# GENERALIZED FIELD OPERATOR ASSOCIATED TO THE FRACTIONAL LÉVY PROCESSES

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ABSTRACT. Using the Kolmogorov decomposition approach for a positive definite kernel, we establish a unitary fractional isomorphism $\mathcal{U}_L^\alpha$ between the Fock space $\Gamma(\mathcal{S}_p \oplus L^2(\nu_\alpha))$ and the fractional Lévy white noise space $L^2(\mathcal{S}', \mu_L^\alpha)$. As a consequence, under the second-order moment condition of the fractional Lévy white noise measure $\nu_\alpha$, we derive explicitly the generalized fractional Lévy white noise field operator in terms of creation, annihilation and preservation operators.

## 1. Introduction

The development of the fractional Lévy white noise calculus is an extension of the classical white noise theory introduced by T. Hida [17] in 1976. During the last decade, many authors used this theory for modeling driving noises in different applications such as mathematical finance, network traffic analysis and quantum probability. Among them, Hu and Øksendal [18] constructed an Itô fractional Black-Scholes model with an European option and proved that the corresponding market is complete, Huang [19] used the characteristic functionals on Hilbertian nuclear spaces to construct an infinitely divisible distributions on Gel'fand triple.

The present paper deals with the fractional Lévy white noise calculus. The first natural problems arises: which Fock space structure serves for the analogs of the Wiener-Itô-Segal isomorphism in the fractional Lévy case. The construction presented here have the advantage to be directly connected to the fractional Lévy distributions which enables us to prove the above mentioned results on the chaotic representation. After having established the chaotic property, we will focus on the expression of the generalized fractional Lévy white noise field operator.

The contents of the paper is organized as follows. In Section 2, we recall some basic results about Kolmogrorov decomposition of a positive definite kernel and the fractional Lévy processes on Gel'fand triple. Section 3 is devoted to the study of the fractional Lévy white noise functionals with special emphasis on the Kolmogorov isomorphism associated to the fractional Lévy white noise measure $\mu_L^\alpha$. In particular, the Kolmogorov decomposition for the positive definite kernel $\Psi_\alpha(\eta-\xi)$

play an essential role to construct an unitary isomorphism $\mathcal{U}_L^\alpha$ between the Fock space $\Gamma(\mathcal{S}_p \oplus L^2(\nu_\alpha))$, where $\nu_\alpha$ is a Lévy measure having a finite second order moment, and the fractional Lévy white noise space $L^2(\mathcal{S}', \mu_L^\alpha)$. In section 4, we consider the generalized fractional Lévy white noise field operator $Q_\xi$ defined as the image of the multiplication operator by the random variable $\langle \cdot, \xi \rangle$ under $\mathcal{U}_L^\alpha$. Moreover, the action of $Q_\xi$ on the total set of exponential vector is used to give explicitly this operator in terms of creation, annihilation and preservation operators.

## 2. Frameworks

First we review from the papers [1, 2, 19] and [28] basic concepts, notations and some results which will be needed in the present paper.

Recall that, given a set $\mathcal{X}$, a function $k : (x, y) \in \mathcal{X} \times \mathcal{X} \longrightarrow k(x, y) \in \mathbb{C}$ is called a positive definite $\mathbb{C}$-valued kernel if, for every finite subset $F \subseteq \mathcal{X}$ the complex square matrix

$$k_{ij} := k(x_i, x_j), \quad x_i, x_j \in F$$

is positive definite, i.e. if for all $d \in \mathbb{N}^*$, $x_1, \cdots, x_d \in \mathcal{X}$ and $\lambda_1, \cdots, \lambda_d \in \mathbb{C}$, one has

$$\sum_{i,j=1}^{d} \lambda_i \bar{\lambda}_j k(x_i, x_j) \geq 0. \tag{2.1}$$

Moreover $k$ is called conditionally positive definite if (2.1) holds whenever the $\lambda_j$'s satisfy the additional condition

$$\sum_{j=1}^{d} \lambda_j = 0.$$

**Definition 2.1.** A positive definite kernel $k$ is called *infinitely divisible* if for each $n \in \mathbb{N}$ there exists a positive definite kernel $k_n$ such that $k = (k_n)^n$.

It is well known (see [28]) that a positive definite kernel $k \geq 0$ on $\mathcal{X} \times \mathcal{X}$ is infinitely divisible kernel $k$ on $\mathcal{X}$ if and only if, for all $t > 0$ the kernel $k^t$ defined by

$$k^t(x, y) := \left( k(x, y) \right)^t$$

is positive definite.

**Proposition 2.2.** *(see [2]) A $\mathbb{C}$–valued kernel $k$ on a set $\mathcal{X}$ is positive definite if and only if there exists an Hilbert space $\mathcal{H}$ and a map*

$$e. : \mathcal{X} \ni x \longmapsto e_x \in \mathcal{H}$$

*such that the following two conditions are satisfied:*

$$k(x, y) = \langle e_x, e_y \rangle_{\mathcal{H}}, \quad \forall x, y \in \mathcal{X} \tag{2.2}$$

*and*

$$\{e_x, x \in \mathcal{X}\} \text{ is total in } \mathcal{H}. \tag{2.3}$$

*The pair $(\mathcal{H}, e.)$ is unique up to unitary isomorphism and is called the Kolmogorov decomposition of kernel $k$.*

It is known that the bosonic Fock space $\Gamma(\mathcal{H})$ can be represented in the form

$$\Gamma(\mathcal{H}) := \bigoplus_{n=0}^{+\infty} \mathcal{H}^{\odot n},$$

where $\mathcal{H}^{\odot n}$ denotes the $n$–th symmetric tensor power of $\mathcal{H}$. For $f \in \mathcal{H}$, we denote by $\mathrm{Exp}(f)$ the exponential vector defined on $\Gamma(\mathcal{H})$ and given by

$$\mathrm{Exp}(f) := \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} f^{\odot n}. \qquad (2.4)$$

