decay of probabilities of rare events on an exponential scale. We focus on deriving the action functionals for dynamical systems with Lévy processes of finite exponential moments. This is achieved with help of the extended contraction principle, Legendre transform and Lévy symbols. We also illustrate the results with an example.

1. Introduction

Stochastic effects are ubiquitous in complex systems from science and engineering [1]. Although random mechanisms may appear to be very small or very fast, their long time impacts on the system evolution may be delicate or even profound [13]. Mathematical modeling of complex systems under uncertainty often leads to stochastic differential equations (SDEs), as seen in, for example, [2, 14, 18, 19]. Fluctuations appeared in these SDEs are often non-Gaussian rather than Gaussian.

The long time large deviation behaviors of slow-fast systems have attracted a lot of attention because of the various applications in statistical physics, biophysics, geophysics, climate dynamics engineering, chemistry and financial mathematics [3, 8, 11]. Large deviations for SDEs driven by Brownian motion are now well-known [5, 10, 17], while certain large deviation results for SDEs with Lévy noise are available more recently [4, 12].

Action functionals play an important role in understanding transitions in the context of large deviations [9, 15, 16]. The main goal of this review article is to derive the action functionals for the following SDE with a Lévy process

$$dX_t^\varepsilon = b(X_{t-}^\varepsilon)dt + \sqrt{\varepsilon}\sigma(X_{t-}^\varepsilon)dB_t + \eta(X_{t-}^\varepsilon)dL_t^\varepsilon,$$

where $L_t^\varepsilon := \varepsilon L_{\frac{t}{\varepsilon}}$ is a scaled Lévy process with finite exponential moments.

We first show that the scaled Lévy process satisfies a large deviation principle, and obtain its action functional. Then we construct continuous mappings to get an exponentially good approximations. Finally, we derive the action functionals for SDEs with Lévy noise by using extended contraction principle, Legendre transform

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and Lévy symbols. For simplicity, we restrict on one-dimensional processes and stochastic dynamical systems. Most of the results can be proved in a similar fashion for multi-dimensional processes and systems.

This article is arranged as follows. In Section 2, we recall some basic concepts, and introduce extensions of the contraction principle. In Section 3, we focus on the action functionals for scaled Brownian motion, and obtain the action functionals for SDEs with Brownian motion (Lemma 3.2 and Theorem 3.3). In Section 4, we derive the action functionals for scaled Lévy processes, and the action functionals for SDEs with Lévy noise (Lemma 4.3, Theorem 4.5 and Corollary 4.7). This article ends with a simple example in Section 5.

2. Prelimilaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We consider Euclidean space $\mathbb{R}$ endowed with the Borel σ-algebra $\mathcal{B}(\mathbb{R})$. Let $\lfloor x \rfloor$ denotes the integer part of $x \in \mathbb{R}$. As usual, $\mathcal{C}[0,1]$ denotes the space of all continuous functions $f : [0,1] \to \mathbb{R}$ such that $f(0) = 0$, equipped with the uniform norm

$$\|f\|_\infty := \sup_{t\in[0,1]} |f(t)|.$$

We denote by $\mathcal{D}[0,1]$ the space of real-valued cádlág (right continuous with finite left limits) functions on $[0,1]$ endowed with the supremum norm topology, and the σ-algebra $\mathcal{B} := \sigma(\pi_t; t \in [0,1])$ generated by the projections $\pi_t : \mathcal{D}[0,1] \to \mathbb{R}$, $f \mapsto f(t), t \in [0,1]$. Note that $\mathcal{B}$ equals the Borel σ-algebra generated by the J_1-metric. The notation $BV[0,1]$ denotes the space of functions with bounded variation. Let $AC[0,1]$ denotes the space of all absolutely continuous functions with value 0 at 0.

Now we introduce the contraction principle and investigate its extensions. They will be a crucial tool for studying action functionals of SDEs with Lévy noise. The following theorem is devoted to transformations that preserves the large deviation principle under continuous mappings.

Theorem 2.1. *(*[5, Theorem 4.2.1]*) Let (M_1, d_1), (M_2, d_2) be metric spaces and $f : M_1 \to M_2$ be a continuous function. Suppose that a family $(\mu^\varepsilon)_{\varepsilon>0}$ of probability measures on M_1 satisfies a large deviation principle with action functional I. Then the sequence of image measures $(\nu^\varepsilon)_{\varepsilon>0}$ defined by $\nu^\varepsilon := \mu^\varepsilon \circ f^{-1}$ on M_2, obeys a large deviation principle with action functional*

$$S(y) := \inf\{I(x) : x \in M_1, y = f(x)\}.$$

Proof. Since I is lower semicontinuous, it attains its minimum on compact sets. This implies that for any $y \in M_2$ and $S(y) < \infty$, there exists $x \in M_1$ such that $f(x) = y$ and $S(y) = I(x)$. Then

$$\Phi_S(r) = \{y \in M_2 : S(y) \le r\} = f(\Phi_I(r)) \text{ for } r \ge 0.$$

In particular, $\Phi_S(r)$ is compact, i.e., S is an action functional. Now let U be an open set in M_1. Since f is continuous, we know $f^{-1}(U)$ is open. Apply the large deviation lower bound to $f^{-1}(U)$ and obtain

$$\liminf_{\varepsilon\to 0} \varepsilon \log \nu^\varepsilon(U) = \liminf_{\varepsilon\to 0} \varepsilon \log \mu^\varepsilon(f^{-1}(U)) \ge - \inf_{x\in f^{-1}(U)} I(x) = - \inf_{y\in U} S(y).$$

When F is a closed set in M_1, the upper bound

$$\limsup_{\varepsilon\to 0}\varepsilon\log\nu^\varepsilon(F)=\limsup_{\varepsilon\to 0}\varepsilon\log\mu^\varepsilon(f^{-1}(F))\le-\inf_{x\in f^{-1}(F)}I(x)=-\inf_{y\in F}S(y)$$

follows in the same way. □

Remark 2.2. Once the large deviation principle with an action functional is established for μ^ε, the contraction principle yields the large deviation principle for $\mu^\varepsilon\circ f^{-1}$, where f is any continuous map. Hence the large deviation principle is preserved under continuous mappings.

Definition 2.3. Let $(X^{\varepsilon,m})_{\varepsilon>0,m\in\mathbb{N}}$ and $(X^\varepsilon)_{\varepsilon>0}$ be families of random variables taking values in a metric space (M,d). If

$$\lim_{m\to\infty}\limsup_{\varepsilon\to 0}\varepsilon\log\mathbb{P}(d(X^{\varepsilon,m},X^\varepsilon)>\delta)=-\infty,\quad\text{for all}\quad\delta>0,\tag{2.1}$$

then $(X^{\varepsilon,m})_{\varepsilon>0,m\in\mathbb{N}}$ is called exponentially good approximation of $(X^\varepsilon)_{\varepsilon>0}$.

The following theorem provides a relation for large deviation principles of exponentially good approximations.

Theorem 2.4. *Let $(X^{\varepsilon,m})_{\varepsilon>0,m\in\mathbb{N}}$ be an exponentially good approximation of $(X^\varepsilon)_{\varepsilon>0}$ such that $X^{\varepsilon,m}$ satisfies a large deviation principle with action functional S_m as $\varepsilon\to 0$.*
(i) $(X^\varepsilon)_{\varepsilon>0}$ satisfies a weak large deviation principle with action functional

$$S(x):=\sup_{\delta>0}\liminf_{m\to\infty}\inf_{y\in B(x,\delta)}S_m(y),\tag{2.2}$$

i.e., S is lower semicontinuous, the large deviation lower bound for $(X^{\varepsilon,m})_{\varepsilon>0}$ holds for all open set in M, and the large deviation upper bound for $(X^{\varepsilon,m})_{\varepsilon>0}$ holds for all compact set in M.
(ii) If S is an action functional and

$$\inf_{x\in F}S(x)\le\sup_{\delta>0}\limsup_{m\to\infty}\inf_{x\in F}S_m(x)\tag{2.3}$$

holds for each closed set $F\subseteq M$, then $(X^\varepsilon)_{\varepsilon>0}$ satisfies a large deviation principle with action functional S.

