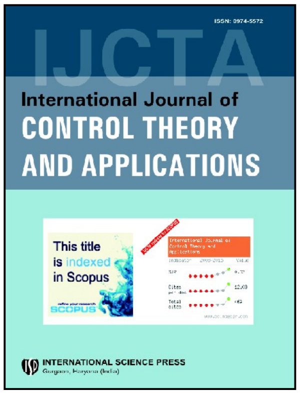

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An Unreliable Batch Arrival Feedback Retrial Queue with Multi Optional stages of Service, Optional Re-service and Delaying Repair under Modified Vacation Policy

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Abstract: This paper deals with the steady state analysis ofbatch arrival retrial queue with $k$optional stages of service under modifiedvacation policy, where each type of service consists of an optional re-service (of the same stage) without joining the orbit with probability r_i or may leave the system with probability $1\text{-}r_i$. After completion of the i^{th},$(i=1,2,\ldots k)$ stage of service, the customer may have the option to choose $(i+1)^{th}$ stage of service with probability θ_i, with probability p_i may join into orbit as feedback customer or may leave the system with probability $q_i = 1 - p_i - \theta_i$, $(i = 1, 2, \ldots k - 1)$ and $q_i = 1 - p_i$, $(i = k)$. If the orbit is empty at the service completion of each stage service, the server takes at most J vacations until at leastone customer appears in the orbit on the server returns from a vacation. Busy server may get to breakdown and the service channel will fail for a short interval of time and considering the waiting time as the delay time. By using the supplementary variable method, steady state probability generating function for system size, some system performance measures and numerical illustrations are discussed.

Keywords: Bernoulli feedback, k-optional service, service interruption, modified vacation policy.

AMS Classification: 60J10, 90B18, 90B22.

1. INTRODUCTION

Retrial queues or queues with repeated attempts have widely used to provide stochastic modeling of many problems arising in telecommunication and computer network. In retrial queues, if the server is busy at the arrival epoch and there is no waiting space is available then the whole batch joins the retrial group known as orbit, whereas if the server is free then one of the arriving units starts its service and the rest join the orbit. There is an extensive literature on the retrial queues. We refer the works by Artelijo,. and Gomez-Corral, [1], Artalejo [2], and Falin., Templeton [6] as a few. Keilson and Servi [11] introduced the concept of Bernoulli vacation. If the system is empty, the assumption for their model is that the server must take another vacation. Ke and Chang [9] and Chen et al. [4] discussed different J vacation queueing models.

In our model, a single server provides K optional stages of service. When the service of a customer is unsatisfied, it may be retried again and again until a successful service completion.Recently many authors developed queueing models with two or more stages of service.Wang and Li[15] have studied the single server retrial queueing system with second multi optional services. Recently, Salehirad and Badmachizadeh [14] and Bagyam and Charika [3] have discussed about the concept of Bernoulli feedback.

Ke and Choudhury [10] discussed about the batch arrival retrial queueing system with two phases of service under the concept of breakdown and delaying repair. Choudhury and Deka [5] considered a single server queue with two phases of service and the server is subject to breakdown while providing service to the customers. Recently, authors like Wang and Li [15], Radha et al. [13] and Mokaddis et al. [8]discussed about the retrial queueing systems with the concept of breakdown and repair. In this paper, we investigate a steady state analysis of retrial queueing system with multi optional stage of service, Bernoulli feedback , optional re-service under modified vacation policy, where the server is subject to breakdown and delaying repair.

The results of this paper finds applications in LAN, telephone systems, electronic mail services on internet, network and software designs of various computer communications systems, packet switched networks, production lines, In the operational model of WWW server for HTTP requests, call centers, inventory and production, maintenance and quality control in industrial organizations, etc.

2. MODEL DESCRIPTION

In this section, we consider a model for batch arrival retrial queue with K optional stages of service, Bernoulli feedback and optional reservice under modified vacation policy, where the server is subject to breakdowns and delaying repair. The detailed description of the model given as follows:

***Arrival process*:** Customers arrive in batches according to a compound Poisson process with arrival rate λ. Let X_k denote the number of customers belonging to the k^{th} arrival batch, where X_k, k = 1, 2, 3,…are with a common distribution $\Pr[X_k = n] = \chi_n$, n= 1,2,3… $X(z)$ denotes the probability generating function of X. The first and second moments are $E(X)$ and $E(X(X-1))$.

***Retrial process*:** Assume that there is no waiting space and therefore if an arriving batch of customers finds the server free, one of the arrivals from the batch begins his service and the rest of them join into the pool of blocked customers called an orbit. If an arriving batch finds the server either busy or on vacation or breakdown, then the batch joins into an orbit. HereInter-retrial times form an arbitrary distribution $R(t)$ with corresponding Laplace-Stieltijes transform (LST) $R^*(\varphi)$.

***Service process*:** The server provides k stages of service in succession. The First Stage Service (FSS) is followed by $i$stages of service. The service time for all the stages has a general distribution. It is denoted by the random variable S_i with distribution function $S_i(t)$ having LST $S_i^*(\varphi)$ and first and second moments are $E(S_i)$ and $E(S_i^2)$, (i= 1,2,…k).

Feedback rule:After completion of i^{th} stage of service the customer may go to $(i+1)^{th}$ stage with probability θ_i or may join into the orbit as feedback customer with probability p_i or leaves the system with probability $q_i = 1 - \theta_i - p_i$ for i = 1,2,…k-1. If the customer in the last k^{th} stage may join to the orbit with probability p_k or leaves the system with probability $q_k = 1 - p_k$. From this model, the service time or the time required by the customer to complete the service cycle is a random variable S is given by $S = \sum_{i=1}^{k} \Theta_{i-1} S_i$ having the LST $S^*(\varphi) = \prod_{i=1}^{k} \Theta_{i-1} S_i^*(\varphi)$ and the expected value is $E(S) = \sum_{i=1}^{k} \Theta_{i-1} E(S_i)$, where $\Theta_i = \theta_1 \theta_2 \ldots \theta_i$ and $\Theta_0 = 1$.

***Vacation process*:** Whenever the orbit is empty, the server leaves for a vacation of random length V. If no customer appears in the orbit when the server returns from a vacation, it leaves again for another vacation with the same length. Such pattern continues until it returns from a vacation to find atleast one customer found in the orbit or it has already taken J vacations. If the orbit is empty at the end of the J^{th} vacation, the server remains idle for new arrivals in the system. At a vacation completion epoch the orbit is non empty, the server waits for the customers in the orbit. The vacation time V has distribution function $V(t)$ and LST $V^*(\varphi)$ with moments $E(V)$ and $E(V^2)$.

Breakdownprocess: While the server is working with any phase of service, it may breakdown at any time and the service channel will fail for a short interval of time. Due to this breakdown, the server is free for a short interval of time. The server's life times are generated by exogenous Poisson processes with rates a_i for i^{th} stage respectively for (i = 1, 2,...k).

Repair process: As soon as breakdown occurs the server is sent for repair. During that time it stops providing service to the primary customers till service channel is repaired. The customer who was just being served before server breakdown waits for the remaining service to get complete.

We define the waiting time as delay time. The delay time D_i of the server for ith type of service follows with density function $D_i(y)$, Laplace- Stieltijes Transform $D_i^*(\theta)$ and finite k^{th} moment $E(D_i^k)$ (i=1,2 ,...k and k=1,2). The customer who was just being served before server breakdown waits for the remaining service to complete. The repair time (denoted by G_i)distributions of the server for i stagesare assumed to be arbitrarily distributed with density function $G_i(y)$ and LST $G_i^*(\varphi)$ for (i=1,2,...k).

Various stochastic processes involved in the system are assumed to be mutually exclusive.

In the steady state, we assume that $R(0)=0, R(\infty)=1, S_i(0)=0, S_i(\infty)=1, V(0)=0, V(\infty)=1, i=1,2,...k$ are continuous at $x=0$ and $D_i(0)=0, D_i(\infty)=1$, $G_i(0)=0, G_i(\infty)=1$ are continuous at $y=0$, $(1 \le i \le k)$. Let $R^0(t)$, $S_i^0(t)$, $V_j^0(t)$, $D_i^0(t)$ and $G_i^0(t)$ be the the elapsed retrial time, the elapsed service time on i^{th} stage,the elapsed vacation time on j^{th} vacation ($for j=1,2,...J$), the elapsed delaying repair time on i^{th} stage,the elapsed repair time on i^{th} stage, $(1 \le i \le k)$ respectively.Further, we introduce a random variable,

$$C(t)=\begin{cases} 0, \text{ if the server is idle at time } t, \\ 1, \text{ if the server is busy on } i^{\text{th}} \text{ stage at time } t, \\ 2, \text{ if the server is on re-service of } i^{\text{th}} \text{ stage at time } t, \\ 3, \text{ if the server is repair on } i^{\text{th}} \text{ stage at time } t. \\ 4, \text{ if the server is on delaying repair of } i^{\text{th}} \text{ stage at time } t, \\ 5, \text{ if the server is on vacation with the first vacation at time } t, \\ 6, \text{if the server is on vacation with the second vacation at time } t, \\ \ldots \\ j+2, \text{if the server is on vacation with the j}^{\text{th}} \text{ vacation at time } t, \\ \ldots \\ J+2, \text{if the server is on vacation with the J}^{\text{th}} \text{ vacation at time } t. \end{cases}$$