**Theorem 2.3.** *(see* [2]*) For a kernel $k$ on a set $\mathcal{X}$ the following statements are equivalent :*

(i) *$k$ is infinitely divisible positive definite.*

(ii) *there exists a conditionally positive definite kernel $q_0$ such that $k$ has the form*

$$k(f,g) = e^{q_0(f,g)}, \quad f,g \in S.$$

(iii) *if there exists a positive definite kernel $q$ on $\mathcal{X}$ and a map $\kappa : \mathcal{X} \ni f \longmapsto \kappa_f \in \mathbb{C}$ such that, denoting $(\mathcal{H}, v) = Kol(k)$ (resp. $(\mathcal{K}, u) = Kol(q)$) the Kolmogorov decomposition of $k$ (resp. $q$), then the map*

$$\mathcal{U} : \mathrm{Exp}(u_f) \in \Gamma(\mathcal{K}) \longmapsto e^{\kappa_f} v_f \in \mathcal{H} \qquad (2.5)$$

*extends to a unitary isomorphism between $\mathcal{H}$ and the Fock space $\Gamma(\mathcal{K})$ over $\mathcal{K}$.*

For a fixed $f_0 \in S$, we observe that the kernel $q$ can be given in term of $q_0$ by

$$q(f,g) = q_0(f,g) - q_0(f,f_0) - q_0(f_0,g). \qquad (2.6)$$

Let $\mathcal{S}(\mathbb{R})$ be the space of rapidly decreasing functions equipped with the canonical topology, and $\mathcal{S}'(\mathbb{R})$ its dual space, i.e., the space of tempered distributions. The real Gel'fand triple:

$$\mathcal{S}(\mathbb{R}) \subset L^2_{\mathbb{R}}(\mathbb{R}, dt) \subset \mathcal{S}'(\mathbb{R}) \qquad (2.7)$$

is our starting point. Since the inner product of $L^2_{\mathbb{R}}(\mathbb{R}, dt)$ and the canonical bilinear form on $\mathcal{S}'(\mathbb{R}) \times \mathcal{S}(\mathbb{R})$ are compatible, they are denoted by the same symbol $\langle \cdot, \cdot \rangle$. For simplicity, the complexification of (2.7) is denoted by

$$\mathcal{S} \subset \mathcal{H} := L^2_{\mathbb{C}}(\mathbb{R}, dt) \subset \mathcal{S}'. \qquad (2.8)$$

(Throughout this paper $L^2(\cdots)$ means the complex $L^2$-space.) The canonical $\mathbb{C}$-bilinear form on $\mathcal{S} \times \mathcal{S}'$ is denoted again by $\langle \cdot, \cdot \rangle$ so the norm of $\mathcal{H}$, denoted by $\| \cdot \|_0$, satisfies $\|\xi\|_0^2 = \langle \overline{\xi}, \xi \rangle$ for $\xi \in \mathcal{H}$.

It is well known that the topology of $\mathcal{S}$ is defined by means of the differential operator $A = 1 + t^2 - d^2/dt^2$ acting in $\mathcal{H}$. For each $p \geq 0, \mathcal{S}_p = Dom(A_p)$ becomes a Hilbert space with norm $\|\xi\|_p = \|A^p \xi\|_0$ and $\mathcal{S}_{-p}$ denotes the completion of $\mathcal{H}$ with respect to the norm $\|\xi\|_{-p} = \|A^{-p} \xi\|_0$. Then we obtain a chain of Hilbert spaces:

$$\cdots \subset \mathcal{S}_p \subset \cdots \subset \mathcal{S}_0 := \mathcal{H} \subset \cdots \mathcal{S}_{-p} \subset \cdots$$

Note that $\mathcal{S}_{-p}$ is identified with the strong dual space of $\mathcal{S}_p$ through the canonical $\mathbb{C}$-bilinear form. Finally, we have topological isomorphisms:

$$\mathcal{S} \cong \underset{p \to \infty}{\text{proj} \lim} \mathcal{S}_p, \qquad \mathcal{S}' \cong ind \lim_{p \to \infty} \mathcal{S}_{-p}.$$

Let $X = \{X_t, t \geq 0\}$ be an $\mathcal{S}'$-valued Lévy process such that the characteristic function of the random variable $X_1$ ( with distribution $\mu_L$) take the form

$$\widehat{\mu}_L(\xi) = \exp\left\{ -\frac{1}{2}\langle B\xi, \xi\rangle + \int_{\mathcal{S}'} [e^{i\langle x,\xi\rangle} - 1 - i\langle x,\xi\rangle]\nu(dx)\right\}, \quad \forall \xi \in \mathcal{S},$$

where $\langle \cdot, \cdot\rangle$ is the $\mathcal{S}' - \mathcal{S}$ dual pairing, $B \in \mathcal{L}^+(\mathcal{S}, \mathcal{S}')$ i.e., $B$ is a continuous positive definite $\mathcal{S}'$-valued operator acting on the Schwartz space $\mathcal{S}$ and $\nu$ is a Lévy measure on $\mathcal{S}'$ satisfying that there exists $p > 0$, such that $\nu$ is supported in $\mathcal{S}_{-p}$ and

$$\int_{\mathcal{S}'} |x|^2_{-p}\,\nu(dx) < \infty. \tag{2.9}$$

(for more details about Lévy processes on Gel'fand triple, see [19]).

Now, for a given infinitely divisible distribution $\mu_L$ on $\mathcal{S}'$ we will gives the associated fractional Lévy white noise measure $\mu_L^\alpha$. First of all, recalling that for $0 < \alpha < \frac{1}{2}$, the Riemann-Liouville fractional integral operator $I_\alpha$ is defined by

$$(I_\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty f(t)(t-x)^{\alpha-1}dt$$

if the integrals exist for almost all $x \in \mathbb{R}$.