Proof. (i) In order to prove (2.2), it suffices to show that for any $x\in M$,

$$S(x)=-\inf_{\delta>0}\limsup_{\varepsilon\to 0}\varepsilon\log\mathbb{P}(X^\varepsilon\in B[x,\delta])=-\inf_{\delta>0}\liminf_{\varepsilon\to 0}\varepsilon\log\mathbb{P}(X^\varepsilon\in B(x,\delta)).\tag{2.4}$$

Fix $\delta>0$ and $x\in M$. From

$$\mathbb{P}(X^{\varepsilon,m}\in B(x,\delta))\le\mathbb{P}(X^\varepsilon\in B(x,2\delta))+\mathbb{P}(d(X^{\varepsilon,m},X^\varepsilon)>\delta),$$

we find, by the large deviation lower bound for $(X^{\varepsilon,m})_{\varepsilon>0}$,

$$\begin{aligned}-\inf_{y\in B(x,\delta)}S_m(y)&\le\liminf_{\varepsilon\to 0}\varepsilon\log\mathbb{P}(X^{\varepsilon,m}\in B(x,\delta))\\&\le\max\{\liminf_{\varepsilon\to 0}\varepsilon\log\mathbb{P}(X^\varepsilon\in B(x,2\delta)),\limsup_{\varepsilon\to 0}\varepsilon\log\mathbb{P}(d(X^{\varepsilon,m},X^\varepsilon)>\delta)\}.\end{aligned}$$

Since $(X^{\varepsilon,m})_{\varepsilon>0,m\in\mathbb{N}}$ is an exponentially good approximation,

$$\inf_{\delta>0}\liminf_{\varepsilon\to 0}\varepsilon\log\mathbb{P}(X^\varepsilon\in B(x,2\delta))\ge\inf_{\delta>0}\limsup_{m\to\infty}(-\inf_{y\in B(x,\delta)}S_m(y))=-S(x).\tag{2.5}$$

By interchanging the roles of $X^{\varepsilon,m}$ and X^{ε}, we get

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\varepsilon\log\mathbb{P}(X^{\varepsilon}\in B[x,\delta])\\&\qquad\leq\max\{\liminf_{\varepsilon\to 0}\varepsilon\log\mathbb{P}(X^{\varepsilon}\in B[x,2\delta]),\limsup_{\varepsilon\to 0}\varepsilon\log\mathbb{P}(d(X^{\varepsilon,m},X^{\varepsilon})>\delta)\}.\end{aligned}$$

Therefore, by the large deviation upper bound for $(X^{\varepsilon,m})_{\varepsilon>0}$ and (2.1), we obtain

$$\inf_{\delta>0}\limsup_{\varepsilon\to 0}\varepsilon\log\mathbb{P}(X^{\varepsilon}\in B[x,\delta])\leq\inf_{\delta>0}\limsup_{m\to\infty}(-\inf_{y\in B[x,2\delta]}S_m(y))=-S(x).\quad(2.6)$$

Combining (2.5) and (2.6) yields (2.4).

(ii) From the first part of this theorem that $(X^{\varepsilon})_{\varepsilon>0}$ satisfies a weak large deviation principle, it remains to show the large deviation upper bound for any closed set $F\subseteq M$. Fix $\delta>0$, the large deviation upper bound for $(X^{\varepsilon,m})_{\varepsilon>0}$ implies

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\varepsilon\log\mathbb{P}(X^{\varepsilon}\in F)\\&\leq\max\{\limsup_{\varepsilon\to 0}\varepsilon\log\mathbb{P}(X^{\varepsilon,m}\in F+B[0,\delta]),\limsup_{\varepsilon\to 0}\varepsilon\log\mathbb{P}(d(X^{\varepsilon,m},X^{\varepsilon})>\delta)\}\\&\leq\max\{-\inf_{x\in F+B[0,\delta]}S_m(x),\limsup_{\varepsilon\to 0}\varepsilon\log\mathbb{P}(d(X^{\varepsilon,m},X^{\varepsilon})>\delta)\}.\end{aligned}$$

Consequently, by (2.1) and (2.3),

$$\begin{aligned}\limsup_{\varepsilon\to 0}\varepsilon\log\mathbb{P}(X^{\varepsilon}\in F)&\leq-\lim_{\delta\to 0}\limsup_{m\to\infty}\inf_{x\in F+B[0,\delta]}S_m(x)\\&\leq-\lim_{\delta\to 0}\inf_{x\in F+B[0,\delta]}S(x)=-\inf_{x\in F}S(x).\end{aligned}$$

This finishes the proof. □

Remark 2.5. If the action functional S of $(X^{\varepsilon})_{\varepsilon>0}$ satisfies

$$\lim_{\delta\to 0}\liminf_{\varepsilon\to 0}\varepsilon\log\mathbb{P}(X^{\varepsilon}\in B(x,\delta))=\lim_{\delta\to 0}\limsup_{\varepsilon\to 0}\varepsilon\log\mathbb{P}(X^{\varepsilon}\in B[x,\delta])=-S(x),$$

then $(X^{\varepsilon})_{\varepsilon>0}$ obeys a weak large deviation principle.

In order to extend the contraction principle beyond the continuous case, we consider the extension of the contraction principle to maps that are not continuous, but that can be approximated well by continuous maps. Now we present the extended contraction principle.

Theorem 2.6. *Let (M_1,d_1), (M_2,d_2) be metric spaces and $(X^{\varepsilon})_{\varepsilon>0}$ denotes a family of random variables obeying a large deviation principle in (M_1,d_1) with action functional I. For $m\in\mathbb{N}$, let $f_m:M_1\to M_2$ be continuous functions and $f:M_1\to M_2$ measurable such that*

$$\limsup_{m\to\infty}\sup_{\{x:I(x)\leq r\}}d_2(f_m(x),f(x))=0\ \textit{for all}\ r\geq 0.\quad(2.7)$$

Then for any family of random variables $(Y^{\varepsilon})_{\varepsilon>0}$ for which $(f_m(X^{\varepsilon}))_{\varepsilon>0,m\in\mathbb{N}}$ is an exponentially good approximation obeys a large deviation principle with action functional

$$S(y)=\inf\{I(x):y=f(x)\}.$$

Proof. Since the functions f_m, $m \in \mathbb{N}$, are continuous, the contraction principle entails that $(f_m(X^\varepsilon))_{\varepsilon>0}$ satisfies a large deviation principle with action functional

$$S_m(y) := \inf\{I(x) : y = f_m(x)\}.$$

Moreover, by (2.7), f is continuous on any sublevel set $\Phi_I(r) := \{x \in M_1 : I(x) \leq r\}, r \geq 0$. Hence, S is an action functional with sublevel sets $f(\Phi_I(r))$. In view of Theorem 2.4, it suffices to check (2.3) and identify the action functional.

Fix $F \subseteq M_2$ closed and $\delta > 0$, and also suppose

$$c := \liminf_{m\to\infty} \inf_{y\in F} S_m(y) < \infty.$$

Then we can choose a sequence $(x_m)_{m\in\mathbb{N}} \subseteq M_1$ and $r > 0$, such that $f_m(x_m) \in F$ and $I(x_m) = \inf_{y\in F} S_m(y) \leq r$. From (2.6) we have $f(x_m) \in F + B[0, \delta]$ for $m = m(\delta)$ sufficiently large. Thus,

$$\inf_{y\in F+B(0,\delta)} S(y) \leq S(f(x_m)) \leq I(x_m) = \inf_{y\in F} S_m(y).$$

Taking $\delta \to 0$ and $m \to \infty$, we infer

$$\inf_{y\in F} S(y) \leq \liminf_{m\to\infty} \inf_{y\in F} S_m(y) = c.$$

Obviously, this inequality is trivially satisfied if $c = \infty$. In particular, (2.3) holds. In order to identify the action functional, we use the preceding inequality for $F := B[y, \delta]$ and let $\delta \to 0$. □

3. Action Functionals for Stochastic Systems with Brownian Noise

Let B_t, $t \in [0, 1]$ denotes a standard Brownian motion in $\mathbb{R}$. The logarithmic moment generating function of B_1 is

$$\Lambda(\xi) := \log \mathbb{E}e^{\xi B_1} = \frac{1}{2}\xi^2,$$

and the Legendre transform [8] of Λ is

$$\Lambda^*(p) := \sup_{\xi\in\mathbb{R}}\{\xi p - \frac{1}{2}\xi^2\} = \sup_{\xi\in\mathbb{R}}\{-\frac{1}{2}(\xi - p)^2 + \frac{1}{2}p^2\} = \frac{1}{2}p^2.$$

Definition 3.1. Let $\phi \in \mathcal{C}[0, 1]$. The functional $S : \mathcal{C}[0, 1] \to [0, \infty]$,

$$S(\phi) = \begin{cases} \frac{1}{2}\int_0^1 |\phi'(t)|^2 dt, & \phi \in AC[0, 1], \\ \infty, & \text{otherwise}, \end{cases} \tag{3.1}$$

is the action functional of the Brownian motion $(B_t)_{t\in[0,1]}$.

