Indian Journal of Mathematics and Mathematical Sciences Vol. 12, No. 1, (January. 2016)

MADHAVA'S CORRECTION FUNCTION TECHNIQUE

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Abstract

In this paper we give a rational approximation to the alternating harmonic series, by applying a correction function to the series. The introduction of correction function certainly improves the value of sum of the series and gives a better approximation to it.

Key Words: *Correction function, error function, remainder term , alternating harmonic series, rational approximation.*

INTRODUCTION

Commenting on the Lilavati rule for finding the value of circumference of a circle from its diameter, the commentator Sankara refers to several important enunciations from the works of earlier and contemporary mathematicians and gives a detailed exposition of various results contained in them. Sankara also refers to various infinite series for computing the circumference from the diameter. One such series attributed to illustrious mathematician Madhava of 14th century is

$$C = \frac{4d}{1} - \frac{4d}{3} + \frac{4d}{5} - \ldots\ldots\ldots\ldots \pm \frac{4d}{2n-1} \mp \frac{4d\left(\frac{2n}{2}\right)}{(2n)^2+1}$$, where + or – indicates that

n is odd or even and C is the circumference of a circle of diameter d.

or more specifically,

$$C = \frac{4d}{1} - \frac{4d}{3} + \frac{4d}{5} - \ldots\ldots\ldots\ldots + (-1)^{n-1}\frac{4d}{2n+1} + (-1)^n \frac{4d(2n)/2}{(2n)^2/1}$$

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$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-...........+(-1)^{n-1}\frac{1}{2n-1}+(-1)^{n}\frac{(2n)/2}{(2n)^{2}+1},$$ for large n.

MADHAVA’S CORRECTION FUNCTION

For the Madhava series, rational approximation to the value of C (and hence for π) may be obtained. The remainder term $(-1)^n$ 4d G_n where $G_n=\frac{(2n)/2}{(2n)^2+1}$ satsfying the condition $G_n + G_{n+1} = \frac{1}{2n+1}$ has been augmented to the series for C by Madhava to get a better approximation. The introduction of the remainder term definitely improves the value of C and is very effective in giving a better approximation for it.

RATIONAL APPROXIMATION OF ALTERNATING HARMONIC SERIES

The Alternating Harmonic Series (abbreviated as AHS) is convergent and converges to log 2.

Thus log 2 = $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+......................... +\frac{(-1)^{n-1}}{n}+..................$

The **remainder term** R_n for AHS is the sum of the series after n terms

ie $R_n=\Sigma_{k=n+1}^{\infty}(-)^{k-1}\frac{1}{k}$

ie $R_n=(-1)^n G_n$

where $G_n=\Sigma_{i=1}^{\infty}(-1)^{i-1}\frac{1}{(n+i)}$ is the **correction function**

after n terms of AHS.

The **error function is** $E_n = G_n + G_{n+1} - \frac{1}{n+1}$.

Theorem 1: The correction function after n terms for Alternating Harmonic Series is $G_n=\frac{1}{2n+1}$

Proof: We have AHS is convergent and converges to log 2.

If G_n denotes the correction function after n terms of A H S,

then it follows that $G_n + G_{n+1} = \frac{1}{n+1}$

Now we define, the error function as $E_n = G_n + G_{n+1} - \frac{1}{n+1}$

We may choose G_n in such a way that $|E_n|$ is a minimum function of n. For a fixed n and for $r \in R$, let $G_n(r) = \frac{1}{(2n+2)-r}$.

Then the error function is a rational function of r, say

$E_n(r) = \frac{N_n(r)}{D_n(r)}$ where

$D_n(r) = (2n+2-r)(2n+4-r)(n+1)$

$\approx 4n^3$, which is a maximum for large values of n.

$$N_n(r) - \begin{cases} (r-1)(2n+2-r)+r, & r \neq 1 \\ 1 & r = 1 \end{cases}$$

Thus $|N_n(r)|$ is a minimum function of n for $r = 1$.

So $|E_n(r)|$ is a minimum function of n for $r = 1$.

For $|r| > 1$, the magnitude of the error function increases

Hence for $r = 1$, E_n and G_n are functions of a single variable n.

The minimum value of $|E_n| = \frac{1}{4n^2 + 12n^2 + 11n + 2}$

Hence the correction function after n terms of AHS is $G_n = \frac{1}{2n+1}$

and the corresponding error function is

$$|E_n| = \frac{1}{4n^3 + 12n^2 + 11n + 3}$$

Hence the proof.

Remark 1

Clearly $G_n = \frac{1}{2n+1} < \frac{1}{n+1}$, the absolute value of the $(n+1)^{th}$ term.

Remark 2

We have $$\frac{1}{2n+2} < \frac{1}{2n+1} < \frac{1}{2n}$$

That is $$\frac{1}{2}\left|\frac{1}{n+1}\right| < G_n < \frac{1}{2}\left|\frac{1}{n}\right|$$

Theorem 2

The correction functions of AHS follow an infinite continued fraction

$$\frac{1}{(2n+1)+}\,\frac{1^2}{(2n+1)+}\,\frac{2^2}{(2n+1)+}\,\frac{3^2}{(2n+1)+\ldots}$$

Proof

The correction function for AHS is $G_n = \frac{1}{2n+1}$

The error function is $E_n = \frac{1}{4(n+1)^2 - (n+1)}$

Here onwards we shall rename this correction function as the first order correction function, denoted by $G_n(1) = \frac{1}{2n+1}$ and

error function as $$E_n(1) = \frac{1}{4(n+1)^2 - (n+1)} = \frac{k_1}{f_1(n)}$$

For further reducing error function we may add fractions of correction divisor to the correction divisor itself.