**Proposition 2.4.** *(see* [19]*) There exists an infinitely divisible probability measure* $\mu_L^\alpha$ *on* $(\mathcal{S}', \mathcal{B}(\mathcal{S}'))$ *such that its characteristic function is given by*

$$\widehat{\mu}_L^\alpha(\xi) = \exp\left\{\Psi_\alpha(\xi)\right\}, \tag{2.10}$$

*with*

$$\Psi_\alpha(\xi) = -\frac{1}{2}\langle B_\alpha\xi, \xi\rangle + \int_{\mathcal{S}'} [e^{i\langle x,\xi\rangle} - 1 - i\langle x,\xi\rangle]\nu_\alpha(dx), \tag{2.11}$$

*where* $B_\alpha = \|I_\alpha\chi_{[0,1]}\|_0^2 B$ *and for any* $A \in \mathcal{B}(\mathcal{S}')$

$$\nu_\alpha(A) = \int_{\mathbb{R}} \int_{\mathcal{S}'} \chi_A(I_\alpha\chi_{[0,1]}(s)x)\nu(dx)ds.$$

*The distribution* $\mu_L^\alpha$ *is then called the Fractional Lévy white noise measure and* $(\mathcal{S}', \mathcal{B}(\mathcal{S}'), \mu_L^\alpha)$ *will serve as the underlying probability space in our study.*

## 3. The Kolmogorov Isomorphism Associated to the Fractional Lévy White Noise Measure

In the following, we fix the measure $\mu_L^\alpha$ defined via its Fourier transform in Eq. (2.10). By direct computation, we find that the kernel $q_0$ on $\mathcal{S}$ defined by

$$q_0(\xi, \eta) := \Psi_\alpha(\eta - \xi)$$

is conditionally positive definite. According to the Schoenberg correspondence [6, Theorem 1.1.13], the kernel

$$k(\xi, \eta) := e^{\Psi_\alpha(\eta - \xi)}$$

is positive definite.

**Lemma 3.1.** *There exists an index $q$ such that for any $p \geq q$, there exists a positive definite self-adjoint operator $\sqrt{B_\alpha}$ on $\mathcal{S}_p$ such that*

$$\langle B_\alpha \phi, \psi \rangle = \left\langle \sqrt{B_\alpha}\phi, \sqrt{B_\alpha}\psi \right\rangle_{\mathcal{S}_p}, \quad \forall \phi, \psi \in \mathcal{S}_p.$$

*Proof.* Let $f(\phi, \psi) = \langle B_\alpha \phi, \psi \rangle$ for $\phi, \psi \in \mathcal{S}$. Then we can see that there exist $C > 0$ and $q \geq 0$ such that

$$f(\phi, \phi) \leq C\|\phi\|_q^2, \quad \forall \phi \in \mathcal{S}.$$

Hence

$$|f(\phi, \psi)| \leq C\|\phi\|_q\|\psi\|_q \leq C\|\phi\|_p\|\psi\|_p, \quad \forall \phi, \psi \in \mathcal{S}, \ p \geq q.$$

Therefore $f$ can be extended to become a symmetric continuous bilinear form on $\mathcal{S}_p \times \mathcal{S}_p$. As $f(\phi, \cdot) \in \mathcal{S}_{-p}$ for any $\phi \in \mathcal{S}_p$, it follows from Riesz's representation theorem that there exist $B_\alpha \phi \in \mathcal{S}_p$ such that

$$f(\phi, \psi) = \langle B_\alpha \phi, \psi \rangle_{\mathcal{S}_p}, \quad \forall \psi \in \mathcal{S}_p.$$

Finally, using the fact that $B_\alpha$ is a positive definite self-adjoint operator on $\mathcal{S}_p$ and hence $\sqrt{B_\alpha}$ is well-defined and we get

$$f(\phi, \psi) = \left\langle \sqrt{B_\alpha}\phi, \sqrt{B_\alpha}\psi \right\rangle_{\mathcal{S}_p}, \quad \forall \phi, \psi \in \mathcal{S}_p. \qquad \square$$

**Notations 3.2.** *Let $\xi \in \mathcal{S}$, we shall use the following notations:*

(i) *$e_\xi \in L^2(\mu_L^\alpha)$ is the function associated to $\xi$ given by*

$$e_\xi(x) := e^{i\langle x,\xi \rangle}, \quad x \in \mathcal{S}' \tag{3.1}$$

(ii) *$\rho_\xi$ be the function defined by*

$$\rho_\xi(x) := e^{i\langle x,\xi \rangle} - 1 \tag{3.2}$$

*and*

$$\mathcal{K}_0 := \text{closed linear span of } \{\rho_\xi \ , \ \xi \in \mathcal{S}\} \subseteq L^2(\nu_\alpha). \tag{3.3}$$

(iii) *$u_\xi$ the vector in $\mathcal{S}_p \oplus \mathcal{K}_0$ given by:*

$$u_\xi = \sqrt{B_\alpha}\xi + \rho_\xi \in \mathcal{S}_p \oplus \mathcal{K}_0. \tag{3.4}$$

**Theorem 3.3.** *The linear operator $\mathcal{U}_\alpha$ such that for all $\xi \in \mathcal{S}$,*

$$\mathcal{U}_\alpha : \Gamma(\mathcal{S}_p \oplus \mathcal{K}_0) \ni e^{\Psi_\alpha(\xi)}\text{Exp}(u_\xi) \longmapsto e_\xi \in L^2(\mu_L^\alpha) \tag{3.5}$$

*is a unitary isomorphism from the Fock space $\Gamma(\mathcal{S}_p \oplus \mathcal{K}_0)$, over $\mathcal{S}_p \oplus \mathcal{K}_0$, onto $L^2(\mu_L^\alpha)$.*

*Proof.* In the notations of Theorem 2.3 we choose:

$$\mathcal{X} = \mathcal{S}, \qquad k(\xi, \eta) = e^{\Psi_\alpha(\eta - \xi)}, \qquad q_0(\xi, \eta) = \Psi_\alpha(\eta - \xi).$$

Then, using the expression (2.11) for the Lévy-Khintchine function, one has

$$
\begin{aligned}
q_0(\xi, \eta) &= \Psi_\alpha(\eta - \xi) \\
&= -\frac{1}{2}\langle B_\alpha(\eta - \xi), \eta - \xi\rangle + \int_{\mathcal{S}'}[e^{i\langle x, \eta - \xi\rangle} - 1 - i\langle x, \eta - \xi\rangle]\nu_\alpha(dx).
\end{aligned}
$$