Lemma 3.2. *The scaled Brownian motion* $B_t^\varepsilon := \varepsilon B_{\frac{t}{\varepsilon}}$ *satisfies a large deviation principle in* $(\mathcal{C}[0, 1], \|\cdot\|_\infty)$ *as* $\varepsilon \to 0$ *with action functional in* (3.1)*, i.e.,*

$$\liminf_{\varepsilon\to 0} \varepsilon \log \mathbb{P}(\varepsilon B(\frac{\cdot}{\varepsilon}) \in U) \geq -\inf_{\phi\in U} S(\phi),$$
$$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}(\varepsilon B(\frac{\cdot}{\varepsilon}) \in F) \leq -\inf_{\phi\in F} S(\phi),$$

for any open set $U \subset \mathcal{C}[0, 1]$ *and closed set* $F \subset \mathcal{C}[0, 1]$

Proof. In order to prove that $(B_t^\varepsilon)_{t\in[0,1]}$ satisfies a large deviation principle, by the scaling property

$$\varepsilon B(\frac{t}{\varepsilon}) \overset{d}{=} \sqrt{\varepsilon}B(t), \ \ t \in [0,1],$$

where " $\overset{d}{=}$ " denotes equivalence (coincidence) in distribution, we may replace B_t^ε by $\sqrt{\varepsilon}B_t$, i.e.,

$$\liminf_{\varepsilon\to 0} \varepsilon \log \mathbb{P}(\sqrt{\varepsilon}B \in U) \geq -\inf_{\phi\in U} S(\phi),$$
$$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}(\sqrt{\varepsilon}B \in F) \leq -\inf_{\phi\in F} S(\phi).$$

For every $\phi_0 \in U$, there is some $\delta_0 > 0$ such that

$$\{\phi \in \mathcal{C}[0,1] : \|\phi - \phi_0\|_\infty < \delta_0\} \subset U.$$

Based on Schilder's theorem in [21],

$$\mathbb{P}(\sqrt{\varepsilon}B \in U) \geq \mathbb{P}(\|\sqrt{\varepsilon}B - \phi_0\|_\infty < \delta_0) \geq \exp[-\frac{1}{\varepsilon}(S(\phi_0) + \gamma)], \ \text{for}\ \gamma > 0.$$

Then

$$\liminf_{\varepsilon\to 0} \varepsilon \log \mathbb{P}(\sqrt{\varepsilon}B \in U) \geq S(\phi_0).$$

Since $\phi_0 \in U$ is arbitrary,

$$\liminf_{\varepsilon\to 0} \varepsilon \log \mathbb{P}(\sqrt{\varepsilon}B \in U) \geq -\inf_{\phi\in U} S(\phi).$$

Denote by

$$\Phi(r) := \{f \in \mathcal{C}[0,1] : I(f) \leq r\}, \ r \geq 0$$

the sub-level sets of the action functional S in (3.1). From Lemma 12.8 in [22], the action functional S is lower semicontinuous. Then sub-level sets are closed. For each $r > 0$, $0 \leq s < t \leq 1$ and $\phi \in \Phi(r)$, by Cauchy-Schwarz inequality,

$$\begin{aligned} |\phi(t) - \phi(s)| &= |\int_s^t \phi'(u)du| \leq (\int_s^t |\phi'(u)|^2 du)^{\frac{1}{2}} \sqrt{t-s} \\ &\leq \sqrt{2S(\phi)}\sqrt{t-s} \leq \sqrt{2r}\sqrt{t-s}. \end{aligned}$$

This implies that the family $\Phi(r)$ is equibounded and equicontinuous. Using Ascoli's theorem, $\Phi(r)$ is compact. By the definition of the sub-level set $\Phi(r)$, we have $\Phi(r) \cap F = \emptyset$ for all $r < \inf_{\phi\in F} S(\phi)$. So

$$d(\Phi(r), F) = \inf_{\phi\in\Phi(r)} d(\phi, F) =: \delta_r > 0.$$

Applying Schilder's theorem, we obtain that

$$\mathbb{P}(\sqrt{\varepsilon}B \in F) \leq \mathbb{P}(d(\sqrt{\varepsilon}B, \Phi(r)) > \delta_r) \leq \exp(-\frac{r-\gamma}{\varepsilon}), \ \text{for}\ \gamma > 0.$$

Hence

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}(\sqrt{\varepsilon}B \in F) \leq -r.$$

Since $r < \inf_{\phi\in F} S(\phi)$ is arbitrary, we get

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}(\sqrt{\varepsilon}B \in F) \leq -\inf_{\phi\in F} S(\phi).$$

□

Theorem 3.3. *Let $b, \sigma : \mathbb{R} \to \mathbb{R}$ be bounded, globally Lipschitz continuous functions such that* $\inf_{x\in\mathbb{R}} \sigma(x) > 0$*, i.e., there exists $K > 0$ such that*

$$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \le K|x - y|, \quad \text{for all } x, y \in \mathbb{R}.$$

Assume that $(X_t^\varepsilon)_{t\in[0,1]}$ is a solution of the stochastic differential equation driven by Brownian motion, i.e., SDE of the form

$$dX_t^\varepsilon = b(X_t^\varepsilon)dt + \sqrt{\varepsilon}\sigma(X_t^\varepsilon)dB_t, \; X_0^\varepsilon = 0. \tag{3.2}$$

Then $(X^\varepsilon)_{\varepsilon>0}$ satisfies a large deviation principle in $(\mathcal{C}[0,1], \|\cdot\|_\infty)$ with action functional

$$S(\phi) = \begin{cases} \frac{1}{2}\int_0^1 |\frac{\phi'(t)-b(\phi(t))}{\sigma(\phi(t))}|^2 dt, & \phi \in AC[0,1], \phi(0) = 0, \\ \infty, & \text{otherwise.} \end{cases} \tag{3.3}$$

Proof. The key point of proof is that the family of solutions $(X_t^{\varepsilon,m})_{t\in[0,1]}$ given by the stochastic differential equation

$$dX_t^{\varepsilon,m} = b(X^{\varepsilon,m}_{\frac{\lfloor mt\rfloor}{m}})dt + \sqrt{\varepsilon}\sigma(X^{\varepsilon,m}_{\frac{\lfloor mt\rfloor}{m}})dB_t \tag{3.4}$$

is an exponentially good approximation of $(X_t^\varepsilon)_{t\in[0,1]}$. Then the stochastic integral can be evaluated pathwise.

Let $\delta, \rho, \varepsilon > 0$. For $m \in \mathbb{N}$, define $F_m : \mathcal{C}[0,1] \longrightarrow \mathcal{C}[0,1]$ via $\phi = F_m(g)$, where

$$\phi(t) = \phi(t_k^m) + b(\phi(t_k^m))(t - t_k^m) + \sigma(\phi(t_k^m))(g(t) - g(t_k^m)),$$

for $t \in (t_k^m, t_{k+1}^m]$, $t_k^m := k/m$, $k = 0, ..., m-1$, and $\phi(0) = 0$, such that $F_m(\sqrt{\varepsilon}B) = X^{\varepsilon,m}$. Define a $\mathcal{F}_t^\varepsilon$-stopping time by

$$\tau := \tau(\rho) := \inf\{t \ge 0 : |X_t^{\varepsilon,m} - X^{\varepsilon,m}_{\frac{\lfloor mt\rfloor}{m}}| > \rho\} \wedge 1,$$

and set

$$b_t := b(X^{\varepsilon,m}_{\frac{\lfloor mt\rfloor}{m}}) - b(X_t^\varepsilon), \quad \sigma_t := \sigma(X^{\varepsilon,m}_{\frac{\lfloor mt\rfloor}{m}}) - \sigma(X_t^\varepsilon),$$

where $\mathcal{F}_t^\varepsilon := \sigma\{B_s^\varepsilon : s \le t\}$ denotes the canonical filtration. By the global Lipschitz continuity,

$$|b_t| + |\sigma_t| \le K|X^{\varepsilon,m}_{\frac{\lfloor mt\rfloor}{m}} - X_t^\varepsilon| \le \sqrt{2}K(\rho^2 + |X_t^{\varepsilon,m} - X_t^\varepsilon|^2)^{\frac{1}{2}}, \; \text{ for any } \; t \in [0, \tau].$$

A calculation shows

$$\varepsilon \log \mathbb{P}(\sup_{t\in[0,\tau]} |X_t^{\varepsilon,m} - X_t^\varepsilon| > \delta) \le C + \log(\frac{\rho^2}{\rho^2 + \delta^2}),$$

where $C > 0$ is a constant that does not depend on m, ε, ρ. So,

$$\lim_{\rho\to 0} \sup_{m\ge 1} \limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}(\sup_{t\in[0,\tau]} |X_t^{\varepsilon,m} - X_t^\varepsilon| > \delta) = -\infty, \quad \text{for all } \; \delta > 0.$$