Choose $G_n(2) = \cfrac{1}{(2n+1) + \cfrac{A_1}{(2n+1)}}$ where A_1 is any real number.

By using the argument of minimizing error in theorem 1, it can be proved that $|E_n|$ is a minimum function of n for $A_1 = k_1 = 1$

The second order correction function is Gn(2) = $\dfrac{1}{(2n+1)+\dfrac{1}{(2n+1)}}$

The error function is $E_n(2) = \dfrac{4}{\{(2n+1)^2+1\}\{(2n+1)^2+1\}\{n+1\}} = \dfrac{k_2}{f_3(n)}$

Again for reducing error ,choose the correction function as

$$G_n(3) = \cfrac{1}{(2n+1)+\cfrac{1}{(2n+1)+\cfrac{A_2}{(2n+1)}}}$$

It can be proved that $|E_n|$ is a minimum function of n for $A_2 = \dfrac{k_2}{k_1} = 4$

The third order correction function is

$$G_n(3) = \cfrac{1}{(2n+1)+\cfrac{1}{(2n+1)+\cfrac{1}{(2n+1)}}}$$

The error function is $E_n(3) = \dfrac{26}{\{(2n+1)^3+5(2n+1)\}\{(2n+3)^3+5(2n+3)\}\{n+1\}}$

$$= \frac{k_3}{f_3(n)}$$

For further reducing error, choose the correction function as

$$G_n(4) = \cfrac{1}{(2n+1)+\cfrac{1}{(2n+1)+\cfrac{4}{(2n+1)+\cfrac{A_3}{(2n+1)}}}}$$

Then it can be shown that $|E_n|$ will be a minimum function of n

when $A_3 = \frac{k_3}{k_2} = \frac{36}{4} = 9$

The fourth order correction function is $G_n(4)$ =

$$\cfrac{1}{(2n+1)+\cfrac{1}{(2n+1)+\cfrac{1}{(2n+1)+\cfrac{4}{(2n+1)+\cfrac{9}{(2n+1)}}}}}$$

In general

The i th order correction function is

$$G_n(i) = \frac{1}{(2n+1)+}\frac{1^2}{(2n+1)+}\frac{2^2}{(2n+1)+}\frac{3^2}{(2n+1)+\ldots+}\frac{(i-1)^2}{(2n+1)}$$

Continuing this process we get the correction function follows an infinite continued fraction pattern as follows.

$$\cfrac{1}{(2n+1)+\cfrac{1^2}{(2n+1)+\cfrac{2^2}{(2n+1)+\cfrac{3^2}{(2n+1)+\cfrac{4^2}{(2n+1)+\ldots}}}}}$$

Corollary

The i^{th} order correction function for AHS is the i^{th} successive convergent of the infinite continued fraction $\frac{1}{(2n+1)+}\frac{1^2}{(2n+1)+}\frac{2^2}{(2n+1)+}\frac{3^2}{(2n+1)+\ldots}$.

Proof

The i^{th} order correction function is

$$G_n(i) = \frac{1}{(2n+1)+}\frac{1^2}{(2n+1)+}\frac{2^2}{(2n+1)+}\frac{3^2}{(2n+1)+\ldots+}\frac{(i-1)^2}{(2n+1)}$$

Clearly $G_n(i)$ is the i[th] successive convergent of the infinite continued fraction

$$\frac{1}{(2n+1)+}\frac{1^2}{(2n+1)+}\frac{2^2}{(2n+1)+}\frac{3^2}{(2n+1)+\ldots}$$

The correction functions and the corresponding error functions are tabulated as follows.

Let $p = n + 1$. Then $G_n = \dfrac{1}{2p-1}$

Correction function G_n	*Error function* $\lvert E_n \rvert$
$\dfrac{1}{2p-1}$	$\dfrac{1}{4p^3-p}$
$\dfrac{1}{(2p-1)+\dfrac{1}{(2p-1)}}$	$\dfrac{4}{16p^5+4p}$
$\dfrac{1}{(2p-1)+\dfrac{1}{(2p-1)+\dfrac{4}{(2p-1)}}}$	$\dfrac{36}{64p^7+112p^5+112p^2-36p}$
$\dfrac{1}{(2p-1)+\dfrac{1}{(2p-1)+\dfrac{4}{(2p-1)+\dfrac{9}{(2p-1)}}}}$	$\dfrac{576}{256p^9+1536p^7+3072p^5-256p^2+576p}$
…………………………………..	…………………………………..
…………………………………..	…………………………………..
…………………………………..	…………………………………..

APPLICATION

If S_n denotes the sequence of partial sums of AHS and G_n denotes the correction function after n terms of AHS, then the approximation of AHS while applying correction function is shown in the following table.

We have **ln2 = 0.6931471806** using a calculator

Number of terms (n)	S_n	$S_n + (-1)^n G_n$
10	**0.6**456349206	**0.693**2530476
100	**0.6**881721793	**0.693147**3037
1000	**0.69**26474306	**0.693147180**7
10000	**0.693**0971831	**0.6931471806**
100000	**0.69314**21806	**0.6931471806**

From the table it is clear that the accuracy can be improved by using correction function.

For **n = 10**, the approximation of series using successive convergents is shown below

Correction function	*ln 2*	*Accuracy*
Without correction function	0.**6**45634920	1
$G_n(1)$	0.**693**0033411	3
$G_n(2)$	0.**69341**87322	5
$G_n(3)$	0.**6931471**432	7
$G_n(4)$	0.**6931471806**	10

The table shows that the successive convergents of the infinite continued fraction of the correction function gives better approximation for the series.

CONCLUSION

The introduction of correction function gives better approximation for the series and hence accuracy can be improved.

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