The state of system at time t can be described by the bivariate Markov process $\{C(t), N(t); t \geq 0\}$ where $C(t)$ denotes the server state and $N(t)$ denotes the number of customers in orbit at time t, so that the functions $a(x)$, $\mu_i(x)$, $\gamma(x)$, $\eta_i(y)$ and $\xi_i(y)$ are the conditional completion rates for retrial, service, vacation, delaying repair and repair respectively $(1 \leq i \leq k)$.

$$a(x)dx = \frac{dR(x)}{1-R(x)}, \mu_i(x)dx = \frac{dS_i(x)}{1-S_i(x)}, \gamma(x)dx = \frac{dV(x)}{1-V(x)}, \eta_i(y)dy = \frac{dD_i(y)}{1-D_i(y)}$$

and $\xi_i(y)dy = \dfrac{dG_i(y)}{1-G_i(y)}$. Define $B_i^* = S_1^* S_2^* ... S_i^*$ and $B_0^* = 1$. The first moment M_{1i} and second moment M_{2i} of B_i^* are given by

$$M_{1i} = \lim_{z\to 1} dB_i^*[A_i(z)]\big/dz = \sum_{j=1}^{i} \lambda b E(X)E(S_j)\left(1+\alpha_j[E(G_j)+E(D_j)]\right),$$

$$M_{2i} = \lim_{z\to 1} d^2 B_i^*[A_i(z)]\big/dz^2 = \sum_{j=1}^{i} \left[\begin{array}{l} -M_{1i} \left\{ \begin{array}{l} \lambda E(X(X-1)) + \alpha_j[\lambda E(X(X-1))[E(G_j)+E(D_j)]] \\ -(\lambda E(X))^2 [E(G_j^2)+E(D_j^2)]] \end{array} \right\} \\ \quad + \left(\lambda E(X)E(S_j)\right)^2 E(S_j^2)\left(1+\alpha_j[E(G_j)+E(D_j)]\right)^2 \end{array} \right],$$

where $A_i(z) = b(z) + \alpha_i\left(1 - G_i^*(b(z))D_i^*(b(z))\right)$ and $b(z) = \lambda\left(1 - X(z)\right)$

Let $\{t_n;\ n = 1,2,\ldots\}$ be the sequence of epochs at which either a service period completion occurs or a vacation time ends or repair period ends. The sequence of random vectors $Z_n = \{C(t_n+), N(t_n+)\}$ forms a Markov chain which is embedded in the retrial queueing system.

Theorem 2.1: The embedded Markov chain $\{Z_n;\ n \in N\}$ is ergodic if and only if, $\rho < 1$, where

$$\rho = E(X)\left(1 - R^*(\lambda)\right) + (1-r_i)\left(\sum_{i=1}^{k}\Theta_{i-1}M_{1i} + \sum_{i=1}^{k} p_i\Theta_{i-1} - \sum_{i=1}^{k-1}\Theta_i M_{1i}\right) - 2r_i\Theta_{i-1}M_{1i}.$$

3. STEADY STATE DISTRIBUTION

For the process $\{N(t),\ t \geq 0\}$, define the probabilities $P_0(t) = P\{C(t) = 0,\ N(t) = 0\}$ and the probability density functions

$$P_n(x,t)dx = P\left\{C(t) = 0, N(t) = n,\ x \leq R^0(t) < x + dx\right\},\ \text{for } t \geq 0,\ x \geq 0 \text{ and } n \geq 1.$$

$$\Pi_{i,n}(x,t)dx = P\left\{C(t) = 1,\ N(t) = n,\ x \leq S_i^0(t) < x + dx\right\},\ \text{for, for } t \geq 0, (x,y) \geq 0, (1 \leq i \leq k) \text{ and } n \geq 0.$$

$$\Psi_{i,n}(x,t)dx = P\left\{C(t) = 2,\ N(t) = n,\ x \leq S_i^0(t) < x + dx\right\},\ \text{for, for } t \geq 0, (x,y) \geq 0, (1 \leq i \leq k) \text{ and } n \geq 0.$$

$$Q_{j,n}(x,t)dx = P\left\{C(t) = j+2,\ N(t) = n,\ x \leq V_j^0(t) < x + dx\right\}\ t \geq 0,\ x \geq 0,\ (1 \leq j \leq J) \text{ and } n \geq 0.$$

$$R_{i,n}(x,y,t)dy = P\left\{C(t) = 3,\ N(t) = n,\ y \leq G_i^0(t) < y + dy\big/S_i^0(t) = x\right\}, \text{for } t \geq 0, (x,y) \geq 0, (1 \leq i \leq k) \text{ and } n \geq 0.$$

$$\Omega_{i,n}(x,y,t)dy = P\left\{C(t) = 4,\ N(t) = n,\ y \leq D_i^0(t) < y + dy\big/S_i^0(t) = x\right\}, \text{for } t \geq 0, (x,y) \geq 0, (1 \leq i \leq k) \text{ and } n \geq 0.$$

The following probabilities are used in sequent sections:

$P_0(t)$ is the probability that the system is empty at time t.

$P_n(x,t)$ is the probability that at time t there are exactly n customers in the orbit with the elapsed retrial time of the test customers undergoing retrial is x

$\Pi_{i,n}(x,t),(1 \le i \le k)$ is the probability that at time t there are exactly n customers in the orbit with the elapsed service time on i^{th} stage of the test customer undergoing service is x.

$\Psi_{i,n}(x,t),(1 \le i \le k)$ is the probability that at time t there are exactly n customers in the orbit with the elapsed re-service time on i^{th} stage of the test customer undergoing re-service is x.

$Q_{j,n}(x,t),(j = 1,2,...J)$ is the probability that at time t there are exactly n customers in the orbit with the elapsed vacation time on j^{th} vacation is x.

$R_{i,n}(x,y,t),(1 \le i \le k)$ is the probability that at time t there are exactly n customers in the orbit with the elapsed service time of the test customer undergoing service is x and the elapsed repair time on i^{th} stage of server is y.

$D_{i,n}(x,y,t),(1 \le i \le k)$ is the probability that at time t there are exactly n customers in the orbit with the elapsed service time of the test customer undergoing service is x and the elapsed delaying repair time on i^{th} stage of server is y.

Assume that the stability condition is fulfilled in the sequence so that we can set for te”0, xe”0, ye”0, ne”0, for i = 1,2,…k. and j=*1,2,…k.*

$$P_0 = \lim_{t \to \infty} P_0(t),\ P_n(x) = \lim_{t \to \infty} P_n(x,t),\ \Pi_{i,n}(x) = \lim_{t \to \infty} \Pi_{i,n}(x,t), \Psi_{i,n}(x) = \lim_{t \to \infty} \Psi_{i,n}(x,t),$$
$$Q_{j,n}(x) = \lim_{t \to \infty} Q_{j,n}(x,t),\ \Omega_{i,n}(x,y) = \lim_{t \to \infty} \Omega_{i,n}(x,y,t),\ \text{for } t \ge 0. R_{i,n}(x,y) = \lim_{t \to \infty} R_{i,n}(x,y,t),\ \text{for } t \ge 0.$$

3.1. Steady state equations

By the method of supplementary variable technique (Kelison and Servi., [11]), we obtain the following system of equations are obtained which govern the dynamics of the system behaviour for (i=1,2,..k)

$$\lambda P_0 = \int_0^\infty Q_{j,0}(x)\gamma(x)dx. \tag{3.1}$$

$$\frac{dP_n(x)}{dx} + [\lambda + a(x)]P_n(x) = 0,\ n \ge 1\ . \tag{3.2}$$

$$\frac{d\Pi_{i,0}(x)}{dx} + [\lambda + \alpha_i + \mu_i(x)]\Pi_{i,0}(x) = \int_0^\infty \xi_i(y)R_{i,0}(x,y)dy, n = 0. \tag{3.3}$$

$$\frac{d\Pi_{i,n}(x)}{dx} + [\lambda + \alpha_i + \mu_i(x)]\Pi_{i,n}(x) = \lambda \sum_{k=1}^{n} \chi_k \Pi_{i,n-k}(x) + \int_0^\infty \xi_i(y)R_{i,n}(x,y)dy, n \ge 1. \tag{3.4}$$

$$\frac{d\Psi_{i,0}(x)}{dx}+[\lambda+\alpha_i+\mu_i(x)]\Psi_{i,0}(x)=\int_0^\infty \xi_i(y)R_{i,0}(x,y)dy, n=0. \tag{3.5}$$

$$\frac{d\Psi_{i,n}(x)}{dx}+[\lambda+\alpha_i+\mu_i(x)]\Psi_{i,n}(x)=\lambda\sum_{k=1}^{n}\chi_k\Psi_{i,n-k}(x)+\int_0^\infty \xi_i(y)R_{i,n}(x,y)dy, n\geq 1. \tag{3.6}$$

$$\frac{dQ_{j,0}(x)}{dx}+[\lambda+\gamma(x)]Q_{j,0}(x)=0,\ n=0,\ j=1,2,...J. \tag{3.7}$$

$$\frac{dQ_{j,n}(x)}{dx}+[\lambda+\gamma(x)]Q_{j,n}(x)=\lambda\sum_{k=1}^{n}\chi_k Q_{j,n-k}(x),\ n\geq 1,\ j=1,2,...J. \tag{3.8}$$

$$\frac{d\Omega_{i,0}(x,y)}{dy}+[\lambda+\xi_i(y)]\Omega_{i,0}(x,y)=0,\ n=0. \tag{3.9}$$

$$\frac{d\Omega_{i,n}(x,y)}{dy}+[\lambda+\xi_i(y)]\Omega_{i,n}(x,y)=\lambda\sum_{k=1}^{n}\chi_k\Omega_{i,n-k}(x,y),\ n\geq 1. \tag{3.10}$$

$$\frac{dR_{i,0}(x,y)}{dy}+[\lambda+\xi_i(y)]R_{i,0}(x,y)=0,\ n=0. \tag{3.11}$$

$$\frac{dR_{i,n}(x,y)}{dy}+[\lambda+\xi_i(y)]R_{i,n}(x,y)=\lambda\sum_{k=1}^{n}\chi_k R_{i,n-k}(x,y),\ n\geq 1. \tag{3.12}$$