Since b and σ are bounded, we find

$$|X^{\varepsilon,m}_{\frac{k}{m}+s} - X^{\varepsilon,m}_{\frac{k}{m}}| \le \tilde{C}(\frac{1}{m} + \sqrt{\varepsilon} \max_{0\le k\le m-1} \sup_{0\le s\le \frac{1}{m}} |B_{\frac{k}{m}+s} - B_{\frac{k}{m}}|), \quad \text{for } \; 0 \le s \le \frac{1}{m},$$

where $\tilde{C} := \max\{\|b\|_\infty, \|\sigma\|_\infty\}$. By the stationarity of the increments, we have

$$\mathbb{P}(\tau < 1) = \mathbb{P}(\bigcup_{k=0}^{m-1}\{\sup_{0\le s\le \frac{1}{m}} |X^{\varepsilon,m}_{\frac{k}{m}+s} - X^{\varepsilon,m}_{\frac{k}{m}}| > \rho\}) \le m\mathbb{P}(\sup_{0\le s\le \frac{1}{m}} |B_s| \ge \frac{\rho - \tilde{C}/m}{2\sqrt{\varepsilon}\tilde{C}}),$$

for all $m > \tilde{C}/\rho$. By Etemadi's inequality [7] and Markov's inequality,

$$\mathbb{P}(\sup_{0\le s\le \frac{1}{m}} |B_s| \ge \frac{\rho - \tilde{C}/m}{2\sqrt{\varepsilon}\tilde{C}}) \le 6\exp(-\frac{\rho - \tilde{C}/m}{6\sqrt{\varepsilon}\tilde{C}} + \hat{C})$$

with a constant $\hat{C} > 0$. Then

$$\lim_{m\to\infty} \limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}(\tau < 1) = -\infty \quad \text{for all} \quad \rho > 0.$$

From

$$\{\|X^{\varepsilon,m} - X^\varepsilon\|_\infty > \delta\} \subseteq \{\tau < 1\} \cup \{\sup_{t\in[0,\tau]} |X^{\varepsilon,m}_t - X^\varepsilon_t| > \delta\},$$

the family of solutions $(X^{\varepsilon,m})_{\varepsilon>0,m\in\mathbb{N}}$ is indeed an exponentially good approximation of $(X^\varepsilon)_{\varepsilon>0}$.

In Lemma 3.2 we have shown that B^ε_t obeys a large deviation principle with action functional S as in (3.1), and $\sqrt{\varepsilon}B_t$ satisfies the same large deviation principle as B^ε_t. The task is now to find a function $F : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ such that the assumptions of Theorem 2.6 are satisfied, and the continuous mappings F_m converges uniformly on the compact sublevel set of S in (3.1) to F.

For absolutely continuous functions $g \in \mathcal{C}[0,1]$ and $x \in \mathbb{R}$, by b and σ are globally Lipschitz continuous, there exists a unique solution $\phi = F(g)$ of the integral equation

$$f(t) = \int_0^t b(f(s))ds + \int_0^t \sigma(f(s))g'(s)ds, \quad t \in [0,1].$$

Fix $g \in \Phi(r) := \{\phi \in \mathcal{C}[0,1] : S(\phi) \le r\}$. Using b, σ are bounded and the Cauchy-Schwarz inequality,

$$\sup_{0\le t\le 1} |F_m(g)(t) - F_m(g)(\frac{\lfloor tm\rfloor}{m})| \le \frac{\|b\|_\infty}{m} + \|\sigma\|_\infty \sqrt{\frac{1}{m}}\sqrt{\int_0^1 g'(s)^2 ds} =: \delta_m \overset{m\to 0}{\longrightarrow} 0.$$

Similarly,

$$\begin{aligned} d(t) :=& |F_m(g)(t) - F(g)(t)| \\ &\le K\int_0^t (1+|g'(s)|)|F_m(g)(\frac{\lfloor ms\rfloor}{m}) - F(g)(s)|ds \\ &\le K(1+\sqrt{2r})\delta_m + L\int_0^t (1+|g'(s)|)d(s)ds. \end{aligned}$$

From Gronwall's lemma,

$$\begin{aligned} d(t) &\le K(1+\sqrt{2r})\delta_m[1 + K\int_0^t (1+|g'(s)|)\exp(K\int_s^t (1+|g'(u)|)du)ds] \\ &\le K(1+\sqrt{2r})\delta_m(1 + K(1+\sqrt{2r})e^{K(1+\sqrt{2r})}). \end{aligned}$$

Because the constants $K, 1+\sqrt{2r}, \delta_m$ do not depend on t and g,

$$\sup_{g\in\Phi(r)} ||F_m(g) - F(g)||_\infty \leq K(1+\sqrt{2r})\delta_m(1+K(1+\sqrt{2r})e^{K(1+\sqrt{2r})}) \overset{m\to 0}{\longrightarrow} 0.$$

Apply Theorem 2.6, $(X^\varepsilon)_{\varepsilon>0}$ satisfies a large deviation principle with action functional

$$S(\phi) = \begin{cases} \frac{1}{2}\int_0^1 |g'(t)|^2 dt, & \phi \in AC[0,1], \phi(0) = 0, \\ \infty, & \text{otherwise}, \end{cases}$$

where the infimum is taken over all functions $g \in AC[0,1]$, $g(0) = 0$, such that

$$\phi(t) = F(g)(t) = \int_0^t b(\phi(s))ds + \int_0^t \sigma(\phi(s))g'(s)ds,$$

that is,

$$g'(t) = \frac{\phi'(t) - b(\phi(t))}{\sigma(\phi(t))}.$$

□

Remark 3.4. For the special case $\sigma = 1$, the solution $(X^\varepsilon)_{\varepsilon>0}$ of the stochastic differential equation

$$dX_t^\varepsilon = b(X_t^\varepsilon)dt + \sqrt{\varepsilon}dB_t,\ X_0^\varepsilon = 0,$$

has the action functional

$$S(\phi) = \begin{cases} \frac{1}{2}\int_0^1 |\phi'(t) - b(\phi(t))|^2 dt, & \phi \in AC[0,1], \phi(0) = 0, \\ \infty, & \text{otherwise}. \end{cases}$$

Corollary 3.5. *The symbol of solution X_t for the SDE driven by Brownian motion*

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t,\ X_0 = 0,$$

is given by

$$q(x,\xi) = ib(x)\xi - \frac{1}{2}\sigma^2(x)\xi^2.$$

Set

$$H(x,\xi) := q(x,-i\xi) = b(x)\xi + \frac{1}{2}\sigma^2(x)\xi^2,$$

then the Legendre transform of $H(x,\xi)$ is

$$\begin{aligned} L(x,\zeta) &= \sup_{\xi\in\mathbb{R}}[\zeta\xi - H(x,\xi)] \\ &= \sup_{\xi\in\mathbb{R}}[\zeta\xi - b(x)\xi - \frac{1}{2}\sigma^2(x)\xi^2] \\ &= \sup_{\xi\in\mathbb{R}}[-\frac{1}{2}\sigma^2(x)\big(\xi^2 - \frac{2}{\sigma^2(x)}(\zeta - b(x))\xi\big)] \\ &= \sup_{\xi\in\mathbb{R}}[-\frac{1}{2}\sigma^2(x)\big(\xi - \frac{\zeta - b(x)}{\sigma^2(x)}\big)^2 + \frac{1}{2}|\frac{\zeta - b(x)}{\sigma^2(x)}|^2] \\ &= \frac{1}{2}|\frac{\zeta - b(x)}{\sigma^2(x)}|^2. \end{aligned}$$

Hence the action functional of solution $(X^\varepsilon)_{\varepsilon>0}$ *for* (3.2) *is*

$$S(\phi) := \begin{cases} \int_0^1 L(\phi(t), \phi'(t))dt, & \phi \in AC[0,1], \phi(0) = 0, \\ \infty, & \textit{otherwise}, \end{cases}$$

where

$$L(\phi(t), \phi'(t)) = \frac{1}{2}|\frac{\phi'(t) - b(\phi(t))}{\sigma(\phi(t))}|^2.$$

4. Action Functionals for Stochastic Systems with Lévy Noise

A stochastic process $L_t \in \mathbb{R}, t \in [0,1]$ is called a Lévy process [6, Chapter 7] if the following properties hold:
(1) $L_0 = 0$(a.s.);
(2) L has independent increments, i.e., for each $n \in \mathbb{N}$ and $0 \leq t_1 < t_2 < ... < t_{n+1} \leq 1$, the random variables $(L_{t_{j+1}} - L_{t_j}, 1 \leq j \leq n)$ are independent;
(3) L has stationary increments, i.e., for each $n \in \mathbb{N}$ and $0 \leq t_1 < t_2 < ... < t_{n+1} \leq 1$, $L_{t_{j+1}} - L_{t_j} \overset{\mathrm{d}}{=} L_{t_{j+1}-t_j}$;
(4) L is stochastically continuous, i.e., $\lim_{t\downarrow 0} \mathbb{P}(|L_t| > \varepsilon) = 0$ for all $\varepsilon > 0$;
(5) the paths $t \mapsto L_t$ are cádlág with probability 1, that is, the trajectories are right continuous with existing left limits.