Thus, using Lemma 3.1 with the above choices, the kernel $q$ defined by (2.6) is given by

$$
\begin{aligned}
q(\xi, \eta) &= \Psi_\alpha(\eta - \xi) - \Psi_\alpha(-\xi) - \Psi_\alpha(\eta) \\
&= \frac{1}{2}\langle B_\alpha\xi, \eta\rangle + \frac{1}{2}\langle \xi, B_\alpha\eta\rangle + \int_{\mathcal{S}'}\overline{\rho_\xi(x)}\rho_\eta(x)\nu_\alpha(dx) \\
&= \left\langle \sqrt{B_\alpha}\xi, \sqrt{B_\alpha}\eta\right\rangle_{\mathcal{S}_p} + \int_{\mathcal{S}'}\overline{\rho_\xi(x)}\rho_\eta(x)\nu_\alpha(dx), \qquad (3.6)
\end{aligned}
$$

where $\rho_\xi(x) = e^{i\langle x, \xi\rangle} - 1$. The right hand side of (3.6) suggests a natural choice for a Kolmogorov decomposition of the kernel $q$. On the other hand there exists a subset $\mathcal{S}_0 \subseteq \mathcal{S}$ such that $\{\rho_\xi, \xi \in \mathcal{S}_0\}$ is a linearly independent. Then one can see that the first term of the sum is a scalar product on $\mathcal{S}_p$ and the second, due to the linear independence of the $\rho_\xi, \xi \in S_0$ extends to a scalar product on the space $\mathcal{K}_0$, defined by (3.3). The complexification of the inner product (3.6) gives a scalar product on the space

$$
\mathcal{K} := \mathcal{S}_p \oplus \mathcal{K}_0 \qquad (3.7)
$$

with inner product

$$
\langle\cdot, \cdot\rangle_\mathcal{K} := \langle\cdot, \cdot\rangle := \langle\cdot, \cdot\rangle_{\mathcal{S}_p} + \langle\cdot, \cdot\rangle_{L^2(\nu_\alpha)}. \qquad (3.8)
$$

From the definition of $\mathcal{K}_0$ it is clear that the range of the map (3.4) is total in $\mathcal{S}_p \oplus \mathcal{K}_0$. Therefore the pair $(\mathcal{K}, u.)$ defined respectively by (3.7) and (3.4) is a Kolmogorov decomposition of the kernel $q$. Passing to the exponential space $\Gamma(\mathcal{K})$ of $\mathcal{K}$ the exponential kernel of the scalar product (3.8) is:

$$
\left\langle \mathrm{Exp}(u_\xi), \mathrm{Exp}(u_\eta)\right\rangle = e^{\langle u_\xi, u_\eta\rangle} = e^{q(\xi, \eta)}.
$$

On the other hand we have

$$
e^{\Psi_\alpha(\eta - \xi)} = \int_{\mathcal{S}'} e^{-i\langle x, \xi\rangle}e^{i\langle x, \eta\rangle}\mu_L^\alpha(dx) = \hat{\mu}_L^\alpha(\eta - \xi) = \langle e_\xi, e_\eta\rangle_{L^2(\mu_L^\alpha)}
$$

and the family $\{e_\xi, \xi \in \mathcal{S}\}$ is total in $L^2(\mu_L^\alpha)$. It follows that, if we define the linear map

$$
\mathcal{U}_\alpha : \Gamma(\mathcal{K}) = \Gamma\big(\mathcal{S}_p \oplus \mathcal{K}_0\big) \longrightarrow L^2(\mu_L^\alpha)
$$

by linear extension of

$$
\mathcal{U}_\alpha\big(e^{\Psi_\alpha(\xi)}\mathrm{Exp}(u_\xi)\big) = e_\xi, \quad \xi \in \mathcal{S},
$$

then we get

$$\langle e_\xi, e_\eta \rangle_{L^2(\mu_L^\alpha)} = \widehat{\mu}_L^\alpha(\eta - \xi) = e^{\Psi_\alpha(\eta - \xi)}$$
$$= e^{q(\xi,\eta) + q_0(\xi,0) + q_0(0,\eta)}$$
$$= e^{\langle u_\xi, u_\eta \rangle + \overline{\Psi_\alpha(\xi)} + \Psi_\alpha(\eta)}$$
$$= \left\langle e^{\Psi_\alpha(\xi)} \mathrm{Exp}(u_\xi), e^{\Psi_\alpha(\eta)} \mathrm{Exp}(u_\eta) \right\rangle_{\Gamma(\mathcal{K})}. \qquad \square$$

**Lemma 3.4.** *If the Lévy measure $\nu_\alpha$ has finite second order moment, i.e.,*

$$\int_{\mathcal{S}'} |\langle x, \eta \rangle|^2 \nu_\alpha(dx) < +\infty, \tag{3.9}$$

*then $\mathcal{K}_0 = L^2(\nu_\alpha)$.*

*Proof.* Let $f \in L^2(\nu_\alpha)$ satisfy

$$\langle \rho_\xi, f \rangle = \int_{\mathcal{S}'} \overline{\rho_\xi(x)} f(x) \nu_\alpha(dx) = 0, \tag{3.10}$$

and consider the function

$$F(\xi) := \int_{\mathcal{S}'} \ell_\xi(x) \nu_\alpha(dx)$$

where $\ell_\xi(x) = (e^{-i\langle x, \xi \rangle} - 1) f(x)$. For $\eta \in \mathcal{S}$, one can see that the Gâteaux derivative of $F$ in direction $\eta$ is given by

$$D_\eta F(\xi) = \lim_{t \to 0} \frac{F(\xi + t\eta) - F(\xi)}{t} := \frac{d}{dt}\Big|_{t=0} G_{\xi,\eta}(t),$$

where $G_{\xi,\eta}(t) = F(\xi + t\eta)$. To prove that $t \longmapsto G_{\xi,\eta}(t)$ is derivable it is sufficient to check the two following conditions

(i) $t \longmapsto \ell_{\xi + t\eta}(x)$ is derivable on $\mathbb{R}$ for $\nu_\alpha$-a.e. $\xi, \eta \in \mathcal{S}$.
(ii) $\frac{\partial}{\partial t} \ell_{\xi + t\eta}(x)$ exists on $\mathbb{R}$ for $\nu_\alpha$-a.e. $\xi, \eta \in \mathcal{S}$ and $|\frac{\partial}{\partial t} \ell_{\xi + t\eta}(x)|$ is dominated by a $\nu_\alpha$-integrable function $\omega_\eta(x)$, independent of $t$.