The steady state boundary conditions at $x=0$ and $y=0$ are

$$P_n(0)=(1-r_i)\left\{\begin{array}{c}\sum_{i=1}^{k-1}q_i\int_0^\infty \Pi_{i,n}(x)\mu_i(x)dx+(1-p_k)\int_0^\infty \Pi_{k,n}(x)\mu_k(x)dx\\ +\sum_{i=1}^{k}p_i\int_0^\infty \Pi_{i,n-1}(x)\mu_i(x)dx\end{array}\right\}+\sum_{j=1}^{J}\int_0^\infty Q_{j,n}(x)\gamma(x)dx,\ n\geq 1. \tag{3.13}$$

$$\Pi_{i,0}(0)=\int_0^\infty P_1(x)a(x)dx+\lambda\chi_1P_0,\ n=0. \tag{3.14}$$

$$\Pi_{1,n}(0)=\int_0^\infty P_{n+1}(x)a(x)dx+\lambda\sum_{k=1}^{n}\chi_k\int_0^\infty P_{n-k+1}(x)dx+\lambda\chi_{n+1}P_0,\ n\geq 1. \tag{3.15}$$

$$\Pi_{i,n}(0)=\theta_{i-1}\int_0^\infty \Pi_{i-1,n}(x)\mu_{i-1}(x)dx,\ n\geq 1,\ (2\leq i\leq k). \tag{3.16}$$

$$\Psi_{i,n}(0)=r_i\int_0^\infty \Pi_{i,n}(x)dx,\ n\geq 1. \tag{3.17}$$

$$Q_{1,0}(0) = (1-r_i)\left\{\sum_{i=1}^{k-1} q_i \int_0^\infty \Pi_{i,0}(x)\mu_i(x)dx + (1-p_k)\int_0^\infty \Pi_{k,0}(x)\mu_k(x)dx\right\} + \int_0^\infty \Psi_{i,0}(x)\mu_i(x)dx,\ n=0. \tag{3.18}$$

$$Q_{j,n}(0) = \int_0^\infty Q_{j-1,n}(x)\gamma(x)dx, n \ge 0,\ j = 2,3,...J. \tag{3.19}$$

$$\Omega_{i,n}(x,0) = \alpha_i[\Pi_{i,n}(x) + \Psi_{i,n}(x)],\ n \ge 1. \tag{3.20}$$

$$R_{i,n}(x,0) = \int_0^\infty \Omega_{i,n}(x,y)\eta_i(y)dy,\ n \ge 0. \tag{3.21}$$

The normalizing condition is

$$P_0 + \sum_{n=1}^\infty \int_0^\infty P_n(x)dx + \sum_{n=0}^\infty \left(\sum_{i=1}^k \left(\begin{array}{l} \int_0^\infty \Pi_{i,n}(x)dx + \int_0^\infty \Psi_{i,n}(x)dx \\ + \int_0^\infty\int_0^\infty R_{i,n}(x,y)dxdy + \int_0^\infty\int_0^\infty \Omega_{i,n}(x,y)dxdy \end{array} \right) + \sum_{j=1}^J \int_0^\infty Q_{j,n}(x)dx \right) = 1. \tag{3.22}$$

3.2. Steady state solutions

To solve the above equations, define the generating functions for $|z| \le 1, i = 1,2,...k$

$$P(x,z) = \sum_{n=1}^\infty P_n(x)z^n;\ P(0,z) = \sum_{n=1}^\infty P_n(0)z^n;\ \Pi_i(x,z) = \sum_{n=0}^\infty \Pi_{i,n}(x)z^n; \Pi_i(0,z) = \sum_{n=0}^\infty \Pi_{i,n}(0)z^n;$$

$$\Psi_i(x,z) = \sum_{n=0}^\infty \Psi_{i,n}(x)z^n; \Psi_i(0,z) = \sum_{n=0}^\infty \Psi_{i,n}(0)z^n; Q_j(x,z) = \sum_{n=0}^\infty Q_{j,n}(x)z^n;\ Q_j(0,z) = \sum_{n=0}^\infty Q_{j,n}(0)z^n;$$

$$R_i(x,y,z) = \sum_{n=0}^\infty R_{i,n}(x,y)z^n;\ R_i(x,0,z) = \sum_{n=0}^\infty R_{i,n}(x,0)z^n; \Omega_i(x,y,z) = \sum_{n=0}^\infty \Omega_{i,n}(x,y)z^n;\ \Omega_i(x,0,z) = \sum_{n=0}^\infty \Omega_{i,n}(x,0)z^n$$

Now multiplying the steady state equation and steady state boundary condition (3.2) to (3.21) by z^n and summing up n, ($n = 0,1,2...,1 \le i \le k$ *and j=1,2,....J*)

$$\frac{\partial P(x,z)}{\partial x} + [\lambda + a(x)]P(x,z) = 0. \tag{3.22}$$

$$\frac{\partial \Pi_i(x,z)}{\partial x} + [\lambda(1-X(z)) + \alpha_i + \mu_i(x)]\Pi_i(x,z) = \int_0^\infty \xi_i(y)R_i(x,y,z)dy. \tag{3.23}$$

$$\frac{\partial \Psi_i(x,z)}{\partial x} + [\lambda(1-X(z)) + \alpha_i + \mu_i(x)]\Psi_i(x,z) = \int_0^\infty \xi_i(y)R_i(x,y,z)dy. \tag{3.24}$$

$$\frac{\partial Q_j(x,z)}{\partial x}+[\lambda(1-X(z))+\gamma(x)]Q_j(x,z)=0. \tag{3.25}$$

$$\frac{d\Omega_i(x,y,z)}{dy}+[\lambda(1-X(z))+\xi_i(y)]\Omega_i(x,y,z)=0. \tag{3.26}$$

$$\frac{dR_i(x,y,z)}{dy}+[\lambda(1-X(z))+\xi_i(y)]R_i(x,y,z)=0. \tag{3.27}$$

The steady state boundary conditions at $x = 0$ and $y = 0$ are

$$P(0,z)=(1-r_i)\sum_{i=1}^{k}\left\{\left(p_i z+q_i\right)\int_0^\infty \Pi_i(x,z)\mu_i(x)dx\right\}+\int_0^\infty Q_j(x,z)\gamma(x)dx-\lambda P_0-Q_{j,0}(0)+\sum_{i=1}^{k}\left\{\int_0^\infty \Psi_i(x,z)\mu_i(x)dx\right\}. \tag{3.28}$$

$$\Pi_1(0,z)=\frac{1}{z}\int_0^\infty P(x,z)a(x)dx+\lambda\frac{X(z)}{z}\int_0^\infty P(x,z)dx+\frac{\lambda X(z)}{z}P_0. \tag{3.29}$$

$$\Pi_i(0,z)=\theta_{i-1}\int_0^\infty \Pi_{i-1,n}(x)\mu_{i-1}(x)dx,\quad (2\le i\le k). \tag{3.30}$$

$$\Psi_i(0,z)=r_i\int_0^\infty \Pi_i(x,z)dx,\ n\ge 1. \tag{3.31}$$

$$\Omega_i(x,0,z)=\alpha_i[\Pi_i(x,z)+\Psi_i(x,z)]. \tag{3.32}$$

$$R_i(x,0,z)=\int_0^\infty \Omega_i(x,y,z)\eta_i(y)dy,\ n\ge 0. \tag{3.33}$$

Solving the partial differential equations (3.22) to (3.27), it follows that for $(1\le i\le k)$

$$P(x,z)=P(0,z)[1-R(x)]e^{-\lambda x}. \tag{3.34}$$

$$\Pi_i(x,z)=\Pi_i(0,z)[1-S_i(x)]e^{-A_i(z)x}. \tag{3.35}$$

$$\Psi_i(x,z)=\Psi_i(0,z)[1-S_i(x)]e^{-A_i(z)x}. \tag{3.36}$$

$$Q_j(x,z)=Q_j(0,z)[1-V(x)]e^{-b(z)x}. \tag{3.37}$$

$$\Omega_i(x,y,z)=\Omega_i(x,0,z)[1-D_i(y)]e^{-b(z)y}. \tag{3.38}$$

$$R_i(x,y,z)=R_i(x,0,z)[1-G_i(y)]e^{-b(z)y}, \tag{3.39}$$

where $A_i(z)=b(z)+\alpha_i\left(1-G_i^*(b(z))D_i^*(b(z))\right)$ and $b(z)=\lambda\left(1-X(z)\right)$.

