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Commutativity of Semiprime Rings with Left Generalized Derivations

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Abstract: Let R be an associative ring. An additive mapping $f: R \to R$is called a left generalized derivation if there exists a derivation $d: R \to R$ such that $f(xy) = xf(y) + d(x)y$, for all $x, y \in R$. In this paper, we extended some results on the commutativity of semiprime rings admitting left generalized derivations f and g such that one the following holds for all$x, y \in R$. Let (f, d) and (g, h) be two left generalized derivations of R. (i) $f(x)y = xg(y)$, (ii) $f([x, y]) = \mp[x, y]$, (iii) $f(xoy) = \mp xoy$ for all $x, y \in R$.

Keywords: Derivations, Generalized derivations, Left generalized derivations, Centralizing mapping, Semiprime rings, Prime rings, Commutativity.

Preliminaries: Throughout this paper R will be denote an associative ring with the center $Z(R)$. For any $x, y \in R$, the symbol $[x, y]$ stands for commutator $xy - yx$ and the symbol xoy denotes the anticommutator $xy + yx$. Recall that a ring R is called prime if for any $x, y \in R$, $xRy = \{0\}$ implies that either $x = 0$ or $y = 0$. And R is a semiprime if $xRx = \{0\}$ implies $x = 0$. An additive mapping $d: R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. For a fixed $a \in R$, the mapping$I_a: R \to R$ given by $I_a(x) = [a, x]$ is a derivation which is said to be an inner derivation. Let S be a nonempty subset of R.A mapping $f: R \to R$ is called centralizing on S if $[f(x), x] \in Z$ for all $x \in S$ and is called commuting on S if $[F(x), x] = 0$ for all $x \in S$..An additive mapping $f: R \to R$ is called a generalized derivation if there exists a derivation $d: R \to R$ such that $f(xy) = f(x)y + xd(y)$, for all for all $x, y \in R$. An additive mapping $f: R \to R$ is called a left generalized derivation if there exists a derivation $d: R \to R$ such that $f(xy) = xf(y) + d(x)y$, for all for all $x, y \in R$. Throughout this paper, we make some extensive use of the basic commutator identities,$[x, yz] = y[x, z] + [x, y]z$and $[xy, z] = [x, z]y + x[y, z]$. We denote a left generalized derivation $f: R \to R$ determined by a derivation d of R by (f, d). If $d = 0$ then $f(xy) = f(x)y$ for all $x, y \in R$ and there exists $q \in Q_r(R_c)$ (a right

Martindale ring of quotients) such that $f(x) = qx$, for all $x \in R$ by[7, Lemma2]. So, we assume that $d \neq 0$. For semiprime rings, this implies f is nonzero.

1. Introduction

Bresar [3] introduced the notion of a generalized derivation in rings.Daif and Bell [4] established that if in a semiprime ring R there exists a nonzero ideal of R and derivation d such that $d([x,y]) = [x,y]$ for all $x, y \in I$, then $I \subset Z$. There has been an ongoing interest concerning the relationship between the commutativity of a ring and the existence of certain specific types of derivations of R. Hvala.B [7] initiated the algebraic study of generalized derivation and extended some results concerning derivation to generalized derivations. H.E.Bell et. al., [2] studied centralizing mapping of semi prime rings. Jaya Subba Reddy et. al. [8] studied centralizing and commuting left generalized derivation on prime ring is commutative. Golbasi [6] extended some well known results concerning on commutativity of semiprime rings with generalized derivations. In this paper we can extended some results on commutativity of semiprime rings with left generalized derivations.

In the following we state a well known fact as:

Remark 1: Let R be a prime ring. For a nonzero element $a \in Z$, if $ab \in Z$, then $b \in Z$.

Theorem 1: Let R be a semiprime ring,(f, d) and (g, h) be two left generalized derivations of R. If $f(x)y = xg(y)$ for all $x, y \in R$, then R has a nonzero central ideal.