Condition $(i)$ is easily checked and we have

$$\left| \frac{\partial}{\partial t} \ell_{\xi + t\eta}(x) \right| = |i\langle x, \eta \rangle e^{-i\langle x, \xi + t\eta \rangle} f(x)| = |\langle x, \eta \rangle f(x)| =: \omega_\eta(x).$$

But

$$\int_{\mathcal{S}'} \omega_\eta(x) \nu_\alpha(dx) \leq \left( \int_{\mathcal{S}'} \langle x, \eta \rangle^2 \nu_\alpha(dx) \right)^{\frac{1}{2}} \left( \int_{\mathcal{S}'} |f(x)|^2 \nu_\alpha(dx) \right)^{\frac{1}{2}} < +\infty.$$

This gives $(ii)$. Then from (3.10), we deduce that $\frac{d}{dt}\Big|_{t=0} G_{\xi,\eta}(t) = 0$. Hence

$$D_\eta F(\xi) = -i \int_{\mathcal{S}'} \langle x, \eta \rangle e^{-i\langle x, \xi \rangle} f(x) \nu_\alpha(dx) = 0$$

which is equivalent to $\hat{\sigma}_\xi(x) = 0$, where the the signed measure $\sigma_\xi$ is given by

$$\sigma_\xi(dx) = \langle x, \xi \rangle f(x) \nu_\alpha(dx).$$

This gives that $\sigma_\xi$ is the null measure which implies that $f = 0$. $\qquad \square$

## 4. Generalized Fractional Lévy White Noise Field Operator

Recall that the *bosonic creation* and *annihilation* operators are defined, on the total set

$$\{v_1 \odot \cdots \odot v_n \in \mathcal{H}^{\odot n}, \ v_1, \ldots, v_n \in \mathcal{H}\}$$

as follows: for $u \in \mathcal{H}$,

$$A^+(u) : v_1 \odot \cdots \odot v_n \in \mathcal{H}^{\odot n} \longmapsto \sqrt{n+1} \, u \odot v_1 \odot \cdots \odot v_n \in \mathcal{H}^{\odot (n+1)}, \quad (4.1)$$

$$A^+(u)\Phi = u,$$

$$A^-(u) : v_1 \odot \cdots \odot v_n \in \mathcal{H}^{\odot n} \longmapsto \frac{1}{\sqrt{n}} \sum_{i=1}^n \langle u, v_i \rangle v_1 \odot \cdots \odot \widehat{v_i} \odot \cdots \odot v_n \in \mathcal{H}^{\odot (n-1)},$$

$$(4.2)$$

$$A^-(u)\Phi = 0,$$

where $\widehat{\ }$ denotes omission of the corresponding variable and $\Phi$ is the vacuum vector.

**Definition 4.1.** The *differential second quantized* $\Lambda(T)$ of a self–adjoint operator $T$ acting on a Hilbert space $\mathcal{H}$ is defined via the Stone theorem by

$$\Gamma(e^{itT}) =: e^{it\Lambda(T)}, \quad t \in \mathbb{R},$$

where for an unitary operator $X, \Gamma(X)$ is the second quantized of $X$.

The creation, annihilation operators and second quantized operator of $T$ act on the domain of the exponential vectors as follows:

$$A^-(u)\mathrm{Exp}(x) := \langle u, x \rangle \mathrm{Exp}(x), \quad A^+(u)\mathrm{Exp}(x) := \frac{d}{ds}\Big|_{s=0} \mathrm{Exp}(x + su) \quad (4.3)$$

and

$$\Gamma(T)\mathrm{Exp}(x) := \mathrm{Exp}(Tx).$$

It follows that if $x \in Dom(T)$

$$\Lambda(T)\mathrm{Exp}(x) = -i\frac{d}{ds}\Big|_{s=0} \mathrm{Exp}(e^{isT}x) = A^+(Tx)\mathrm{Exp}(x). \quad (4.4)$$

**Definition 4.2.** For $\xi \in \mathcal{S}$, let $q_\xi$ be the *multiplication operator* by the random variable $\langle \cdot, \xi \rangle$ in $L^2(\mathcal{S}', \mu_L^\alpha)$, i.e.,

$$(q_\xi f)(x) := \langle x, \xi \rangle f(x), \quad f \in L^2(\mathcal{S}', \mu_L^\alpha) \,, \ x \in \mathcal{S}'.$$

Define the operator $Q_\xi$ on $\Gamma(\mathcal{S}_p \oplus \mathcal{K}_0)$ by

$$Q_\xi := \mathcal{U}_\alpha^{-1} q_\xi \mathcal{U}_\alpha$$

where $\mathcal{U}_\alpha$ is the isomorphism defined by (3.5). Since $\mu_L^\alpha$ is a probability measure on $\mathcal{S}'$, $q_\xi$ is self–adjoint (see [29] Proposition 1, chapter VIII. 3) and

$$e^{itQ_\xi} = \mathcal{U}_\alpha^{-1} e^{itq_\xi} \mathcal{U}_\alpha, \quad t \in \mathbb{R}.$$

Moreover $Q_\xi$ is called the *generalized fractional Lévy white noise field operator*.

**Lemma 4.3.** *The one–parameter unitary group*

$$t \mapsto e^{itQ_\xi}$$

*acts on the total set* $\{\mathrm{Exp}(u_\eta), \eta \in \mathcal{S}\}$ *as follows:*

$$e^{itQ_\xi}\mathrm{Exp}(u_\eta) = e^{\Psi_\alpha(\eta+t\xi)-\Psi_\alpha(\eta)}\mathrm{Exp}(u_{t\xi+\eta}). \qquad (4.5)$$