From (3.7) we obtain, $$Q_{j,0}(x)=Q_{j,0}(0)[1-V(x)]e^{-b\lambda x}. \tag{3.40}$$

Multiplying with equation (3.40) by $\gamma(x)$ on both sides for $j = J$ and integrating with respect to x from 0 to ∞,

then from (3.1) we have,
$$Q_{j,0}(0) = \frac{\lambda P_0}{V^*(\lambda)}. \tag{3.41}$$

Integrating (3.40) from 0 to ∞ and using (3.41), (3.19)over the range $j = J-1, J-2, \ldots 1$, we get

$$Q_{j,0}(0) = \frac{\lambda P_0}{\left[V^*(\lambda)\right]^{J-j+1}},\ j = 1,2,\ldots J. \tag{3.42}$$

Note that $Q_{j,0}$ represents the steady-state probability that no customer appear while the server is on the j th vacation. Let us define Q_0 as the probability that no customer appear in the system while the server is on vacation. Then

$$Q_0 = \frac{P_0\left(1-\left[V^*(\lambda)\right]^J\right)}{\left[V^*(\lambda)\right]^J},\ j = 1,2,\ldots J. \tag{3.43}$$

Iserting (3.34) in (3.40), we obtain

$$\Pi_1(0,z) = \frac{P(0,z)}{z}\left[R^*(\lambda) + X(z)\left(1-R^*(\lambda)\right)\right] + \frac{\lambda X(z)}{z}P_0\ . \tag{3.44}$$

Inserting (3.44) in (3.30), we obtain

$$\Pi_i(0,z) = \Theta_{i-1}\Pi_1(0,z)\left(B^*_{i-1}\left[A_{i-1}(z)\right]\right),\ (i = 2,3,\ldots k). \tag{3.45}$$

Similarly,

$$\Psi_i(0,z) = r_i\Pi_i(0,z)\frac{S_i^*\left[A_i(z)\right]}{A_i(z)},\ (i = 2,3,\ldots k), \tag{3.46}$$

and

$$\Omega_i(x,0,z) = \alpha_i\left(\Pi_i(0,z) + \Psi_i(0,z)\right)\frac{S_i^*\left[A_i(z)\right]}{A_i(z)}. \tag{3.47}$$

Inserting (3.47) in (3.33), we obtain

$$R_i(x,0,z) = \Omega_i(x,0,z)D_i^*(b(z)). \tag{3.48}$$

Using (3.35) and (3.36) and (3.37) in (3.28), finally we get,

$$P(0,z) = (1-r_i)\sum_{i=1}^{k}\left\{\left(p_i z + q_i\right)\Pi_i(0,z)\left(S_i^*\left[A_i(z)\right]\right)\right\} + Q_j(0,z)V^*\left[b(z)\right] - \lambda P_0 - \frac{\lambda P_0}{\left[V^*(\lambda)\right]^{J-j+1}} + \Psi_i(0,z)\left(S_i^*\left[A_i(z)\right]\right). \tag{3.49}$$

Solving (3.44) to (3.47) and (3.49), we get

$$P(0,z) = \lambda P_0 \times \left\{\frac{X(z)\left\{(1-r_i)\Sigma + r_i\Theta_{i-1}B_i^*\left[A_i(z)\right]S_i^*\left[A_i(z)\right]\right\} + z\left(N(z)-1\right)}{z - \left[R^*(\lambda) + X(z)\left(1-R^*(\lambda)\right)\right]\left\{(1-r_i)\Sigma + r_i\Theta_{i-1}B_i^*\left[A_i(z)\right]S_i^*\left[A_i(z)\right]\right\}}\right\}, \tag{3.50}$$

where
$$N(z)=\frac{\left(1-\left[V^*(\lambda)\right]^J\right)}{\left[V^*(\lambda)\right]^J\left(1-\left[V^*(\lambda)\right]\right)}\left(V^*(b(z))-1\right),$$

Using (3.50) in (3.45), we get,

$$\Pi_1(0,z)=\lambda P_0\left\{\frac{\left(N(z)-1\right)\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]+X(z)}{z-\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]\left\{(1-r_i)\Sigma+r_i\Theta_{i-1}B_i^*\left[A_i(z)\right]S_i^*\left[A_i(z)\right]\right\}}\right\}. \tag{3.51}$$

$$\Pi_i(0,z)=\lambda P_0\Theta_{i-1}(\mathrm{B}_{i-1}^*(\mathrm{A}_{i-1}(z)))\left\{\frac{\left(N(z)-1\right)\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]+X(z)}{z-\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]\left\{(1-r_i)\Sigma+r_i\Theta_{i-1}B_i^*\left[A_i(z)\right]S_i^*\left[A_i(z)\right]\right\}}\right\}. \tag{3.52}$$

$$\Psi_i(0,z)=\lambda P_0r_i\Theta_{i-1}(\mathrm{B}_{i-1}^*(\mathrm{A}_{i-1}(z)))\left\{\frac{\left(N(z)-1\right)\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]+X(z)}{z-\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]\left\{(1-r_i)\Sigma+r_i\Theta_{i-1}B_i^*\left[A_i(z)\right]S_i^*\left[A_i(z)\right]\right\}}\right\}. \tag{3.53}$$

$$\Omega_i(x,0,z)=\alpha_i\Theta_{i-1}(\mathrm{B}_i^*(\mathrm{A}_i(z)))\frac{\Pi_1(0,z)}{\mathrm{A}_i(z)}[1+r_iS_i^*(\mathrm{A}_i(z))]. \tag{3.54}$$

From (3.42), we get
$$Q_{j,0}(0)=\frac{\lambda P_0}{\left[V^*(\lambda)\right]^{J-j+1}},\ j=1,2,\ldots J. \tag{3.55}$$

Using (3.54) in (3.48), we get

$$R_i(x,0,z)=\alpha_i\Theta_{i-1}(\mathrm{B}_i^*(\mathrm{A}_i(z)))D_i^*(b(z))\frac{\Pi_1(0,z)}{\mathrm{A}_i(z)}[1+r_iS_i^*(\mathrm{A}_i(z))]. \tag{3.56}$$

Using (3.34) to (3.39) and (3.50) to (3.55), we get the limiting probability generating functions $P(x,z),\Pi_i(x,z),\Psi_i(x,z),Q_j(x,z),\ \Omega_i(x,y,z)$ *and* $R_i(x,y,z)$.

Theorem 3.1. Under the stability condition ρ < 1, the stationary distributions of the number of customers in the orbit and the server's state has the following PGF's (for 1 ≤ *i* ≤ k)

$$P(x,z)=\lambda P_0\times\left\{\frac{X(z)\left\{(1-r_i)\Sigma+r_i\Theta_{i-1}B_i^*\left[A_i(z)\right]S_i^*\left[A_i(z)\right]\right\}+z\left(N(z)-1\right)}{z-\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]\left\{(1-r_i)\Sigma+r_i\Theta_{i-1}B_i^*\left[A_i(z)\right]S_i^*\left[A_i(z)\right]\right\}}\right\}\left(1-R(x)\right)e^{-\lambda x} \tag{3.43}$$

$$\Pi_i(x,z)=\lambda P_0\left\{\frac{\Theta_{i-1}\left(\left(N(z)-1\right)\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]+X(z)\right)\left(B_{i-1}^*\left[A_{i-1}(z)\right]\right)\left(1-S_i(x)\right)e^{-A_i(z)x}}{z-\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]\left\{(1-r_i)\Sigma+r_i\Theta_{i-1}B_i^*\left[A_i(z)\right]S_i^*\left[A_i(z)\right]\right\}}\right\} \tag{3.44}$$

$$Q_j(x,z) = \frac{\lambda P_0}{\left[V^*(\lambda)\right]^{J-j+1}}\left(1-V(x)\right)e^{-b(z)x},\ j=1,2,\ldots J \tag{3.45}$$

$$\Psi_i(z) = \lambda P_0 r_i \Theta_{i-1} S_i^*(A_i(z)) \mathrm{B}_{i-1}^*(\mathrm{A}_{i-1}(z))\left(1-S_i(x)\right)e^{-A_i(z)x}\left\{\frac{\left(N(z)-1\right)\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]+X(z)}{z-\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]\left\{(1-r_i)\Sigma+r_i\Theta_{i-1}B_i^*\left[A_i(z)\right]S_i^*\left[A_i(z)\right]\right\}}\right\}. \tag{3.45.1}$$

$$\Omega_i(z) = \alpha_i \Theta_{i-1}\left(1-S_i^*(A_i(z))\right)e^{-A_i(z)x}\left(1-D_i^*(b(z))\right)e^{-b(z)x}B_{i-1}^*\left[A_{i-1}(z)\right][1+r_i S_i^*(\mathrm{A}_i(z))]\Pi_1(0,z). \tag{3.45.2}$$

$$R_i(x,y,z) = \alpha_i \Pi_i(0,z) B_{i-1}^*\left[A_{i-1}(z)\right]D_i^*(b(z))[1+r_i S_i^*(\mathrm{A}_i(z))]\Pi_1(0,z)[1-S_i(x)]e^{-A_i(z)x}\times[1-G_i(y)]e^{-b(z)y} \tag{3.46}$$

Where $A_i(z) = b(z) + \alpha_i\left(1-G_i^*(b(z))D_i^*(b(z))\right)$ and $b(z) = \lambda\left(1-X(z)\right)$.

Next the marginal orbit size distributions due to system state of the server is investigated.