Proof: By the hypothesis, we get $f(x)y = xg(y)$ and $f(y)x = yg(x)$ for all $x, y \in R$. Combining these two equations, we have

$$f(x)y + f(y)x = xg(y) + yg(x) \text{for all } x, y \in R. \quad (1)$$

Replacing x by yx in equation (1), we obtain

$$f(yx)y + f(y)yx = yxg(y) + yg(yx).$$

$$yf(x)y + d(y)xy + f(y)yx = yxg(y) + yyg(x) + yh(y)x$$

$$yf(x)y + d(y)xy + f(y)yx = y(xg(y) + yg(x)) + yh(y)x$$

Using the hypothesis in the above relation, we get

$$yf(x)y + d(y)xy + f(y)yx = yf(x)y + yf(y)x + yh(y)x$$

$$f(y)yx - yf(y)x = yh(y)x - d(y)xy$$

$$[f(y), y]x = yh(y)x - d(y)xy, \text{ for all } x, y \in R. \quad (2)$$

Let $r \in R$. Replacing x by rx in equation (2), we have

$$[f(y), y]rx = yh(y)rx - d(y)rxy, \text{ for all } x, y \in R.$$

Using equation (2), we get

$$yh(y)rx - d(y)ryx = yh(y)rx - d(y)rxy.$$

$$d(y)rxy - d(y)ryx = 0$$

$$d(y)r(xy - yx) = 0$$

$$d(y)\, r\, [x\, , y] = 0, \text{for all } x, y, r \in R. \quad (3)$$

Replacing x by $d(y)$ in equation (3) yields $d(y)r[d(y), y] = 0$.

In particular, $d(y)yr[d(y), y] = 0$ and also, $yd(y)r[d(y), y] = 0$.Hence combining the last two relations, we conclude that $[d(y), y]R[d(y), y] = \{0\}$, for all $y \in R$. By the semiprimeness of R, we get $[d(y), y] = 0$, for all $y \in R$. Therefore R has a nonzero central ideal by [2, Theorem 3].

Corollary 1: Let R be a prime ring,(f, d) and (g, h) be two left generalized derivations of R. If $f(x)y = xg(y)$ for all $x, y \in R$, then R is a commutative.

Theorem 2: Let R be a semiprime ring,(f, d) and (g, h) be two left generalized derivations of R. If $f(x)x = xg(x)$ for all $x \in R$, then R has a nonzero central ideal.

Proof: Replacing $x + y$ by x in the hypothesis, we have

$$f(x + y)(x + y) = (x + y)g(x + y)$$

$$\big(f(x) + f(y)\big)(x + y) = (x + y)(g(x) + g(y))$$

$$f(x)y + f(y)x = xg(y) + yg(x), \text{ for all } x, y \in R. \quad (4)$$

Replacing x byyxinequation(4), we get

$$f(yx)y + f(y)yx = yxg(y) + yg(yx)$$

$$yf(x)y + d(y)xy + f(y)yx = yxg(y) + yyg(x) + yh(y)x$$

$$yf(x)y + d(y)xy + f(y)yx = y(xg(y) + yg(x)) + yh(y)x$$

Using equation (4), we obtain

$$[f(y),y]x = yh(y)x - d(y)xy, \text{ for all } x,y \in R. \quad (5)$$

The equation (5) is same as equation (2) in theorem1.Thus, by same argument of theorem 1, we can conclude the result is there. Thus the proof is completed.

Corollary 2: Let R be a prime ring,(f,d) and (g,h) be two left generalized derivations of R. If $f(x)x = xg(x)$ for all $x \in R$, then R is commutative.

Theorem 3: Let R be a semiprime ring, (f,d) be left generalized derivation of R. If (f,d) satisfies one of the following conditions then R has a nonzero central ideal.

(i) $f([x,y]) = [x,y]$for all $x,y \in R$.

(ii) $f([x,y]) = -[x,y]$for all $x,y \in R$.

(iii) For each$x,y \in R$, either $f([x,y]) = [x,y]$ or $f([x,y]) = -[x,y]$.