The Lévy-Itô decomposition [20] of Lévy process $(L_t)_{t\in[0,1]}$ with Lévy triplet (a, σ^2, ν) is

$$L_t = at + \sigma B_t + \int_0^t \int_{|z|>1} zN(dz, ds) + \int_0^t \int_{0<|z|\leq 1} z\tilde{N}(dz, ds),$$

where $(B_t)_{t\in[0,1]}$ is a Brownian motion, N denotes the jump counting measure, and $\tilde{N}$ is the compensated jump counting measure. The characteristic function of $(L_t)_{t\in[0,1]}$ is given by the Lévy-Khintchine formula [6] :

$$\mathbb{E}e^{i\xi L_t} = e^{t\psi(\xi)}, \quad \xi \in \mathbb{R}, \, t \in [0,1],$$

where ψ is the Lévy symbol

$$\psi(\xi) = ia\xi - \frac{1}{2}\sigma^2\xi^2 + \int_{\mathbb{R}\setminus\{0\}} (e^{i\xi y} - 1 - i\xi y\chi_{\{|y|\leq 1\}})\nu(dy).$$

There is a one-to-one correspondence between ψ and (a, σ^2, ν) consisting of the drift parameter $a \in \mathbb{R}$, the diffusion coefficient $\sigma \geq 0$, and the Lévy measure ν on $(\mathbb{R} \setminus \{0\}, \mathcal{B}(\mathbb{R} \setminus \{0\}))$ satisfying $\int_{\mathbb{R}\setminus\{0\}} (y^2 \wedge 1)\nu(dy) < \infty$. Denote the logarithmic moment generating function of L_1 by

$$\Psi(\xi) = \psi(-i\xi) = a\xi + \frac{1}{2}\sigma^2\xi^2 + \int_{\mathbb{R}\setminus\{0\}} (e^{\xi y} - 1 - \xi y\chi_{\{|y|\leq 1\}})\nu(dy). \tag{4.1}$$

If $\sigma = 0$, we say that $(L_t)_{t\in[0,1]}$ is a Lévy process without Gaussian component.

Example 4.1. The Lévy measure of the a tempered stable Lévy process $(L_t)_{t\in[0,1]}$ is

$$\nu(dy) = \frac{1}{2}\frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}e^{-my}\frac{dy}{|y|^{1+y}}, \quad \text{for } \alpha \in (1,2), \, m > 0.$$

The Lévy symbol of $(L_t)_{t\geq 0}$ is given by

$$\psi(\xi) = -(|\xi|^2 + m^2)^{\frac{\alpha}{2}} \cos(\alpha \arctan \frac{|\xi|}{m}) + m^\alpha.$$

Definition 4.2. Assume that Lévy process $(L_t)_{t\in[0,1]}$ with Lévy triplet (a, σ^2, ν) satisfies $\mathbb{E}e^{\lambda|L_1|} < \infty$, for all $\lambda \geq 0$. The action functional of $(L_t)_{t\in[0,1]}$ on $(\mathcal{D}[0,1], \|\cdot\|_\infty)$ is defined by

$$S(\phi) = \begin{cases} \int_0^1 \Psi^*(\phi'(t))dt, & \phi \in AC[0,1], \phi(0) = 0, \\ \infty, & \text{otherwise}, \end{cases} \tag{4.2}$$

where $\Psi^*(\cdot)$ is the Legendre transform of $\Psi(\cdot)$ in (4.1).

Lemma 4.3. *The scaled Lévy process $L_t^\varepsilon := \varepsilon L_{\frac{t}{\varepsilon}}, t \in [0,1]$ satisfies a large deviation principle in $(\mathcal{D}[0,1], \|\cdot\|_\infty)$ as $\varepsilon \to 0$ with action functional in (4.2), i.e.,*

$$\liminf_{\varepsilon\to 0} \varepsilon \log \mathbb{P}(L^\varepsilon \in U) \geq -\inf_{\phi\in U} S(\phi),$$
$$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}(L^\varepsilon \in F) \leq -\inf_{\phi\in F} S(\phi),$$

for any open set $U \in \mathcal{B}$ and closed set $F \in \mathcal{B}$.

Proof. In order to prove that $(L_t^\varepsilon)_{t\in[0,1]}$ satisfies a large deviation principle as $\varepsilon \to 0$ with the action functional S in (4.2), we split the proof into several steps:

(i) The sequence of discretizations $(Z_n^L)_{n\in\mathbb{N}}$ defined by

$$\frac{Z_n^L(t,\omega)}{n} := \frac{1}{n} L(\lfloor n.t \rfloor, \omega) = \frac{1}{n}[\sum_{j=0}^{n-1} L(j,\omega)\chi_{[\frac{j}{n}, \frac{j+1}{n})}(t) + L(n,\omega)\chi_{\{1\}}(t)]$$

is exponentially tight in $(\mathcal{D}[0,1], \|\cdot\|_\infty)$.

Since the mapping

$$(\mathbb{R}^n, |\cdot|) \ni x \mapsto (T_n x)(t) := \sum_{j=1}^{n-1} x_j \chi_{[\frac{j}{n}, \frac{j+1}{n})}(t) + x_n \chi_{\{1\}}(t) \in (\mathcal{D}[0,1], \|\cdot\|_\infty)$$

is continuous, we obtain that $T_n(K)$ is compact for any compact set $K \subseteq \mathbb{R}^n$. For $K \subseteq \mathbb{R}$ compact, we have

$$\mathbb{P}(\frac{Z_n^L}{n} \notin T_n(K^n)) \leq \sum_{j=1}^{n-1} \mathbb{P}(\frac{L_j}{n} \notin K).$$

The distribution of $\frac{L_j}{n}$ is a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, hence $\frac{L_j}{n}$ is tight. For $j = 1, ..., n$, we conclude that $\frac{Z_n^L}{n}$ is tight in $(\mathcal{D}[0,1], \|\cdot\|_\infty)$. Fix $r > 0$ and $\epsilon > 0$, for $K \subseteq \mathbb{R}$ and $n \geq m$, we get

$$\begin{aligned} \mathbb{P}(d(\frac{Z_n^L}{n}, T_m(K^m)) > \epsilon) &\leq \mathbb{P}(\frac{Z_n^L}{n} \notin T_n(K^n)) + \mathbb{P}(\frac{Z_n^L}{n} \in T_n(K^n), d(\frac{Z_n^L}{n}, T_m(K^m)) > \epsilon) \\ &=: I_1 + I_2. \end{aligned} \tag{4.3}$$

We choose $K := [-r, r]$ and estimate the terms separately. Applying Etemadi's inequality and Markov's inequality yields

$$I_1 = \mathbb{P}(\sup_{1\leq j\leq n} |\frac{L_j}{n}| > r) \leq 3 \sup_{1\leq j\leq n} \mathbb{P}(|L_j| > \frac{nr}{3}) \leq 3 \sup_{1\leq j\leq n} \mathbb{E}e^{|L_j| - nr/3} \leq 3e^{-nr/3}\beta_1^n,$$

where $\beta_1 := \mathbb{E}e^{|L_1|} < \infty$ because $(L_t)_{t\in[0,1]}$ has finite exponential moments. If we set $f_m := f(\frac{\lfloor m.\rfloor}{m})$, then

$$d(f, T_m(K^m)) \le \|f - f_m\|_\infty, \quad \text{for all} \quad f \in T_n(K^n). \tag{4.4}$$

Moreover,

$$\begin{aligned}\|f - f_m\|_\infty &= \max_{1\le i\le m-1} \sup_{t\in[\frac{i}{m},\frac{i+1}{m})} |f(t) - f_m(t)| \\ &= \max_{1\le i\le m-1} \sup_{t\in[\frac{i}{m},\frac{i+1}{m})} |f(\frac{\lfloor nt\rfloor}{n}) - f(\frac{\lfloor mt\rfloor}{m})| \\ &\le \max_{1\le i\le m-1} \sup_{1\le j\le\lfloor\frac{n}{m}\rfloor+1} |f(\frac{\lfloor n\frac{i}{m}\rfloor}{n} + \frac{j}{n}) - f(\frac{\lfloor n\frac{i}{m}\rfloor}{n})|. \end{aligned} \tag{4.5}$$

Combining (4.4) and (4.5),

$$\begin{aligned} I_2 &\le \mathbb{P}(\sup_{1\le i\le m-1} \sup_{1\le j\le\lfloor\frac{n}{m}\rfloor+1} |Z_n^L(\frac{\lfloor n\frac{i}{m}\rfloor}{n} + \frac{j}{n}) - Z_n^L(\frac{\lfloor n\frac{i}{m}\rfloor}{n})| > n\epsilon) \\ &\le \sum_{i=1}^{m-1} \mathbb{P}(\sup_{1\le j\le\lfloor\frac{n}{m}\rfloor+1} |L(\lfloor n\frac{i}{m}\rfloor + j) - L(\lfloor n\frac{i}{m}\rfloor)| > n\epsilon). \end{aligned}$$

By the stationarity and independence of the increments of L and Markov's inequality,

$$\begin{aligned} I_2 &\le m\mathbb{P}(\sup_{1\le j\le\lfloor\frac{n}{m}\rfloor+1} |L_j| > n\epsilon) \le 3m \sup_{1\le j\le\lfloor\frac{n}{m}\rfloor+1} \mathbb{P}(|L_j| > \frac{n\epsilon}{3}) \\ &\le 3m \sup_{1\le j\le\lfloor\frac{n}{m}\rfloor+1} \mathbb{E}e^{r|L_j|-nr\varepsilon/3} \le 3m\beta_2^{\lfloor\frac{n}{m}\rfloor+1} e^{-nr\varepsilon/3}, \end{aligned}$$

where $\beta_2 := \mathbb{E}e^{r|L_1|} < \infty$. Then

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}(d(\frac{Z_n^L}{n}, T_m(K^m)) > \epsilon) &\le \max\{\log\beta_1 - \frac{r}{3}, \frac{1}{m}\log\beta_2 - \frac{r\epsilon}{3}\} \\ &\overset{r,m\to\infty}{\longrightarrow} -\infty. \end{aligned}$$

Consequently, $(Z_n^L)_{n\in\mathbb{N}}$ is exponentially tight in $(\mathcal{D}[0,1], \|\cdot\|_\infty)$.