Theorem 3.2. *Under the stability condition* $\rho < 1$*, the stationary distributions of the number of customers in the system when server being idle, busy on* i^{th} *stage, vacation on* j^{th} *stage of vacation, under repair on* i^{th} *stage (for* $1 \le i \le k, j=1,2,..J$*) are given by*

$$P(z) = P_0\left(1-R^*(\lambda)\right)\times\left\{\frac{X(z)\left\{(1-r_i)\Sigma+r_i\Theta_{i-1}B_i^*\left[A_i(z)\right]S_i^*\left[A_i(z)\right]\right\}+z\left(N(z)-1\right)}{z-\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]\left\{(1-r_i)\Sigma+r_i\Theta_{i-1}B_i^*\left[A_i(z)\right]S_i^*\left[A_i(z)\right]\right\}}\right\}. \tag{3.57}$$

$$\Pi_i(z) = \lambda P_0 \Theta_{i-1}(\mathrm{B}_{i-1}^*(\mathrm{A}_{i-1}(z)))\frac{\left(1-S_i^*(A_i(z))\right)}{A_i(z)}\left\{\frac{\left(N(z)-1\right)\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]+X(z)}{z-\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]\left\{(1-r_i)\Sigma+r_i\Theta_{i-1}B_i^*\left[A_i(z)\right]S_i^*\left[A_i(z)\right]\right\}}\right\}. \tag{3.58}$$

$$Q_j(z) = \frac{P_0\left(1-V^*(b(z))\right)}{\left[V^*(\lambda)\right]^{J-j+1}(1-X(z))}. \tag{3.59}$$

$$\Psi_i(z) = \lambda P_0 r_i \Theta_{i-1} S_i^*(A_i(z)) \mathrm{B}_{i-1}^*(\mathrm{A}_{i-1}(z))\frac{\left(1-S_i^*(A_i(z))\right)}{A_i(z)}\left\{\frac{\left(N(z)-1\right)\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]+X(z)}{z-\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]\left\{(1-r_i)\Sigma+r_i\Theta_{i-1}B_i^*\left[A_i(z)\right]S_i^*\left[A_i(z)\right]\right\}}\right\}. \tag{3.60}$$

$$\Omega_i(z) = \frac{\alpha_i\Theta_{i-1}\left(1-S_i^*(A_i(z))\right)\left(1-D_i^*(b(z))\right)}{A_i(z)b(z)}B_{i-1}^*\left[A_{i-1}(z)\right][1+r_i S_i^*(\mathrm{A}_i(z))]\Pi_1(0,z). \tag{3.61}$$

$$R_i(z) = \frac{\alpha_i\Theta_{i-1}\left(1-S_i^*(A_i(z))\right)\left(1-G_i^*(b(z))\right)}{A_i(z)b(z)}B_{i-1}^*\left[A_{i-1}(z)\right]D_i^*(b(z))[1+r_i S_i^*(\mathrm{A}_i(z))]\Pi_1(0,z), \tag{3.62}$$

where $$P_0 = \frac{\left\{1-E(X)\left(1-R^*(\lambda)\right)-(1-r_i)\left(\sum_{i=1}^{k}\Theta_{i-1}M_{1i}-\sum_{i=1}^{k}p_i\Theta_{i-1}+\sum_{i=1}^{k-1}\Theta_iM_{1i}\right)-2r_i\Theta_{i-1}M_{1i}\right\}}{\left\{\begin{array}{l}\left(1+\frac{N^1(1)}{E(X)}\right)(1-E(X)\left(1-R^*(\lambda)\right)-(1-r_i)\left(\sum_{i=1}^{k}\Theta_{i-1}M_{1i}-\sum_{i=1}^{k}p_i\Theta_{i-1}+\sum_{i=1}^{k-1}\Theta_iM_{1i}\right)-2r_i\Theta_{i-1}M_{1i})\\+\sum_{i=1}^{k}\lambda\Theta_{i-1}E(S_i)(1+r_i)\left(1+\alpha_j[E(G_j)+E(G_j)]\right)(E(X)+N'(1)-E(X)\left(1-R^*(\lambda)\right))\end{array}\right\}}. \quad (3.63)$$

Proof: Integrating the limiting probability generating functions with respect to x and define the partial probability generating functions as, for (1 d" *i*d" *k and j=1,2,…J*)

$P(z)=\int_0^\infty P(x,z)dx,\ \Pi_i(z)=\int_0^\infty \Pi_i(x,z)dx,\ \Psi_i(z)=\int_0^\infty \Psi_i(x,z)dx, Q_j(z)=\int_0^\infty Q_j(x,z)dx.$ Integrating the equation (3.46) with respect to x and y define the partial probability generating functions as, for (1d"*i*d"*k*), $R_i(x,z)=\int_0^\infty R_i(x,y,z)dy,\ R_i(z)=\int_0^\infty R_i(x,z)dx,\ \int_0^\infty \Omega_i(x,z)=\int_0^\infty \Omega_i(x,y,z)dy,\ \Omega_i(z)=\int_0^\infty \Omega_i(x,z)dx.$ Since, the only unknown is P_0 the probability that the server is idle when no customer in the orbit and it can be determined using the normalizing condition ($1\le i\le k$). Thus, by setting $z=1$ in (3.57) to (3.62) and applying L – Hospitals rule whenever necessary and we get $P_0+P(1)+\sum_{j=1}^{J}Q_j(1)+\sum_{i=1}^{k}\left(\Pi_i(1)+\Psi_i(1)+\Omega_i(1)+R_i(1)\right)=1.$

Theorem 3.2: Under the stability condition ρ < 1, probability generating function of number of customers in the system and orbit size distribution at stationary point of time is

$$K(z)=\frac{Nr(z)}{Dr(z)}, \quad (3.64)$$

$$\text{where}\quad Nr(z)=P_0\left\{\begin{array}{l} z\left\{\sum_{i=1}^{k}\Theta_{i-1}\left(B_{i-1}^*\left[A_{i-1}(z)\right]\right)\left(1-S_i^*(A_i(z))\right)[1+r_iS_i^*(A_i(z))]\left(\begin{array}{l}(N(z)-1)(R^*(\lambda)\\+X(z)\left(1-R^*(\lambda)\right)+X(z)\end{array}\right)\right\}\\ -N(z)\left(z-\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]\left\{(1-r_i)\Sigma+r_i\Theta_{i-1}B_i^*\left[A_i(z)\right]S_i^*\left[A_i(z)\right]\right\}\right)\\ +[1-X(z)]\left(\begin{array}{l}z-\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]\left\{(1-r_i)\Sigma+r_i\Theta_{i-1}B_i^*\left[A_i(z)\right]S_i^*\left[A_i(z)\right]\right\}\\+\left(X(z)\left\{(1-r_i)\Sigma+r_i\Theta_{i-1}B_i^*\left[A_i(z)\right]S_i^*\left[A_i(z)\right]\right\}+z(N(z)-1)\right)\left(1-R^*(\lambda)\right)\end{array}\right)\end{array}\right\},$$

$$Dr(z)=[1-X(z)]\left(z-\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]\left\{(1-r_i)\Sigma+r_i\Theta_{i-1}B_i^*\left[A_i(z)\right]S_i^*\left[A_i(z)\right]\right\}\right),$$

$$\text{and}\quad \Sigma=\sum_{i=1}^{k}\Theta_{i-1}M_{1i}-\sum_{i=1}^{k}p_i\Theta_{i-1}+\sum_{i=1}^{k-1}\Theta_iM_{1i}.$$

Also $$H(z)=\frac{NR(z)}{Dr(z)}, \quad (3.65)$$

$$where \quad NR(z)=P_0\left\{\begin{array}{l}\left\{\sum_{i=1}^{k}\Theta_{i-1}\left(B_{i-1}^*\left[A_{i-1}(z)\right]\right)\left(1-S_i^*(A_i(z))\right)[1+r_iS_i^*(A_i(z))]\begin{pmatrix}(N(z)-1)(R^*(\lambda)\\+X(z)\left(1-R^*(\lambda)\right))+X(z)\end{pmatrix}\right\}\\-N(z)\left(z-\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]\left\{(1-r_i)\Sigma+r_i\Theta_{i-1}B_i^*\left[A_i(z)\right]S_i^*\left[A_i(z)\right]\right\}\right)\\+\left[1-X(z)\right]\begin{pmatrix}z-\left[R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right]\left\{(1-r_i)\Sigma+r_i\Theta_{i-1}B_i^*\left[A_i(z)\right]S_i^*\left[A_i(z)\right]\right\}\\+\left(X(z)\left\{(1-r_i)\Sigma+r_i\Theta_{i-1}B_i^*\left[A_i(z)\right]S_i^*\left[A_i(z)\right]\right\}+z\left(N(z)-1\right)\right)\left(1-R^*(\lambda)\right)\end{pmatrix}\end{array}\right\},$$

where P_0 is given in Eq. (3.57).

Proof: Theprobability generating function of the number of customer in the system ($K(z)$) is obtained by using $K(z)=P_0+P(z)+\sum_{j=1}^{J}Q_j(z)+z\sum_{i=1}^{k}\left(\Pi_i(z)+\Psi_i(z)+\Omega_i(z)+R_i(z)\right)$. The PGF of number of customer in the orbit ($H(z)$) is obtained by using $H(z)=P_0+P(z)+\sum_{j=1}^{J}Q_j(z)+\sum_{i=1}^{k}\left(\Pi_i(z)+\Psi_i(z)+\Omega_i(z)+R_i(z)\right)$ Substituting (3.57) to (3.62) in the above results, then the equations (3.64) and (3.65) can be obtained by direct calculation.

4. PERFORMANCE MEASURES

In this section, we obtain some probabilities when the system is in different status. Various performance measures like the mean number of customers in the orbit (L_q) and the reamaining measures like the mean number of customers in the system (L_s), the average time a customer spends in the system (W_s) and the average time a customer spends in the queue (W_q) are required during the analysis of unreliable queueing model.