Proof: (i) For any $x,y \in R$, we have $f([x,y]) = [x,y]$ which gives

$$xf(y) + d(x)y - yf(x) - d(y)x = [x,y], \text{ for all } x,y \in R. \quad (6)$$

Replacing y byzy,$z \in R$ in equation (6), we get

$$xf(zy) + d(x)zy - zyf(x) - d(zy)x = [x,zy]$$

$$xzf(y) + xd(z)y + d(x)zy - zyf(x) - zd(y)x - d(z)yx$$
$$= z[x,y] + [x,z]y$$

$$xzf(y) + xd(z)y + d(x)zy + z(-yf(x) - d(y)x) - d(z)yx$$
$$= z[x,y] + [x,z]y$$

Using equation (6) to substitute $yf(x) + d(y)x$ in the last equation, we obtain

$$xzf(y) + xd(z)y + d(x)zy - zxf(y) - zd(x)y + z[x,y] - d(z)yx$$
$$= z[x,y] + [x,z]y.$$

This can be written as

$$(xz - zx)f(y) + \big(d(x)z - zd(x)\big)y + xd(z)y - d(z)yx = [x,z]y$$

$$[x,z]f(y) + [d(x),z]y + [x,d(z)y] = [x,z]y, \text{ for all } x,y \in R. \quad (7)$$

Replacing x by$z$ in equation (7), we get

$$[z,z]f(y) + [d(z),z]y + [z,d(z)y] = [z,z]y$$

$$d(z)[z,y] + [z,d(z)]y = 0$$

$$d(z)[z,y] = 0 \text{ for all } y,z \in R. \quad (8)$$

Replacing y byyr in equation (8) and using equation (8), we get

$$d(z)[z,yr] = 0$$

$$d(z)y[z,r] + d(z)[z,y]r = 0$$

$$d(z)y[z,r] = 0 \text{ for all} y,z,r \in R.$$

Replacing r by $d(z)$ in this equation, we have $d(z)y[z,d(z)] = 0$. In particular $zd(z)y[z,d(z)] = 0$ and also $d(z)zy[z,d(z)] = 0$. Hence combining the last two equations, we get $[z,d(z)]R[z,d(z)] = \{0\}$. By the semiprimeness of R, we obtain that $[z,d(z)] = 0$, for all $z \in R$ and so, R has a nonzero central ideal by [2,Theorem 3].

(ii) can be proved by using the same techniques.

(iii) For each $x \in R$, we put

$R_x = \{y \in R / f([x,y]) = [x,y]\}$ and $R_x^* = \{y \in R / f([x,y]) = -[x,y]\}$.

The sets of x for which $R = R_x$and $R = R_x^*$ are additive subgroups of$R$, so one must be equal to R and therefore R satisfies (i) or (ii). We have completed the proof.

Corollary 3: Let R be a semiprime ring, (f,d) be left generalized derivation of R. If (f,d) satisfies one of the following conditions then R has a nonzero central ideal.

(i) $f(xy) = xy$for all $x,y \in R$.

(ii) $f(xy) = yx$for all $x,y \in R$.

(iii) For each$x,y \in R$, either $f(xy) = xy$ or $f(xy) = yx$.

Corollary 4: Let R be a prime ring, (f,d) be left generalized derivation of R. If (f,d) satisfies one of the following conditions then R is a commutative.

(i) $f(xy) = xy$for all $x,y \in R$.

(ii) $f(xy) = yx$for all $x,y \in R$.

(iii) For each$x,y \in R$, either $f(xy) = xy$ or $f(xy) = yx$.

Theorem 4: Let R be a semiprime ring, (f,d) be left generalized derivation of R. If (f,d) satisfies one of the following conditions then R has a nonzero central ideal.

(i) $f(xoy) = xoy$for all $x,y \in R$.

(ii) $f(xoy) = -xoy$for all $x,y \in R$.

(iii) For each $x,y \in R$, either $f(xoy) = xoy$ or $f(xoy) = -xoy$.