(ii) $(Z_n^L)_{n\in\mathbb{N}}$ satisfies a large deviation principle in $(\mathcal{D}[0,1], \|\cdot\|_\infty)$ with respect to $\mathcal{B}$ as $n\to\infty$ with action functional

$$I(\phi) = \sup_{\alpha\in BV[0,1]\cap\mathcal{D}[0,1]} (\int_0^1 \phi d\alpha - \frac{1}{2}\int_0^1 \Psi(\alpha(1) - \alpha(s))ds). \tag{4.6}$$

where $\Psi(\cdot)$ as in (4.1). Note that

$$Z_n^L = \sum_{j=1}^{n-1} L_j \chi_{[\frac{j}{n},\frac{j+1}{n})} + L_n\chi_{\{1\}} = \sum_{j=1}^{n} (L_j - L_{j-1})\chi_{[\frac{j}{n},1]}.$$

By the stationarity and independence of the increments,

$$\mathbb{E}e^{\langle \alpha, Z_n^L\rangle} = \mathbb{E}\exp(\sum_{j=1}^{n}(L_j - L_{j-1})(\alpha(1)-\alpha(\frac{j}{n}))) = \prod_{j=1}^{n}\mathbb{E}\exp(L_1(\alpha(1)-\alpha(\frac{j}{n}))).$$

Since

$$\mathbb{E}e^{\lambda L_1} = e^{\Psi(\lambda)}, \quad \text{for all} \quad \lambda \in \mathbb{R},$$

we have

$$\Lambda(\alpha) := \lim_{n\to\infty}\frac{1}{n}\log\mathbb{E}e^{\langle\alpha, Z_n^L\rangle} = \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\Psi(\alpha(1)-\alpha(\frac{j}{n})) = \int_0^1 \Psi(\alpha(1)-\alpha(s))ds.$$

Pick $\beta \in BV[0,1]\cap\mathcal{D}[0,1]$ and set

$$u(t,s) := \Psi((\alpha(1)-\alpha(s)) + t(\beta(1)-\beta(s))), \ \ t\in[-1,1], \, s\in[0,1].$$

From α, $\beta \in BV[0,1]$, it follows that $\|\alpha\|_\infty + \|\beta\|_\infty \le C < \infty$. By $\mathbb{E}e^{\lambda|L_1|} < \infty$, for all $\lambda \ge 0$, we have

$$-\infty < \log\mathbb{E}e^{-2C|L_1|} \le |u(t,s)| \le \log\mathbb{E}e^{2C|L_1|} < \infty.$$

And then

$$|\partial_t u(t,s)| \le 2C\frac{1}{\mathbb{E}e^{-2C|L_1|}}\sqrt{\mathbb{E}(L_1^2)}\sqrt{\mathbb{E}e^{2C|L_1|}} < \infty, \quad \text{for all} \ \ t\in[-1,1].$$

We get

$$\frac{\Lambda(\alpha+t\beta)-\Lambda(\alpha)}{t} \overset{t\to 0}{\longrightarrow} \int_0^1 \partial_t u(0,s)ds = \int_0^1(\beta(1)-\beta(s))\frac{1}{\mathbb{E}e^{L_1(\alpha(1)-\alpha(s))}}\mathbb{E}(L_1e^{L_1(\alpha(1)-\alpha(s))})ds.$$

So Λ is $\mathcal{D}[0,1]$-Gâteaux differentiable at α, and its derivative equals

$$D_\alpha(t) := \int_0^t \frac{1}{\mathbb{E}e^{L_1(\alpha(1)-\alpha(s))}}\mathbb{E}(L_1e^{L_1(\alpha(1)-\alpha(s))})ds, \ \ t\in[0,1].$$

We defer the rest proof of (ii) to (iv).

(iii) $(Z^L_{\lfloor\frac{1}{\varepsilon}\rfloor}/\lfloor\frac{1}{\varepsilon}\rfloor)_{\varepsilon>0}$ and $\varepsilon L(\frac{\cdot}{\varepsilon})$ are exponentially equivalent.

Let $\epsilon > 0$ and $r \ge 0$. We have

$$\|\frac{Z^L_{\lfloor\frac{1}{\varepsilon}\rfloor}}{\lfloor\frac{1}{\varepsilon}\rfloor} - \varepsilon L(\frac{\cdot}{\varepsilon})\|_\infty \le (\frac{1}{\lfloor\frac{1}{\varepsilon}\rfloor}-\varepsilon)\|Z^L_{\lfloor\frac{1}{\varepsilon}\rfloor}\|_\infty + \|\varepsilon Z^L_{\lfloor\frac{1}{\varepsilon}\rfloor} - \varepsilon L(\frac{\cdot}{\varepsilon})\|_\infty := C_\varepsilon + D_\varepsilon. \quad (4.7)$$

We find

$$\mathbb{P}(C_\varepsilon > \epsilon) \le \mathbb{P}(\sup_{0\le k\le\lfloor\frac{1}{\varepsilon}\rfloor}|L_k| > \frac{1}{\varepsilon}(\frac{1}{\varepsilon}-1)\epsilon) \le 3\exp(-\frac{1}{\varepsilon}(\frac{1}{\varepsilon}-1)\frac{\epsilon}{3})\beta_1^{\lfloor\frac{1}{\varepsilon}\rfloor}, \quad (4.8)$$

where $\beta_1 = \mathbb{E}e^{|L_1|}$ as in (i). Note that

$$\sup_{t\in[0,1]}|Z^L_{\lfloor\frac{1}{\varepsilon}\rfloor}(t) - L(\frac{t}{\varepsilon})| \le \sup_{0\le k\le\lfloor\frac{1}{\varepsilon}\rfloor}\sup_{l\in[0,2]}|L_{k+l}-L_k| \ \ \text{when} \ \ \frac{t}{\varepsilon} - \lfloor\lfloor\frac{1}{\varepsilon}\rfloor t\rfloor \le 2.$$

By Etemadi's inequality and the stationarity of the increments for L,

$$\mathbb{P}(D_\varepsilon > \epsilon) \leq \mathbb{P}(\sup_{0\leq k\leq \lfloor\frac{1}{\varepsilon}\rfloor} \sup_{l\in[0,2]} |L_{k+l} - L_k| > \frac{\epsilon}{\varepsilon}) \leq 3(\lfloor\frac{1}{\varepsilon}\rfloor + 1) \sup_{l\in[0,2]} \mathbb{P}(|L_l| > \frac{\epsilon}{3\varepsilon}).$$

Since $(L_t - t\mathbb{E}L_1)_{t\geq 0}$ is a martingale, we know that $(e^{r|L_t - t\mathbb{E}L_1|})_{t\geq 0}$ is a submartingale. By Markov's inequality,

$$\sup_{l\in[0,2]} \mathbb{P}(|L_l - l\mathbb{E}L_1| > \frac{\epsilon}{3\varepsilon}) \leq e^{-r\epsilon/3\varepsilon}\mathbb{E}e^{r|L_2-2\mathbb{E}L_1|} =: \beta_3 e^{-r\epsilon/3\varepsilon}. \tag{4.9}$$

Combining (4.7), (4.8) and (4.9) implies

$$\begin{aligned}&\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}(\|\frac{Z^L_{\lfloor\frac{1}{\varepsilon}\rfloor}}{\lfloor\frac{1}{\varepsilon}\rfloor} - \varepsilon L(\frac{\cdot}{\varepsilon})\|_\infty > 2\epsilon)\\ &\quad \leq \max\{\limsup_{\varepsilon\to 0} \varepsilon\log\mathbb{P}(C_\varepsilon > \epsilon), \limsup_{\varepsilon\to 0}\varepsilon\log\mathbb{P}(D_\varepsilon > \epsilon)\} \leq -\frac{r\epsilon}{3} \overset{r\to\infty}{\longrightarrow} -\infty.\end{aligned}$$

Hence $(Z^L_{\lfloor\frac{1}{\varepsilon}\rfloor}/\lfloor\frac{1}{\varepsilon}\rfloor)_{\varepsilon>0}$ and $\varepsilon L(\frac{\cdot}{\varepsilon})$ are exponentially equivalent.