Theorem 4.1. If the system satisfies the stability condition $\rho < 1$, then we get the following probabilities of the server state,

1) Let P be the steady state probability that the server is idle during the retrial time

$$P=\frac{\left(1-R^*(\lambda)\right)}{\beta_1}\left(E(X)+N'(1)+(1-r_i)\left(\sum_{i=1}^{k}\Theta_{i-1}M_{1i}-\sum_{i=1}^{k}p_i\Theta_{i-1}+\sum_{i=1}^{k-1}\Theta_iM_{1i}\right)-2r_i\Theta_{i-1}M_{1i}-1\right).$$

2) Let Π_i be the steady-state probability that the server is busy on i^{th} stage,

$$\Pi_i=\sum_{i=1}^{k}\Pi_i=\frac{1}{\beta_1}\sum_{i=1}^{k}\{\Theta_{i-1}\lambda E(S_i)\}(N'(1)+E(X)R^*(\lambda)).$$

3) Let ψ_i be the steady-state probability that the server is on re-service of i i^{th} stage,

$$\Psi_i=\sum_{i=1}^{k}\Psi_i=\frac{1}{\beta_1}\sum_{i=1}^{k}r_i\{\Theta_{i-1}\lambda E(S_i)\}(N'(1)+E(X)R^*(\lambda)).$$

4) Let Q_j be the steady state probability that the server is on vacation with j th stage

$$Q_j=\sum_{j=1}^{J}Q_j=\frac{1}{\beta_1}\left\{1-E(X)\left(1-R^*(\lambda)\right)-(1-r_i)\left(\sum_{i=1}^{k}\Theta_{i-1}M_{1i}-\sum_{i=1}^{k}p_i\Theta_{i-1}+\sum_{i=1}^{k-1}\Theta_iM_{1i}\right)-2r_i\Theta_{i-1}M_{1i}\right\}\frac{N'(1)}{E(X)}.$$

5) Let Ω_i be the steady state probability that the server is under delaying repair on i^{th} stage,

$$\Omega_i = \sum_{i=1}^{k} R_i = \frac{1}{\beta_1}\sum_{i=1}^{k}(1+r_i)\alpha_i E(D_i)\left\{\Theta_{i-1}\lambda E(S_i)\right\}(N'(1)+E(X)R^*(\lambda)).$$

6) Let R_i be the steady state probability that the server is under repair on i^{th} stage,

$$R_i = \sum_{i=1}^{k} R_i = \frac{1}{\beta_1}\sum_{i=1}^{k}(1+r_i)\alpha_i E(G_i)\left\{\Theta_{i-1}\lambda E(S_i)\right\}(N'(1)+E(X)R^*(\lambda)).$$

Proof. Noting that

$$P = \lim_{z\to 1} P(z),\quad \sum_{i=1}^{k}\Pi_i = \lim_{z\to 1}\sum_{i=1}^{k}\Pi_i(z),\quad \sum_{i=1}^{k}\Psi_i = \lim_{z\to 1}\sum_{i=1}^{k}\Psi_i(z),\quad \sum_{j=1}^{J}Q_j = \lim_{z\to 1}\sum_{j=1}^{J}Q_j(z),$$

$$\sum_{i=1}^{k}\Omega_i = \lim_{z\to 1}\sum_{i=1}^{k}\Omega_i(z) \; and \; \sum_{i=1}^{k}R_i = \lim_{z\to 1}\sum_{i=1}^{k}R_i(z).$$

The stated formula follows by direct calculation.

Theorem 4.2. Let L_s, L_q, W_s and W_q be the mean number of customers in the system, the mean number of customers in the orbit, average time a customer spends in the system and average time a customer spends in the orbit using Little's formula respectively, then under the stability condition, we have

$$L_q = P_0\left[\frac{Nr_q'''(1)Dr_q''(1) - Dr_q'''(1)Nr_q''(1)}{3\left(Dr_q''(1)\right)^2}\right],$$

where,

$$Nr_q''(1) = -2\left\{\begin{array}{l}\left\{\sum_{i=1}^{k}\Theta_{i-1}(1-r_i)M_{2i}\left(N'(1)+E(X)R^*(\lambda)\right)\right\}\\ -(E(X))^2\left(1-R^*(\lambda)\right) - 2N'(1)+E(X)(2R^*(\lambda)-1)((1-r_i)\omega - 2r_i\Theta_{i-1}M_{1i}-1)\end{array}\right\},$$

$$Dr_q''(1) = -2E(X)\left(1-\rho\right),$$

$$Dr_q''' = 3\left\{\begin{array}{l}E(X)\left[\begin{array}{l}\left(1-R^*(\lambda)\right)\left(E(X(X-1))+2E(X)((1-r_i)\omega - 2r_i\Theta_{i-1}M_{1i})\right)\\ +(1-r_i)\tau + 2r_i\Theta_{i-1}\left((M_{1i})^2 - M_{2i}\left(\lambda E(X)\right)^2\left(1+\alpha_j[E(G_j)+E(D_j)]\right)^2\right)\end{array}\right]\\ -E(X(X-1))(1-\rho)\end{array}\right\},$$

$$Nr_q'''(1) = 3\left\{\begin{array}{l}-\sum_{i=1}^{k}\Theta_{i-1}M_{1i}\left[N''(1)+E(X(X-1))+E(X)\left(1-R^*(\lambda)\right)(2N'(1)-1)\right]\\ +N'(1)\left[2E(X)\left(1-R^*(\lambda)\right)-1\right]((1-r_i)\omega - 2r_i\Theta_{i-1}M_{1i}) + E(X(X-1))\left[E(X)+R^*(\lambda)N'(1)-1\right]\\ +[N'(1)+E(X(X-1))]\left(\begin{array}{l}1-E(X)\left(1-R^*(\lambda)\right)+\sum_{i=1}^{k}\Theta_{i-1}\left[M_{2i}+2M_{1i}(1+r_i)M_{1i-1}+2r_i(M_{1i})^2\right]\\ -(1-r_i)\tau - 2r_i\Theta_{i-1}\left[\left((M_{1i})^2 - M_{2i}\left(\lambda E(X)\right)^2\left(1+\alpha_j[E(G_j)+E(D_j)]\right)^2\right)\right]\end{array}\right)\end{array}\right\},$$

$$\omega = \sum_{i=1}^{k} \Theta_{i-1} M_{1i} - \sum_{i=1}^{k} p_i \Theta_{i-1} + \sum_{i=1}^{k-1} \Theta_i M_{1i},$$

$$\tau = \sum_{i=1}^{k} \Theta_{i-1} M_{2i} + 2\sum_{i=1}^{k} p_i \Theta_{i-1} M_{1i} - \sum_{i=1}^{k-1} \Theta_i M_{2i},$$

$$and \quad \rho = E(X)\left(1 - R^*(\lambda)\right) - (1 - r_i)\omega - 2r_i \Theta_{i-1} M_{1i}.$$

$$L_s = P_0 \left[\frac{Nr_s'''(1) Dr_q''(1) - Dr_q'''(1) Nr_q''(1)}{3\left(Dr_q''(1)\right)^2} \right],$$

Where, $$Nr_s'''(1) = Nr_q'''(1) - 6\sum_{i=1}^{k} \Theta_{i-1}(1 + r_i) M_{1i} \left(N'(1) + E(X) R^*(\lambda) \right).$$

$$W_s = \frac{L_s}{\lambda E(X)} \text{ and } W_q = \frac{L_q}{\lambda E(X)}.$$

Proof. The mean number of customers in the orbit (L_q) under steady state condition is obtained by differentiating (3.65) with respect to z and evaluating at $z = 1$

$$L_q = \frac{Nr(z)}{Dr(z)} = \lim_{z \to 1} \frac{d}{dz} H(z) = H'(1) = P_0 \left[\frac{Nr_q'''(1) Dr_q''(1) - Dr_q'''(1) Nr_q''(1)}{3\left(Dr_q''(1)\right)^2} \right].$$

The mean number of customers in the system (L_s) under steady state condition is obtained by differentiating (3.64) with respect to z and evaluating at $z = 1$

$$L_s = \frac{Nr(z)}{Dr(z)} = \lim_{z \to 1} \frac{d}{dz} K(z) = K'(1) = P_0 \left[\frac{Nr_s'''(1) Dr_q''(1) - Dr_q'''(1) Nr_q''(1)}{3\left(Dr_q''(1)\right)^2} \right].$$

The average time a customer spends in the system (W_s) and orbit (W_q) under steady- state condition due to Little's formula is, $L_s = \lambda W_s$ and $L_q = \lambda W_q$.

5. STOCHASTIC DECOMPOSITION AND SPECIAL CASES

Stochastic decomposition has been widely observed among M/G/1 type queueing models with server vacations by Fuhrman and Cooper [7]. A key result in these analysis is that the number of customers in the system in steady-state at a random point in time 't' is distributed as the sum of two independent random variables. One of the variable is the number of customers in the corresponding standard queueing system (in steady-state) at a random point in time 't' and the other random variable may have different probabilistic interpretations in specific cases depending on how the vacations are scheduled

Let $K(z)$ be the stationary system size distribution of $M^{[X]}/G/1$ feedback retrial queue with multi stage service, balking & reneging, at most J vacation policyand random breakdown is the convolution of two independent random variables $\chi(z)$ and $\phi(z)$.

The mathematical version of the stochastic decomposition law is $K(z) = \chi(z).\ \phi(z)$.

(i) The system size distribution of $M^{[x]}/G/1$ feedback queueing system with multi stage service and service interruption. (represented in first term of $K(z)$),

(ii) The conditional distribution of the number of customers in the vacation system at random point in time given the server is idle (represented in second term of $K(z)$).