Proof: (i) Suppose that $f(xoy) = xoy$ for all $x,y \in R$. Then we have

$$f(xy + yx) = xy + yx$$

$$xf(y) + d(x)y + yf(x) + d(y)x = xy + yx, \text{ for all } x,y \in R. \tag{9}$$

Replacing y by xy in equation (9), we get

$$xf(xy) + d(x)xy + xyf(x) + d(xy)x = xxy + xyx$$

$$xxf(y) + xd(x)y + d(x)xy + xyf(x) + xd(y)x + d(x)yx = x^2y + xyx \tag{10}$$

Left multiply equation (9) by x, we obtain

$$xxf(y) + xd(x)y + xyf(x) + xd(y)x = x^2y + xyx \tag{11}$$

Combining equations (10) and (11) , we get

$$d(x)xy + d(x)yx = 0$$

$$d(x)(xy + yx) = 0, \text{ for all } x,y \in R. \tag{12}$$

Replacing y by yr in equation (12) yields that $d(x)(xyr + yrx) = 0$. Right multiply equation (12) by$r$, we have $d(x)(xyr + yxr) = 0$. Combining the last two equations, we find that $d(x)y[x,r] = 0$, for all $x,y,r \in R$. As in the proof of theorem 3(i), we can see that d is commuting on R. So, R has a nonzero central ideal by [2, theorem 3].

(ii) Can be proved by using the same techniques.

(iii) The argument used to prove Theorem 3(iii) works here as well.

Corollary 5: Let R be a semiprime ring, (f,d) be a left generalized derivation of R. If (f,d) satisfies one of the following conditions then Rhas a nonzero central ideal or R is commutative.

(i) $f(x^2) = x^2$ for all $x \in R$.

(ii) $f(x^2) = -x^2$ for all $x \in R$.

Proof: (i) By the hypothesis $f((x+y)^2) = (x+y)^2$ for all $x, y \in R$. That is

$$f(x^2 + xy + yx + y^2) = x^2 + xy + yx + y^2$$

$$f(x^2) + f(xy + yx) + f(y^2) = x^2 + xy + yx + y^2$$

$$x^2 + f(xy + yx) + y^2 = x^2 + xy + yx + y^2$$

$$f(xy + yx) = xy + yx$$

$$f(xoy) = xoy, \text{ for all } x, y \in R.$$

Now, if $f = 0$, then $oy = xy + yx = 0$, for all $x, y \in R$. Replacing y by yz in this relation, we get $xyz + yzx = 0$ and also $y[x, z] = 0$,for all $x, y \in R$. Since R is semiprime ring, we get $[x, z] = 0$, and so R is commutative. Hence, we may assume $f \neq 0$. By Theorem 4(i), R has a nonzero central ideal.

(ii) Can be proved by using the same techniques.

We now propose to extend these results on Ua nonzero ideal of semiprimering R. Parallel results are to be obtained using the same techniques.

Theorem 5: Let R be a semiprime ring, U a nonzero ideal of R, (f, d) and (g, h) be two left generalized derivations of R such that $d(U) \neq 0$. If $f(x)y = xg(y)$ for all $x, y \in U$, then R contains a nonzero central ideal.

Proof: By the hypothesis, we get $f(x)y = xg(y)$ and $f(y)x = yg(x)$ for all $x, y \in U$. Combining these two equations, we have