(iv) $(L^\varepsilon_t)_{t\in[0,1]}$ satisfies a large deviation principle with action functional I in (4.6) that equals the action functional S defined in (4.2).

Fix $\varepsilon > 0$, $0 < s_1 < t_1 \leq ...... \leq s_n < t_n \leq 1$, and $c = (c_1, ..., c_n) \in \mathbb{R}^n$. We define

$$\alpha(t) := \sum_{j=1}^n c_j \chi_{[s_j,t_j)}(t), \quad t\in[0,1]. \tag{4.10}$$

Then $\alpha \in BV[0,1]\cap D[0,1]$ and

$$\int_0^1 \phi d\alpha = \sum_{j=1}^n c_j(\phi(s_j) - \phi(t_j)). \tag{4.11}$$

Moreover,

$$\begin{aligned}\int_0^1 \log\mathbb{E}e^{L_1(\alpha(1)-\alpha(s))}ds &= \sum_{j=1}^n \int_0^1 \log\mathbb{E}e^{-c_jL_1}\chi_{[s_j,t_j)}(s)ds\\ &\leq \log\mathbb{E}e^{||c||_\infty|L_1|}\sum_{j=1}^n (t_j - s_j).\end{aligned} \tag{4.12}$$

By (4.6), we obtain

$$\int_0^1 \phi d\alpha \leq I(\phi) + \int_0^1 \log\mathbb{E}e^{L_1(\alpha(1)-\alpha(s))}ds.$$

Using (4.11) and (4.12), we find

$$\sum_{j=1}^n c_j(\phi(s_j)-\phi(t_j)) \leq I(\phi) + \log\mathbb{E}e^{||c||_\infty|L_1|}\sum_{j=1}^n (t_j - s_j).$$

In particular, for $c_j := r\mathrm{sgn}(f(s_j) - f(t_j))$ and $r > 0$,

$$\sum_{j=1}^n |\phi(t_j) - \phi(s_j)| \leq \frac{I(\phi)}{r} + \frac{\log\mathbb{E}e^{|L_1|r}}{r}\sum_{j=1}^n (t_j - s_j).$$

Choosing $r > 0$ sufficiently large and $\delta > 0$ sufficiently small, we see that

$$\sum_{j=1}^{n}(t_j - s_j) < \delta \Longrightarrow \sum_{j=1}^{n}|\phi(t_j) - \phi(s_j)| < \epsilon,$$

i.e., ϕ is absolutely continuous. Letting $t \to 0$ and $r \to \infty$,

$$|\phi(t)| \leq \frac{I(\phi)}{r} + t\frac{\log \mathbb{E}e^{|L_1|r}}{r}$$

yields $\phi(0) = 0$. Hence $I(\phi) < \infty$ implies that ϕ is absolutely continuous and $\phi(0) = 0$. Then there exists $f \in L^1[0,1]$ such that

$$\phi(t) = \int_0^t f(s)ds, \quad t \in [0,1].$$

So

$$\begin{aligned}\int_0^1 \phi d\alpha - \int_0^1 \Psi(\alpha(1) - \alpha(s))ds &= \int_0^1 [f(s)(\alpha(1) - \alpha(s)) - \Psi(\alpha(1) - \alpha(s))]ds \\ &\leq \int_0^1 \Psi^*(f(s))ds = \int_0^1 \Psi^*(\phi'(s))ds = S(\phi),\end{aligned}$$

for any $\alpha \in BV[0,1] \cap D[0,1]$. Now we prove $I(\phi) \geq S(\phi)$ for $\phi \in AC[0,1]$, $\phi(0) = 0$. By the monotone convergence theorem, it suffices to show

$$\int_0^1 \Lambda_k(\phi'(s))ds \leq I(\phi) \text{ where } \Lambda_k(x) := \sup_{|\alpha|\leq k}(\alpha x - \Psi(\alpha)), x \in \mathbb{R}, k \in \mathbb{N}.$$

Note that Λ_k is convex and locally bounded, hence continuous. From

$$\sum_{j=0}^{n-1}\frac{\phi(\frac{j+1}{n}) - \phi(\frac{j}{n})}{\frac{1}{n}}\chi_{[\frac{j}{n},\frac{j+1}{n})}(t) \longrightarrow \phi'(t) \quad a.s.$$

and the dominated convergence theorem, we get

$$\int_0^1 \Lambda_k(\phi'(t))dt = \lim_{n\to\infty}\sum_{j=0}^{n-1}\frac{1}{n}\Lambda_k(n[\phi(\frac{j+1}{n}) - \phi(\frac{j}{n})]).$$

As $\alpha \mapsto \alpha x - \Psi(\alpha)$ is continuous, we can choose $|\alpha(x)| \leq k$ such that

$$\Lambda_k(x) = \alpha(x)x - \Psi(\alpha(x)).$$

For suitable $\alpha_0^n, ..., \alpha_{n-1}^n$,

$$\begin{aligned}\int_0^1 \Lambda_k(\phi'(t))dt &= \lim_{n\to\infty}\sum_{j=0}^{n-1}[\alpha_j^n(\phi(\frac{j+1}{n}) - \phi(\frac{j}{n})) - \frac{1}{n}\Psi(\alpha_j^n)] \\ &= \lim_{n\to\infty}(\int_0^1 \phi d\alpha^n - \int_0^1 \Psi(\alpha^n(1) - \alpha^n(t))dt) \leq I(\phi),\end{aligned}$$

where $\alpha^n \in BV[0,1] \cap \mathcal{D}[0,1]$, $n \in \mathbb{N}$, is a step function of the form (4.10). Consequently, the action funtionals (4.2) and (4.6), i.e., $S(\phi) = I(\phi)$. □

Remark 4.4. Lemma 3.2 does not apply to Lévy processes with infinite moments of order n, for some $n \in \mathbb{N}$. In particular, α-stable process with symbol $\psi(\xi) = |\xi|^\alpha$ is not covered because it has finite first order moment for $\alpha \in (1, 2]$. But Lemma 3.2 is valid for the tempered stable Lévy process $(L_t)_{t\in[0,1]}$ in Example 4.1.

Theorem 4.5. *Let $b, \sigma, \eta : \mathbb{R} \to \mathbb{R}$ be bounded, Lipschitz continuous functions. There exists $K > 0$ such that*

$$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| + |\eta(x) - \eta(y)| \leq K|x - y|, \quad \text{for all } x, y \in \mathbb{R}.$$

Let $(B_t)_{t\geq 0}$ be a Brownian motion and $(L_t)_{t\geq 0}$ be an independent Lévy process with Lévy triplet $(a, 0, \nu)$ and symbol ψ such that $\mathbb{E}e^{\lambda|L_1|} < \infty$, for all $\lambda \geq 0$. Define a scaled Lévy process as $L^\varepsilon_t := \varepsilon L_{\frac{t}{\varepsilon}}$. Then the family of solutions $(X^\varepsilon)_{\varepsilon>0}$ of

$$dX^\varepsilon_t = b(X^\varepsilon_{t-})dt + \sqrt{\varepsilon}\sigma(X^\varepsilon_{t-})dB_t + \eta(X^\varepsilon_{t-})dL^\varepsilon_t \tag{4.13}$$

satisfies a large deviation principle in $(\mathcal{D}[0, 1], \|\cdot\|_\infty)$ as $\varepsilon \to 0$ with action funtional

$$S(\phi) := \begin{cases} \inf\{\frac{1}{2}\int_0^1 |g'(t)|^2 dt + \int_0^1 \Psi^*(h'(t))dt\}, & \phi \in AC[0, 1], \phi(0) = 0, \\ \infty, & \textit{otherwise}, \end{cases} \tag{4.14}$$

where $\Psi^*(\cdot)$ denotes the Legendre transform of $\Psi(\cdot)$ defined by

$$\Psi(\xi) := \psi(-i\xi) = a\xi + \int_{\mathbb{R}\setminus\{0\}} (e^{\xi y} - 1 - \xi y\chi_{\{|y|\leq 1\}})\nu(dy), \ \xi \in \mathbb{R},$$

and the infimum is taken over all functions $g, h \in AC[0, 1]$, $g(0) = h(0) = 0$, such that

$$\phi(t) = F(g, h)(t) = \int_0^t b(\phi(s))ds + \int_0^t \sigma(\phi(s))g'(s)ds + \int_0^t \eta(\phi(s))h'(s)ds.$$