*Proof.* By using the action of the isomorphism $\mathcal{U}_\alpha$ on the exponential vector we get

$$
\begin{aligned}
e^{itQ_\xi}\mathrm{Exp}(u_\eta) &= \mathcal{U}_\alpha^{-1}e^{itq_\xi}\mathcal{U}_\alpha\mathrm{Exp}(u_\eta) = \mathcal{U}_\alpha^{-1}e^{itq_\xi}\left(e^{-\Psi_\alpha(\eta)}e_\eta\right) \\[2mm]
&= e^{-\Psi_\alpha(\eta)}\mathcal{U}_\alpha^{-1}\left(e^{itq_\xi}e^{i\langle\cdot,\eta\rangle}\right) \\[2mm]
&= e^{-\Psi_\alpha(\eta)}\mathcal{U}_\alpha^{-1}\left(e^{i\langle\cdot,t\xi+\eta\rangle}\right) = e^{-\Psi_\alpha(\eta)}\mathcal{U}_\alpha^{-1}\left(e_{t\xi+\eta}\right) \\[2mm]
&= e^{-\Psi_\alpha(\eta)}e^{\Psi_\alpha(t\xi+\eta)}\mathcal{U}_\alpha^{-1}\left(e^{-\Psi_\alpha(t\xi+\eta)}e_{t\xi+\eta}\right) \\[2mm]
&= e^{-\Psi_\alpha(\eta)}e^{\Psi_\alpha(t\xi+\eta)}\mathrm{Exp}(u_{t\xi+\eta}) \\[2mm]
&= e^{\Psi_\alpha(\eta+t\xi)-\Psi_\alpha(\eta)}\mathrm{Exp}(u_{t\xi+\eta}),
\end{aligned}
$$

which completes the proof. $\qquad\qquad\square$

**Lemma 4.4.** *The following statements are equivalent:*
  (i)   *The second moment of* $\mu_L^\alpha$ *is finite.*
  (ii)  *The vacuum vector is in the domain* $\mathcal{D}(Q_\xi)$ *of* $Q_\xi$.
  (iii) *There exists* $\eta \in \mathcal{S}$ *such that* $\mathrm{Exp}(u_\eta)$ *is in the domain* $\mathcal{D}(Q_\xi)$ *of* $Q_\xi$.
  (iv)  *The total set* $\{\mathrm{Exp}(u_\eta), \eta \in \mathcal{S}\}$ *is in the domain of* $Q_\xi$.

*Proof.* The domain $\mathcal{D}(Q_\xi)$ of the generalized fractional Lévy white noise field operator is defined by

$$\mathcal{D}(q_\xi) := \left\{ F \in L^2(\mu_L^\alpha),\ \langle\cdot,\xi\rangle F \in L^2(\mu_L^\alpha) \right\}.$$

Therefore, given $\eta \in \mathcal{S}$, $\mathrm{Exp}(u_\eta) \in \mathcal{D}(Q_\xi)$ if and only if

$$
\begin{aligned}
+\infty > \|Q_\xi\big(\mathrm{Exp}(u_\eta)\big)\|^2 &= \|\mathcal{U}_\alpha^{-1}Q_\xi\mathcal{U}_\alpha\big(\mathrm{Exp}(u_\eta)\big)\|^2 = \|q_\xi\big(e^{-\Psi_\alpha(\eta)}e_\eta\big)\|^2 \\[2mm]
&= e^{-2\Re(\Psi_\alpha(\eta))}\langle e_\eta, q_\xi^2 e_\eta\rangle \\[2mm]
&= e^{-2\Re(\Psi_\alpha(\eta))}\int_{\mathcal{S}'}\langle x,\xi\rangle^2\mu_L^\alpha(dx) \\[2mm]
&= e^{-2\Re(\Psi_\alpha(\eta))}\langle \Phi, Q_\xi^2\Phi\rangle_{\Gamma(\mathcal{K})},
\end{aligned}
$$

from which the lemma immediately follows. $\qquad\qquad\square$

**Proposition 4.5.** *If the second moment of $\mu_L^\alpha$ is finite, the generalized fractional Lévy white noise field operator $Q_\xi$ acts on the total set $\{\mathrm{Exp}(u_\eta), \ \eta \in \mathcal{S}\}$ as follows:*

$$Q_\xi\big(\mathrm{Exp}(u_\eta)\big) = \big(A^+(h_{\xi,\eta}) + A^-(h_{\xi,\eta}) + \gamma(\xi,\eta)\big)\mathrm{Exp}(u_\eta),$$

*where*

$$h_{\xi,\eta} := -i\sqrt{B_\alpha}\xi + q_\xi e_\eta \qquad and \qquad \gamma(\xi,\eta) = -2\Re(\langle h_{\xi,\eta}, u_\eta\rangle).$$

*Proof.* By using Eq. (2.11) we have

$$\begin{aligned}
\Psi_\alpha(\eta + t\xi) - \Psi_\alpha(\eta) &= t^2\langle B_\alpha\xi, \xi\rangle - t\langle B_\alpha\xi, \eta\rangle \\
&\quad + \int_{\mathcal{S}'} (e^{i\langle x, \eta + t\xi\rangle} - e^{i\langle x, \eta\rangle} - it\langle x, \xi\rangle)\nu_\alpha(dx).
\end{aligned}$$

Thus one can take the derivative at $t = 0$ of equation (4.5) to obtain

$$\begin{aligned}
iQ_\xi\mathrm{Exp}(u_\eta) &= \Big[-\langle B_\alpha\xi, \eta\rangle + \int_{\mathcal{S}'}\Big(i\langle x, \xi\rangle e^{i\langle x, \eta\rangle} - i\langle x, \xi\rangle\Big)\nu_\alpha(dx)\Big]\mathrm{Exp}(u_\eta) \\
&\quad + \frac{d}{dt}\Big|_{t=0}\mathrm{Exp}(u_{\eta+t\xi}). \hspace{4cm} (4.6)
\end{aligned}$$

But with notation $f_{\xi,\eta}(t) = u_{\eta+t\xi} - u_\eta$ one has

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0}\mathrm{Exp}(u_{\eta+t\xi}) &= \sum_{n=0}^{+\infty}\frac{1}{\sqrt{n!}}\lim_{t\to 0}\frac{(u_{\eta+t\xi})^{\otimes n} - (u_\eta)^{\otimes n}}{t} \\
&= \sum_{n=0}^{+\infty}\frac{1}{\sqrt{n!}}\lim_{t\to 0}\sum_{k=1}^{n}\binom{n}{k}\frac{f_{\xi,\eta}(t)}{t}\widehat{\otimes}(f_{\xi,\eta}(t))^{\widehat{\otimes}(k-1)}\widehat{\otimes}(u_\eta)^{\widehat{\otimes}(n-k)} \\
&= \sum_{n=0}^{+\infty}\frac{\sqrt{n}}{\sqrt{(n-1)!}}f'_{\xi,\eta}(0)\widehat{\otimes}(u_\eta)^{\widehat{\otimes}(n-1)}.
\end{aligned}$$