The number of arrivals in the variant vacation system at a random point in time given that the server is on vacation or idle. In fact the second term can be also obtained through the vacation definition of our system,

$$\text{i.e., } \phi(z) = \frac{N2(z)}{D2(z)} = \left(P_0 + P(z) + \sum_{j=1}^{J} Q_j(z) \right) \Big/ \left(P_0 + P(1) + \sum_{j=1}^{J} Q_j(1) \right).$$

where,

$$N2(z) = P_0 \left\{ \begin{array}{l} z\left\{ \sum_{i=1}^{k} \Theta_{i-1}\left(B_{i-1}^*\left[A_{i-1}(z)\right]\right)\left(1 - S_i^*(A_i(z))\right)[1 + r_i S_i^*(A_i(z))]\left((N(z)-1)(R^*(\lambda) + X(z)\left(1 - R^*(\lambda)\right)) + X(z) \right) \right\} \\ -N(z)\left(z - \left[R^*(\lambda) + X(z)\left(1 - R^*(\lambda)\right)\right] \left\{ (1-r_i)\Sigma + r_i\Theta_{i-1} B_i^*\left[A_i(z)\right] S_i^*\left[A_i(z)\right] \right\} \right) \\ +\left[1 - X(z)\right] \left(\begin{array}{l} z - \left[R^*(\lambda) + X(z)\left(1 - R^*(\lambda)\right)\right] \left\{ (1-r_i)\Sigma + r_i\Theta_{i-1} B_i^*\left[A_i(z)\right] S_i^*\left[A_i(z)\right] \right\} \\ +\left(X(z)\left\{ (1-r_i)\Sigma + r_i\Theta_{i-1} B_i^*\left[A_i(z)\right] S_i^*\left[A_i(z)\right] \right\} + z\left(N(z)-1\right)\right)\left(1 - R^*(\lambda)\right) \end{array} \right) \end{array} \right\},$$

$$\times\left(z - \sum_{i=1}^{k} \Theta_{i-1}\left(B_{i-1}^*\left[A_{i-1}(z)\right]\right)\left(1 - S_i^*(A_i(z))\right)[1 + r_i S_i^*(A_i(z))] \right)$$

$$D2(z) = (1-\omega)(1-z)\sum_{i=1}^{k} \Theta_{i-1}\left(B_{i-1}^*\left[A_{i-1}(z)\right]\right)\left(1 - S_i^*(A_i(z))\right)[1 + r_i S_i^*(A_i(z))] \times Dr(z).$$

The first term can be obtained through the without vacation definition of our system.

$$\chi(z) = \left\{ \frac{(1-\omega)(1-z)\sum_{i=1}^{k} \Theta_{i-1}\left(B_{i-1}^*\left[A_{i-1}(z)\right]\right)\left(1 - S_i^*(A_i(z))\right)[1 + r_i S_i^*(A_i(z))]}{\left(z - \sum_{i=1}^{k}\left\{ (p_i z + q_i)\Theta_{i-1}\left(B_i^*\left[A_i(z)\right]\right)[1 + r_i S_i^*(A_i(z))] \right\} \right)} \right\}.$$

From above stochastic decomposition law, we observe that $K(z) = c(z).\tilde{O}(z)$ which conform that the decomposition result of Fuhrman and Cooper [11], also valid for this special vacation system.

5.1. Special case

In this section a special case of the above model, which are consistent with the existing literature is analysed.

Case (i): *Single phase, No retrial,, No Vacation and No breakdown, No delaying repair*

Let $P[X = 1] = 1$, $R^*(\lambda) \to 1$, $P[V = 0] = 1, V^*(\lambda) \to 1, r = 1$ and $a_i = 0$. Our model can be reduced to multi stage M/G/1 queueing system with Bernoulli feedback. The following results agree with Salehirad and Badamchizadeh [23].

$$K(z) = P_0 \left\{ \frac{\left(1 - S_1^*(A_1(z))\right) + \sum_{i=2}^{k} \Theta_{i-1}\left(B_{i-1}^*\left[A_{i-1}(z)\right]\right)\left(1 - S_i^*(A_i(z))\right)}{z - \sum_{i=1}^{k} \left\{\left(p_i z + q_i\right)\Theta_{i-1}\left(B_i^*\left[A_i(z)\right]\right)\right\}} \right\}.$$

6. NUMERICAL ILLUSTRATION

In this section, some numerical examples are given using MATLAB in order to illustrate the effect of various parameters in the system performance are presented. Here all retrial times, service times, vacation times and repair times are exponentially, Erlangianly and hyper-exponentially distributed. We assume arbitrary values to the parameters such that the steady state condition is satisfied. The following tables give the computed values of various characteristics of our model like, probability that the server is idle P_0, the mean orbit size L_q, probability that server is idle during retrial rime, busy on k stages, vacation on j th stage of vacation and under repair on k stages respectively, P, Π_i, Q_j and R_i for (i=1,2,…k, j=1,2,…J) where exponential distribution is $f(x)=\nu e^{-\nu x}, x>0$ Erlang-2stage distribution is $f(x)=\nu^2 x e^{-\nu x}, x>0$ and hyper-exponential distribution is $f(x)=c\nu e^{-\nu x}+(1-c)\nu^2 e^{-\nu^2 x}, x>0$.

Table 1 shows that when mean batch size ($E(X)$) increases, then the probability that server is idle P_0 decreases, the mean orbit size L_q increasing and probability that the waiting time of the customer in the orbit during retrial time (W_q) also increasing for the values of $\lambda = 0.75$; $p_1 = 0.2$; $\mu_1 = 5$; $\alpha_1 = 0.2$; $\xi_1 = 3$; $\gamma = 5$; k =1.; $r_1 = 0.5$ $\theta_1 = 0.2$; J =1; a = 4. Table 2 shows that when the delaying repair rate (·) increases, then the probability that server is idle P_0 decreases, the mean orbit size L_q increasing and probability that the server on delaying repair (R_s) also increasing for the values of $\lambda = 2$; $p_1 = 0.2$; $\mu_1 = 5$; $\alpha_1 = 0.2$; $\xi_1 = 3$; $\gamma = 5$; $\theta_1 = 0.2$; $p_2 = 0.4$; $\mu_2 = 7$; $\alpha 2 = 0.4$; $\xi_2 = 5$; $\eta_2 = 5$; $\eta_1 = 10$; k =2.;$\theta_2 = 0.4$; J =2; $E(X)=1$.

Table 3 shows that when the feedback (p_1) increases, then the probability that server is idle P_0 decreasing, the mean orbit size L_q increasing and probability that server is idle during retrial time P increasingfor the values of $\lambda = 0.3$; $\mu_1 = 5$; $\alpha_1 = 0.2$; $\xi_1 = 3$; $\gamma = 5$; $\theta_1 = 0.2$; $\mu_2 = 7$; $\alpha 2 = 0.4$; $\xi_2 = 5$; $\eta_2 = 5$; $\eta_1 = 10$; k =1; $\theta_2 = 0.4$; J =1; $E(X)=2$; $r_1=0.3$; $r_2=0.5$.

Table 1
The effect of Mean batch size E(X) on P_0, Lq and W_q

Retrial distribution	*Exponential*			*Erlang – 2 stage*			*Hyper – Exponential*		
E(X)	P_0	L_q	W_q	P_0	L_q	W_q	P_0	L_q	W_q
Mean batch size									
0.70	0.6881	0.1158	0.2205	0.3777	0.5176	0.9858	0.7124	0.1034	0.1970
0.80	0.6597	0.1350	0.2250	0.3027	0.8583	1.4305	0.6867	0.1189	0.1981
0.90	0.6299	0.1603	0.2375	0.2164	1.6205	2.4008	0.6598	0.1389	0.2058
1.00	0.5984	0.1937	0.2582	0.1164	4.1135	5.4847	0.6316	0.1647	0.2197

Table 2
The effect of different re service probabilities (r_1) on P_0, Lq and R_s

Re service probability	*Exponential*			*Erlang – 2 stage*			*Hyper – Exponential*		
r_1	P_0	L_q	R_s	P_0	L_q	R_s	P_0	L_q	R_s
0.10	0.5151	0.2918	0.2810	0.2075	0.5212	2.0291	0.6095	1.7624	0.1483
0.20	0.5129	0.3722	0.3423	0.2067	0.6360	2.2615	0.6068	1.8431	0.1928
0.30	0.5109	1.4489	0.4013	0.2059	0.7448	2.4748	0.6042	1.9201	0.2356
0.40	0.5090	1.5221	0.4582	0.2052	0.8482	2.6723	0.6018	1.9938	0.2768

Table 3
The effect of feedback probability P_1 on $P_{0,}$ L_q and P

Feedback probability	*Exponential*			*Erlang – 2 stage*			*Hyper – Exponential*		
P_1	P_0	L_q	P	P_0	L_q	P	P_0	L_q	P
0.10	0.7248	0.0946	0.0564	0.3609	0.6661	0.1639	0.7489	0.0819	0.0507
0.20	0.7104	0.1088	0.0624	0.3161	0.8927	0.1832	0.7360	0.0937	0.0561
0.30	0.6939	0.1261	0.0692	0.2629	1.2764	0.2060	0.7213	0.1081	0.0622
0.40	0.6749	0.1479	0.0771	0.1985	2.0412	0.2336	0.7044	0.1258	0.0692

Table 4
The effect of vacation Rate (g) on $P_{0,}$ L_q and Q

Vacation rate	*Exponential*			*Erlang – 2 stage*			*Hyper – Exponential*		
γ	P_0	L_q	Q	P_0	L_q	Q	P_0	L_q	Q
1.00	0.2873	2.4204	0.5387	0.1864	6.7646	1.3630	0.3081	1.3242	0.5777
2.00	0.4874	1.6963	0.3427	0.5035	5.6750	1.0079	0.5513	0.8603	0.3372
3.00	0.5854	1.3709	0.2466	0.7173	4.6776	0.7684	0.6572	0.6774	0.2324
4.00	0.6414	1.1902	0.1917	0.8557	3.9865	0.6134	0.7139	0.5825	0.1419

Table 4 shows that when vacation rate (γ) increases, then the probability that server is idle P_0-increases, the mean orbit size L_q and probability that server is Vacation on j^{th} stage of vacation Q also decreasing for the values $\lambda = 0.5$; $p_1 = 0.2$; $\mu_1 = 5$; $\alpha_1 = 0.2$; $\xi_1 = 3$; $\theta_1 = 0.2$; $p_1 = 0.4$; $\mu_2 = 7$; $\alpha 2 = 0.4$; $\xi_2 = 5$; $\eta_2 = 5$; $\eta_1 = 10$; $k = 2$; $\theta_2 = 0.4$; $J = 2$; $E(X) = 1$.