$$f(x)y + f(y)x = xg(y) + yg(x) \text{for all } x, y \in U. \tag{13}$$

Replacing $x$by yxin equation (13), we get

$$f(yx)y + f(y)yx = yxg(y) + yg(yx).$$

$$yf(x)y + d(y)xy + f(y)yx = yxg(y) + yyg(x) + yh(y)x$$

$$yf(x)y + d(y)xy + f(y)yx = y(xg(y) + yg(x)) + yh(y)x$$

Using the hypothesis in the above relation, we get

$$yf(x)y + d(y)xy + f(y)yx = yf(x)y + yf(y)x + yh(y)x$$

$$f(y)yx - yf(y)x = yh(y)x - d(y)xy$$

$$[f(y), y]x = yh(y)x - d(y)xy, \text{ for all } x, y \in U. \quad (14)$$

Let $u \in U$. Replacing x by ux in equation (14), we have

$$[f(y), y]ux = yh(y)ux - d(y)uxy, \text{ for all } x, y \in U.$$

Using equation (14), we get

$$yh(y)ux - d(y)uyx = yh(y)ux - d(y)uxy.$$

$$d(y)uxy - d(y)uyx = 0$$

$$d(y)u(xy - yx) = 0$$

$$d(y)\, u\, [x\, , y] = 0, \text{for all } x, y \in U.$$

So that

$$[d(y), y]\, U\, [x\, , y] = \{0\}, \text{for all } x, y \in U. \quad (15)$$

Replacing x by $d(y)x$ in equation (15), we obtain

$$[d(y), y]U[d(y)x, y] = \{0\}.$$

$$[d(y), y]U[d(y), y]x + [d(y), y]Ud(y)[x, y] = \{0\}$$

$[d(y), y]U[d(y), y]U = \{0\}$, so that $[d(y), y]U$ is a nilpotent right ideal, hence is trivial. It follows that $[d(y), y]U[d(y), y] = \{0\}$, for all $y \in U$. But an ideal of semiprime ring is a semiprime ring, and $[d(y), y] \in U$, for all $y \in U$. Therefore $[d(y), y] = 0$ for all$y \in U$, and theorem follows by [2, Theorem 3].

As an immediate consequence of the theorem, we have

Corollary 6: Let R be a prime ring, U a nonzero ideal of R, (f, d) and (g, h) be two left generalized derivations of R such that $d(U) \neq 0$. If $f(x)y = xg(y)$ for all $x, y \in U$, then R is commutative.

Theorems 2, 3 and 4 have similar extensions.

Theorem 6: Let R be a semiprime ring, U a nonzero ideal of R, (f, d) and (g, h) be two left generalized derivations of R such that $d(U) \neq 0$. If $f(x)x = xg(x)$ for all $x \in U$, then R contains a nonzero central ideal.

Corollary 7: Let R be a prime ring, U a nonzero ideal of$R$, (f, d) and (g, h) be two left generalized derivations of R such that $d(U) = \{0\}$. If $f(x)x = xg(x)$ for all $x \in U$, then R is commutative.

Theorem 7: Let R be a semiprime ring, U a nonzero ideal of R, (f,d) be a left generalized derivation of R. If (f,d) satisfies one of the following conditions then R has a nonzero central ideal.

(i) $f([x,y]) = [x,y]$for all $x,y \in U$.

(ii) $f([x,y]) = -[x,y]$for all $x,y \in U$.

(iii) For each $x,y \in U$, either $f([x,y]) = [x,y]$ or $f([x,y]) = -[x,y]$.

Corollary 8: Let R be a semiprime ring, U a nonzero ideal of R, (f,d) be a left generalized derivation of R. If (f,d) satisfies one of the following conditions then R has a nonzero central ideal.

(i) $f(xy) = xy$for all $x,y \in U$.

(ii) $f(xy) = yx$for all $x,y \in U$.

(iii) For each $x,y \in U$, either $f(xy) = xy$ or $f(xy) = yx$.

Corollary 9: Let R be a prime ring, U a nonzero ideal of R, (f,d) be a left generalized derivation of R. If (f,d) satisfies one of the following conditions then R is commutative.

(i) $f(xy) = xy$for all $x,y \in U$.

(ii) $f(xy) = yx$for all $x,y \in U$.

(iii) For each $x,y \in U$, either $f(xy) = xy$ or $f(xy) = yx$.

Theorem 8: Let R be a semiprime ring, U anonzero ideal of R, (f,d) be a left generalized derivation of R. If (f,d) satisfies one of the following conditions then R has a nonzero central ideal.

(i) $f(xoy) = xoy$for all $x,y \in U$.

(ii) $f(xoy) = -xoy$for all $x,y \in U$.

(iii) For each $x,y \in U$, either $f(xoy) = xoy$ or $f(xoy) = -xoy$.

Corollary 10: Let R be a semiprime ring, U a nonzero ideal of R, (f,d) be a left generalized derivation of R. If (f,d) satisfies one of the following conditions then R has a nonzero central ideal.

(i) $f(x^2) = x^2$for all $x \in U$.

(ii) $f(x^2) = -x^2$for all $x \in U$.

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