Note that

$$\begin{aligned}
f'_{\xi,\eta}(0) &= \lim_{t\to 0}\frac{u_{\eta+t\xi} - u_\eta}{t} \\
&= \lim_{t\to 0}\frac{\sqrt{B_\alpha}(\eta + t\xi) - \sqrt{B_\alpha}\eta}{t}\oplus\frac{\rho_{\eta+t\xi} - \rho_\eta}{t} \\
&= \sqrt{B_\alpha}\xi \oplus iq_\xi e_\eta \\
&= ih_{\xi,\eta}.
\end{aligned}$$

Then from (4.1) we conclude that

$$\frac{d}{dt}\Big|_{t=0}\mathrm{Exp}(u_{\eta+t\xi}) = \sum_{n=0}^{+\infty}\frac{\sqrt{n}}{\sqrt{(n-1)!}}ih_{\xi,\eta}\widehat{\otimes}(u_\eta)^{\otimes(n-1)} = iA^+(h_{\xi,\eta})\mathrm{Exp}(u_\eta).$$

On the other hand, using the fact that

$$\langle u_\eta, h_{\xi,\eta}\rangle = -i\langle B_\alpha\xi, \eta\rangle + \int_{\mathcal{S}'}\Big(\langle x, \xi\rangle - \langle x, \xi\rangle e^{i\langle x, \eta\rangle}\Big)\nu_\alpha(dx),$$

we get

$$
\begin{aligned}
iQ_\xi \mathrm{Exp}(u_\eta) &= \left.\frac{d}{dt}\right|_{t=0} \mathrm{Exp}(u_{\eta+t\xi}) - i\langle u_\eta, h_{\xi,\eta}\rangle \mathrm{Exp}(u_\eta) \\
&= \left[-2i\Re(\langle h_{\xi,\eta}, u_\eta\rangle) + i\langle h_{\xi,\eta}, u_\eta\rangle\right]\mathrm{Exp}(u_\eta) \\
&\quad + \left.\frac{d}{dt}\right|_{t=0} \mathrm{Exp}(u_{\eta+t\xi}) \\
&= \left(i\gamma(\xi,\eta) + i\langle h_{\xi,\eta}, u_\eta\rangle\right)\mathrm{Exp}(u_\eta) + \left.\frac{d}{dt}\right|_{t=0}\mathrm{Exp}(u_{\eta+t\xi}).
\end{aligned}
$$

Hence we obtain

$$
\begin{aligned}
Q_\xi \mathrm{Exp}(u_\eta) &= \gamma(\xi,\eta)\mathrm{Exp}(u_\eta) + \langle h_{\xi,\eta}, u_\eta\rangle \mathrm{Exp}(u_\eta) + A^+(h_{\xi,\eta})\mathrm{Exp}(u_\eta) \\
&= A^+(h_{\xi,\eta})\mathrm{Exp}(u_\eta) + A^-(h_{\xi,\eta})\mathrm{Exp}(u_\eta) + \gamma(\xi,\eta)\mathrm{Exp}(u_\eta).
\end{aligned}
$$

$\square$

**Theorem 4.6.** *Assume that the second moment of $\mu_L^\alpha$ is finite. Then under the identification*

$$\Gamma(\mathcal{S}_p \oplus L^2(\nu_\alpha)) \equiv \Gamma(\mathcal{S}_p) \otimes \Gamma(L^2(\nu_\alpha))$$

$$\mathrm{Exp}(g \oplus f) \equiv \mathrm{Exp}(g) \otimes \mathrm{Exp}(f), \tag{4.7}$$

*the generalized fractional Lévy white noise field operator $Q_\xi$ takes the form*

$$Q_\xi = Q_{G,\xi,\alpha} \otimes 1 + 1 \otimes Q_{CP,\xi,\alpha},$$

*where*

$$Q_{G,\xi,\alpha} = A^+(-i\sqrt{B_\alpha}\xi) + A^-(-i\sqrt{B_\alpha}\xi)$$

$$Q_{CP,\xi,\alpha} = A^+_{\nu_\alpha}(q_\xi \cdot 1) + A^-_{\nu_\alpha}(q_\xi \cdot 1) + \Lambda_{\nu_\alpha}(q_\xi) \tag{4.8}$$

*and $A^+_{\nu_\alpha}, A^-_{\nu_\alpha}, \Lambda_{\nu_\alpha}$ are respectively the creation, annihilation and preservation operators in the Fock representation of $L^2(\nu_\alpha)$.*

*Proof.* By using Proposition 4.5 and the identification (4.7), we have

$$
\begin{aligned}
Q_\xi \mathrm{Exp}(u_\eta) &= \left.\frac{d}{ds}\right|_{s=0} \mathrm{Exp}(u_\eta + sh_{\xi,\eta}) \\
&\quad + \langle -i\sqrt{B_\alpha}\xi + q_\xi e_\eta, \sqrt{B_\alpha}\eta + \rho_\eta\rangle \mathrm{Exp}(\sqrt{B_\alpha}\eta + \rho_\eta) \\
&\quad - 2\Re(\langle h_{\xi,\eta}, u_\eta\rangle)\mathrm{Exp}(\sqrt{B_\alpha}\eta + \rho_\eta) \\
&= \left.\frac{d}{ds}\right|_{s=0} \mathrm{Exp}\big((\sqrt{B_\alpha}\eta + s(-i\sqrt{B_\alpha}\xi)) + (\rho_\eta + sq_\xi e_\eta)\big) \\
&\quad + \big(\langle -i\sqrt{B_\alpha}\xi, \sqrt{B_\alpha}\eta\rangle + \langle q_\xi e_\eta, \rho_\eta\rangle\big)\mathrm{Exp}(\sqrt{B_\alpha}\eta + \rho_\eta) \\
&\quad - 2\Re(\langle h_{\xi,\eta}, u_\eta\rangle)\mathrm{Exp}(\sqrt{B_\alpha}\eta + \rho_\eta)
\end{aligned}
$$