For the effect of the parameters a, p_1, γ, η, r, and ξ_1 on the system performance measures, two dimensional graphs are drawn in Figure 1-4. Figure 1 shows that the mean orbit size L_q decreases for the increasing the value of the retrial rate (a). Figure 2 shows that the mean orbit size L_q decreasing for the increasing the value of the vacation rate γ. Figure 3 shows that the idle probability P_0 increases for the increasing the value of repair rate on FSS (ξ_1). Figure 4 shows that an idle probability P_0 increases for the increasing the value of delaying repair rate on FSS $\eta_{1.}$

Three dimensional graphs are illustrated in Figure 5 and 6. In Figure 5, the surface displays upward trend as expected for increasing the value of the repair rate on FSS ξ_1 and the feedback probability p_1 against the mean

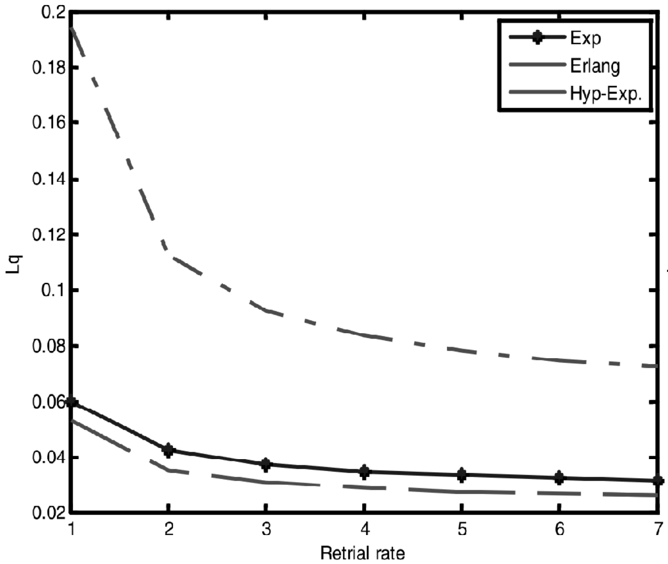

Figure 1: L_q verses a

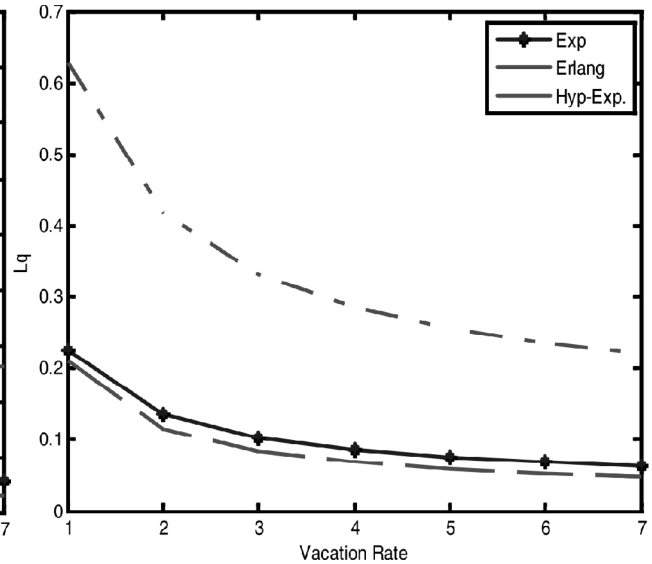

Figure 2: L_q verses γ

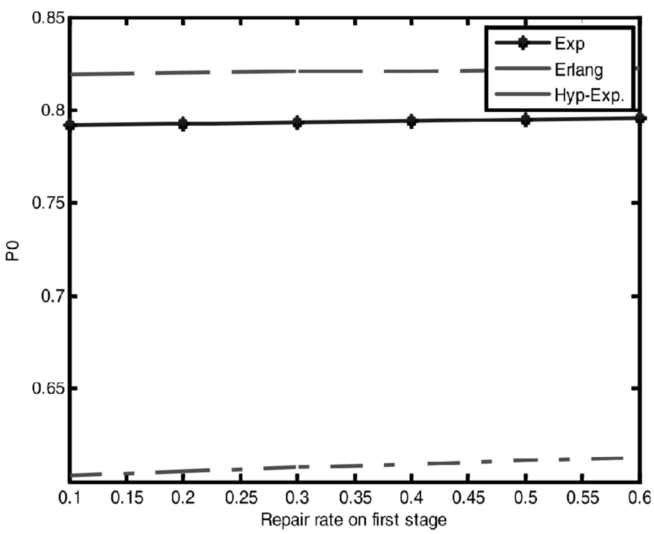

Figure 3: P_0 verses ξ_1

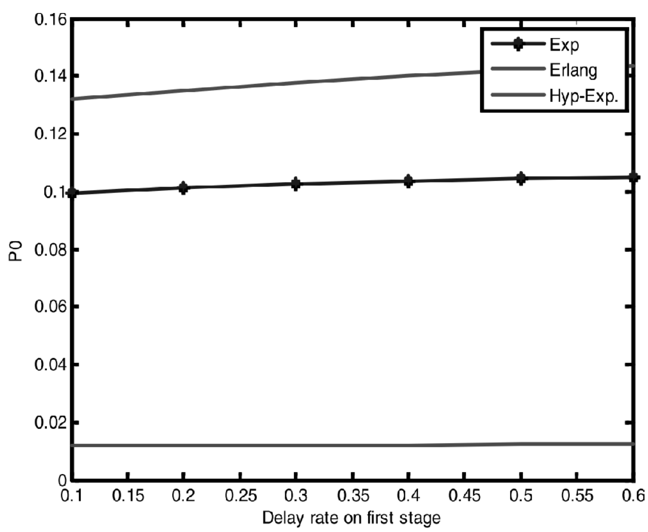

Figure 4 P_0 verses η

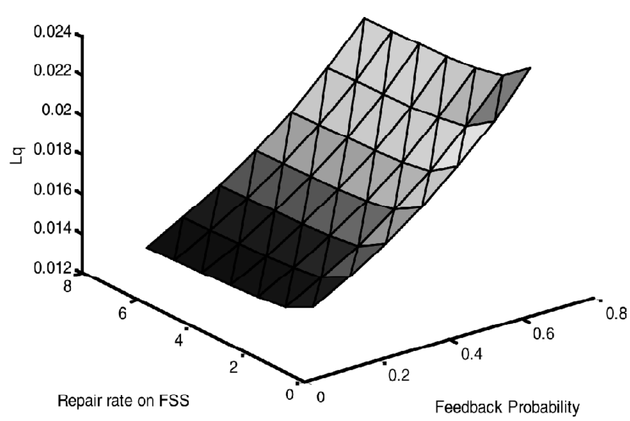

Figure 5: L_q verses ξ_1 and p_1

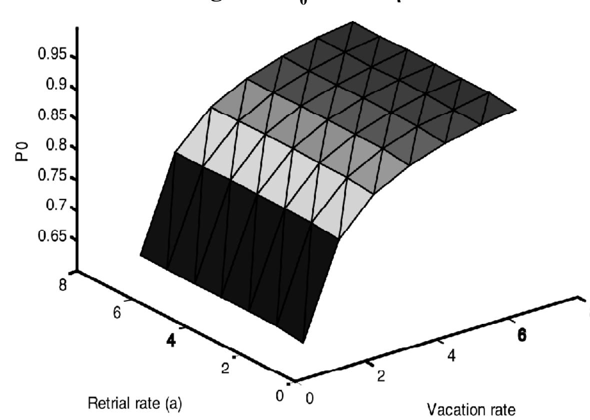

Figure 6: P_0 verses a and γ

orbit size L_q.The idle probability P_0 decreases for increasing the value of retrial rate *a* and the increasing value of the vacation rate*g* is shown in Figure 6.

7. CONCLUSION

In this paper, a batch arrival retrial queue with *k* optional stages of service under modifiedvacation policy, where each type of service consists of an optional re-service and the server is subject to server breakdowns and repair are meticulously studied.The probability generating functions of the number of customers in the system and orbit are found by using the supplementary variable technique. The performance measures like, the mean number of customers in the system/orbit, the average waiting time of customer in the system/orbit and some system probabilities were obtained. Finally, the general decomposition law is shown to hold good for this model. The analytical results are validated with the help of numerical illustrations.

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