Proof. Since $(B_t)_{t\geq 0}$ and $(L_t)_{t\geq 0}$ are independent, $(B_t, L_t)_{t\geq 0}$ is a Lévy process. Denote by $(Z^B_{\lfloor\frac{1}{\varepsilon}\rfloor}/\lfloor\frac{1}{\varepsilon}\rfloor)_{\varepsilon>0}$ and $(Z^L_{\lfloor\frac{1}{\varepsilon}\rfloor}/\lfloor\frac{1}{\varepsilon}\rfloor)_{\varepsilon>0}$ of the approximations of $(B^\varepsilon_t)_{t\in[0,1]}$ and $(L^\varepsilon_t)_{t\in[0,1]}$. By straightforward modifications for the proof of Lemma 3.2, $(\sqrt{\varepsilon}B, L^\varepsilon)_{\varepsilon>0}$ satisfies a large deviation principle in $\mathcal{D}[0, 1] \times \mathcal{D}[0, 1]$ endowed with the norm

$$\|(f_1, f_2)\| := \|f_1\|_\infty + \|f_2\|_\infty, \quad f, g \in \mathcal{D}[0, 1],$$

as $\varepsilon \to 0$ with action funtional

$$S(g, h) = \begin{cases} \frac{1}{2}\int_0^1 |g'(t)|^2 dt + \int_0^1 \Psi^*(h'(t))dt, & g, h \in AC[0, 1], g(0) = h(0) = 0, \\ \infty, & \text{otherwise}. \end{cases} \tag{4.15}$$

For $m \in \mathbb{N}$ define continuous mappings $F_m : \mathcal{D}[0, 1] \times \mathcal{D}[0, 1] \longrightarrow \mathcal{D}[0, 1]$ via $\phi = F_m(g, h)$, where

$$\phi(t) = \phi(t^m_k) + b(\phi(t^m_k))(t - t^m_k) + \sigma(\phi(t^m_k))(g(t) - g(t^m_k)) + \eta(\phi(t^m_k))(h(t) - h(t^m_k))$$

with $t \in (t^m_k, t^m_{k+1}]$, $t^m_k := \frac{k}{m}$, $k = 0, ..., m - 1$, and $\phi(0) := \phi(0-) = 0$. Using similar arguments as in the proof of Theorem 3.3, the solutions $(X^{\varepsilon,m}_t)_{m\in\mathbb{N},\varepsilon>0} = (F_m(\sqrt{\varepsilon}B_t, L^\varepsilon_t))_{m\in\mathbb{N},\varepsilon>0}$ of the stochastic differential equation

$$dX^{\varepsilon,m}_t = b(X^{\varepsilon,m}_{\frac{\lfloor mt\rfloor}{m}-})dt + \sqrt{\varepsilon}\sigma(X^{\varepsilon,m}_{\frac{\lfloor mt\rfloor}{m}-})dB_t + \eta(X^{\varepsilon,m}_{\frac{\lfloor mt\rfloor}{m}-})dL^\varepsilon_t, \quad X^{\varepsilon,m}_0 = 0,$$

are an exponential approximation of $(X^\varepsilon)_{\varepsilon>0}$. For absolutely continuous functions $g, h \in \mathcal{D}[0,1]$, by b and σ are globally Lipschitz continuous, there exists a unique solution $F(g,h)$ of the integral equation

$$\phi(t) = \int_0^t b(\phi(s))ds + \int_0^t \sigma(\phi(s))g'(s)ds + \int_0^t \eta(\phi(s))h'(s)ds, \quad t \in [0,1],$$

such that

$$\lim_{m\to\infty} \sup_{(g,h)\in\Phi(r)} \|F_m(g,h) - F(g,h)\|_\infty = 0, \quad \text{for all } \ r \geq 0,$$

where $\Phi(r) := \{(g,h) \in \mathcal{D}[0,1] \times \mathcal{D}[0,1] : S(g,h) \leq r\}$ is the sublevel set of the action functional S defined in (4.15). From Theorem 2.6, it follows that $(X^\varepsilon)_{\varepsilon>0}$ satisfies a large deviation principle with action functional S as in (4.14). □

Remark 4.6. For the special case $\eta = 0$, Theorem 4.5 concides with Theorem 3.3.

Corollary 4.7. *The symbol of the solution of the stochastic differential equation*

$$dX_t = b(X_{t-})dt + \sigma(X_{t-})dB_t + \eta(X_{t-})dL_t$$

is given by

$$\begin{aligned} q(x,\xi) &= ib(x)\xi - \frac{1}{2}\sigma^2(x)\xi^2 + \psi(\eta(x)\xi) \\ &= i(b(x) + a\eta(x))\xi - \frac{1}{2}\sigma^2(x)\xi^2 + \int_{\mathbb{R}\setminus 0} (e^{iy\eta(x)\xi} - 1 - iy\eta(x)\xi\chi_{\{|y|\leq 1\}})\nu(dy). \end{aligned}$$

Set

$$\begin{aligned} H(x,\xi) := q(x,-i\xi) &= (b(x) + a\eta(x))\xi \\ &+ \frac{1}{2}\sigma^2(x)\xi^2 + \int_{\mathbb{R}\setminus 0} (e^{y\eta(x)\xi} - 1 - y\eta(x)\xi\chi_{\{|y|\leq 1\}})\nu(dy), \end{aligned}$$

and denote the Legendre transform of $H(x,\xi)$ by

$$\begin{aligned} L(x,\zeta) &= \sup_{\xi\in\mathbb{R}} [\zeta\xi - H(x,\xi)] \\ &= \sup_{\xi\in\mathbb{R}} [\zeta\xi - (b(x) + a\eta(x))\xi - \frac{1}{2}\sigma^2(x)\xi^2 - \int_{\mathbb{R}\setminus 0} (e^{y\eta(x)\xi} - 1 - y\eta(x)\xi\chi_{\{|y|\leq 1\}})\nu(dy)]. \end{aligned}$$

Suppose that $L(x,\zeta)$ satisfies
(H1) The function $(x,\zeta) \mapsto L(x,\zeta)$ is finite, i.e., for all $x, \zeta \in \mathbb{R}$, $L(x,\zeta) < \infty$. For any $r > 0$, there exist constants $C_1, C_2 > 0$ such that

$$L(x,\zeta) + |\frac{\partial}{\partial\zeta} L(x,\zeta)| \leq C_1 \quad \text{and} \quad \frac{\partial^2}{\partial\zeta^2} L(x,\zeta) > C_2, \quad \text{for all } \ x \in \mathbb{R}, |\zeta| \leq r.$$

(H2) Continuity condition:

$$\sup_{|x-y|<\delta} \sup_{\zeta\in\mathbb{R}} \frac{L(x,\zeta) - L(y,\zeta)}{1 + L(y,\zeta)} \xrightarrow{\delta\to 0} 0.$$

Then the family of solutions $(X^\varepsilon)_{\varepsilon>0}$ of (4.13) has the action functional:

$$S(\phi) := \begin{cases} \int_0^1 L(\phi(t),\phi'(t))dt, & \phi \in AC[0,1], \phi(0) = x, \\ \infty, & \text{otherwise}, \end{cases} \tag{4.16}$$

where

$$L(\phi(t),\phi'(t)) = \sup_{\xi\in\mathbb{R}}[\phi'(t)\xi - b((\phi(t)) + a\eta(\phi(t)))\xi - \frac{1}{2}\sigma^2(\phi(t))\xi^2 - \int_{\mathbb{R}\setminus 0}(e^{y\eta(\phi(t))\xi} - 1 - y\eta(\phi(t))\xi\chi_{\{|y|\le 1\}})\nu(dy)].$$

Remark 4.8. The action functionals (4.15) and (4.16) are the same. The Lévy symbol ψ is twice differentiable because $(L_t)_{t\ge 0}$ has finite exponential moments. Then $\xi \mapsto H(x,\xi)$ is twice differentiable and so is its Legendre transform $\zeta \mapsto L(x,\zeta)$.

5. An Example

We now present the action functional for a simple stochastic differential equation with non-Gaussian Lévy noise.

Example 5.1. Let $U:\mathbb{R}\to\mathbb{R}$ be a smooth enough function with a global point of minimum $0\in\mathbb{R}$. We consider the stochastic differential equation

$$dX_t^\varepsilon = -\nabla U(X_t^\varepsilon)dt + \varepsilon dL_t^\varepsilon,\ \ X_0^\varepsilon = 0.$$

Here, the scaled Lévy process L^ε is given by

$$L_t^\varepsilon = \int_0^t\int_{\mathbb{R}} z\tilde{N}^{\frac{1}{\varepsilon}}(ds,dz),\ t\in[0,1],$$

where $\tilde{N}^{\frac{1}{\varepsilon}}$ is the compensated Poisson random measure with compensator $\frac{1}{\varepsilon}ds\bigotimes\nu$. The intensity measure ν has the form

$$\nu(dz) = e^{-|z|^\alpha}dz, \quad \text{for some } \alpha > 0.$$

The action functional of $(X_t^\varepsilon)_{t\in[0,1]}$ is

$$S(\phi) := \begin{cases} \inf\{\int_0^1\int_{\mathbb{R}}(g(t,z)\ln g(t,z) - g(t,z) + 1)\nu(dz)dt\}, & \phi\in AC[0,1], \phi(0)=0, \\ \infty, & \text{otherwise,} \end{cases}$$

such that

$$\phi(t) = -\int_0^t \nabla U(\phi(s))ds + \int_0^t\int_{\mathbb{R}} z(g(s,z)-1)\nu(dz)ds.$$

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