$$
\begin{aligned}
= \ & \frac{d}{ds}\Big|_{s=0}\Big(\mathrm{Exp}\big(\sqrt{B_\alpha}\eta + s(-i\sqrt{B_\alpha}\xi)\big) \otimes \mathrm{Exp}(\rho_\eta + sq_\xi e_\eta)\Big) \\
& + \Big(\langle -i\sqrt{B_\alpha}\xi,\ \sqrt{B_\alpha}\eta\rangle + \langle q_\xi e_\eta, \rho_\eta\rangle\Big)\mathrm{Exp}\big(\sqrt{B_\alpha}\eta + \rho_\eta\big) \\
& - 2\Re\big(\langle h_{\xi,\eta}, u_\eta\rangle\big)\mathrm{Exp}\big(\sqrt{B_\alpha}\eta + \rho_\eta\big) \\
= \ & \Big(\frac{d}{ds}\Big|_{s=0}\mathrm{Exp}\big(\sqrt{B_\alpha}\eta + s(-i\sqrt{B_\alpha}\xi)\big)\Big) \otimes \mathrm{Exp}(\rho_\eta) \\
& + \mathrm{Exp}\big(\sqrt{B_\alpha}\eta\big) \otimes \Big(\frac{d}{ds}\Big|_{s=0}\mathrm{Exp}(\rho_\eta + sq_\xi e_\eta)\Big) \\
& + \langle -i\sqrt{B_\alpha}\xi, \sqrt{B_\alpha}\eta\rangle \mathrm{Exp}(\sqrt{B_\alpha}\eta) \otimes \mathrm{Exp}(\rho_\eta) \\
& + \langle q_\xi e_\eta, \rho_\eta\rangle \mathrm{Exp}(\sqrt{B_\alpha}\eta) \otimes \mathrm{Exp}(\rho_\eta) \\
& - 2\Re(\langle h_{\xi,\eta}, u_\eta\rangle)\mathrm{Exp}(\sqrt{B_\alpha}\eta) \otimes \mathrm{Exp}(\rho_\eta).
\end{aligned}
$$

Hence by Eq. (4.3) we have

$$
\begin{aligned}
Q_\xi \mathrm{Exp}(u_\eta) = \ & \Big(A^+(-i\sqrt{B_\alpha}\xi)\mathrm{Exp}\big(\sqrt{B_\alpha}\eta\big)\Big) \otimes \mathrm{Exp}(\rho_\eta) \\
& + \mathrm{Exp}\big(\sqrt{B_\alpha}\eta\big) \otimes \Big(A^+_{\nu_\alpha}(q_\xi e_\eta)\mathrm{Exp}(\rho_\eta)\Big) \\
& + \Big(A^-(-i\sqrt{B_\alpha}\xi)\mathrm{Exp}\big(\sqrt{B_\alpha}\eta\big)\Big) \otimes \mathrm{Exp}(\rho_\eta) \\
& + \mathrm{Exp}\big(\sqrt{B_\alpha}\eta\big) \otimes \Big(A^-_{\nu_\alpha}(q_\xi e_\eta)\mathrm{Exp}(\rho_\eta)\Big) \\
& + \mathrm{Exp}\big(\sqrt{B_\alpha}\eta\big) \otimes \Big(-2\Re(\langle h_{\xi,\eta}, u_\eta\rangle)\mathrm{Exp}(\rho_\eta)\Big) \\
= \ & \Big[\big(A^+(-i\sqrt{B_\alpha}\xi) + A^-(-i\sqrt{B_\alpha}\xi)\big)\mathrm{Exp}(\sqrt{B_\alpha}\eta)\Big] \otimes \mathrm{Exp}(\rho_\eta) \\
& + \mathrm{Exp}\big(\sqrt{B_\alpha}\eta\big) \otimes \Big[\big(A^+_{\nu_\alpha}(q_\xi e_\eta) + A^-_{\nu_\alpha}(q_\xi e_\eta) \\
& - 2\Re(\langle h_{\xi,\eta}, u_\eta\rangle)\big)\mathrm{Exp}(\rho_\eta)\Big].
\end{aligned} \tag{4.9}
$$

Notice that in general the constant function $1 \notin L^2(\nu_\alpha)$. However, if the second moment of $\mu_L^\alpha$ is finite, then $q_\xi \cdot 1 \in L^2(\nu_\alpha)$ and by using the fact that

$$
-2\Re\big(\langle h_{\xi,\eta}, u_\eta\rangle\big) = -(\langle q_\xi e_\eta, \rho_\eta\rangle + \langle \rho_\eta, q_\xi e_\eta\rangle),
$$

$$
q_\xi e_\eta = q_\xi \cdot 1 + q_\xi \rho_\eta, \quad \langle \rho_\eta, q_\xi e_\eta\rangle = -\langle q_\xi \cdot 1, \rho_\eta\rangle
$$

and with Eq. (4.4), we obtain

$$
\begin{aligned}
& \big(A^+_{\nu_\alpha}(q_\xi e_\eta) + A^-_{\nu_\alpha}(q_\xi e_\eta) + \gamma(\xi, \eta)\big)\mathrm{Exp}(\rho_\eta) \\
& = A^+_{\nu_\alpha}(q_\xi \cdot 1 + q_\xi \rho_\eta)\mathrm{Exp}(\rho_\eta) + \langle q_\xi e_\eta, \rho_\eta\rangle \mathrm{Exp}(\rho_\eta) \\
& \quad - \Big(\langle q_\xi e_\eta, \rho_\eta\rangle + \langle \rho_\eta, q_\xi e_\eta\rangle\Big)\mathrm{Exp}(\rho_\eta) \\
& = A^+_{\nu_\alpha}(q_\xi \cdot 1)\mathrm{Exp}(\rho_\eta) + \Lambda_{\nu_\alpha}(q_\xi)\mathrm{Exp}(\rho_\eta) + A^-_{\nu_\alpha}(q_\xi \cdot 1)\mathrm{Exp}(\rho_\eta).
\end{aligned}
$$

Finally, the previous equation and (4.9) yields

$$Q_\xi = \left( A^+(-i\sqrt{B_\alpha}\xi) + A^-(-i\sqrt{B_\alpha}\xi) \right) \otimes 1$$
$$+ 1 \otimes \left( A^+_{\nu_\alpha}(q_\xi \cdot 1) + A^-_{\nu_\alpha}(q_\xi \cdot 1) + \Lambda_{\nu_\alpha}(q_\xi) \right)$$
$$= Q_{G,\xi,\alpha} \otimes 1 + 1 \otimes Q_{CP,\xi,\alpha}. \qquad \square$$

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