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TOPOLOGY HAVING MAXIMUM NUMBER OF COMPLEMENTS IN LATTICE OF TOPOLOGIES

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Abstract: Let T(X) denote the set of all topologies defined on a fixed set X of size n. In this paper, we show that an infraspace topology in T(X) has the largest number of complements in T(X), which answer an open problen in lattice of topologies.

Keywords: Lattice of topologies,ultraspaces, infraspaces, complementation.

INTRODUCTION

In 1936, Birkhoff (Birkhoff, 1936) proved that the set of all topologies T(X) on a fixed set X forms a lattice. Since 1936, many topologists worked on the entire complex structure of T(X) and brought out many beautiful and interesting properties. The study of T(X) is extremely important in the basis pursuit of point-set topology and combinatorial topology. In the article of Birkhoff (Birkhoff, 1936), he defined partial order on T(X) by letting $\tau_1 < \tau_2$ if and only if $\tau_1 \subset \tau_2$ for $\tau_1, \tau_2 \in T(X)$ and proved that it is a lattice. Infact, T(X) is a complete lattice possessing a largest and a smallest element namely the discrete and the indiscrete topologies on X. T(X) possesses ultraspaces and infraspaces. An ultraspace is is a maximal proper topology. A topology τ is an ultraspace if and only if $\tau = \tau(X, \Upsilon) = \{\upsilon \subset X : x \in \upsilon \to \upsilon \in \Upsilon\}$ where Υ is an ultrafilter on X different from $\Upsilon(x)$, the principal ultrafilter generated by *x*. Hence $\tau(x, \Upsilon)$ is a topology on X such that for every $x' \in X, x' \neq x$ the set $\{x'\}$ is open and the open sets containing x are the sets in Υ which contains x. An ultraspace $\tau(X, \Upsilon)$ is principal if and only if $\Upsilon = \Upsilon(y)$ for some $y \in X \setminus \{x\}$. Every topology τ on X is the infimum of the ultraspaces on X which are finer than τ. An infraspace is a minimal proper topology. A topology τ is an infraspace if and only if $\tau = \{\phi, A, X\}$ where $A \neq \phi, A \neq X$ and $A \subset X$.

Two topologies τ_1 and τ_2 are complements of each other if and only if if the following two conditions are satisfied.

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1. If $U \in \tau_1 \cap \tau_2$ and $U \neq \phi$, then $U = X$.
2. For every $x \in X$ there exists $U_1 \in \tau_1, U_2 \in \tau_2$ with $\{x\} = U_1 \cap U_2$.

In $T(X)$ if $|X| = n$, Steiner(Steiner, 1966) proved that there are $n(n-1)$ ultraspaces(all principal) and $2^n - 2$ infraspaces. Also it is shown that if $\tau = \tau(x, \Upsilon(y))$ is an ultraspace in T(X) then the largest complement for τ is given by $\tau' = \{v \subset X : x \in v \subset X \setminus \{y\}\} \cup \{\phi, X\}$. Watson (Watson, 1994) proved that the topology $\tau = \{\phi, \{x\}, X \setminus \{x\}, X\}$ has the least number of complements in T(X) and are of the form $\tau(x, \Upsilon(y)) \wedge \tau(y, \Upsilon(x)$ where $y \in X \setminus \{x\}$.

’Which topology on a set of size n has the largest number of complements?’ was an open problem posed byWatson (Van Mill, Reed et al., 1990) in lattice of topology. This paper partially answer the question.

For basic definitions and notations references cited are Davey and Priestley (Davey and Priestley, 2002), and Thron (Thron, 1966).

INFRASPACE TOPOLOGIES IN T(X)

Theorem 2.1. Let $\tau = \{\phi, A, X\}$ be an infraspace in T(X) such that $|A| = 1$. The τ has $2^{n-1} - 1$ complements in T(X) where n = jXj.

Proof. Let $A = \{x_1\} \subset X$. Consider the family of ultraspace $\{\tau(x_1, \Upsilon(x_i)) : i = 2, 3, \ldots, n\}$ on T(X). The above family contains $(n-1)$ ultraspaces, all are complementsof τ. Since, for each $x \in X$ there exists open sets $G_1 \in \tau_1$ and $G_2 \in \tau(x_1, \Upsilon(x_i))$ for all $i = 2, 3, \ldots, n$ such that $G_1 \cap G_2 = \{x\}$. If $U \in \tau \cap \tau(x_1, \Upsilon(x_i))$ for all $i = 2, 3, \ldots, n$ and $U \neq \phi$, then *U* = *X*.

Again consider the collection of all largest topologies contained in $\{\tau(x_1, \Upsilon(x_i)) \wedge \tau(x_1, \Upsilon(x_j)) : \forall i \neq j = 2, 3, \ldots, n\}$. Each member of the above collection is a topology which is the intersection of two ultraspaces in which for each $x \in X$ if $x \neq x_1$, the set $\{x\}$ is open and the open set containing x_1 are the sets containing $\Upsilon(x_i) \cap \Upsilon(x_j)$. The above collection contains $\frac{(n-1)(n-2)}{2}$members, all of them are complements of “.

Proceeding, similarly we can see that topologies which is the intersection of 3 ultraspaces, 4 ultraspaces and so on $(n-1)$ ultraspaces are complements of τ. Thus, we get $2^{n-1} - 1$ complements for τ.

Corollary 2.1. The topology $\tau_c = \{\phi, X \setminus A, X\}$ has the same number of complements as τ in *T(X)*.

Theorem 2.2. Let $\tau = \{\phi, A, X\}$ be an infraspace in T(X) such that $|A| = 2$. Then τ has $2^{2n-4} - 1$ complements in T(X) where $n = |X|$.

Proof. Let $A = \{x_1, x_2\} \subset X$. Consider the collection of ultraspace $\tau_1 = \{\tau(x_1, \Upsilon(x_i)) : i = 3, 4, \ldots, n\}$ and $\tau_2 = \{\tau(x_2, \Upsilon(x_j)) : j = 3, 4, \ldots, n\}$ in T(X). Each member of τ_1 and τ_2 are complements of τ and they are $(2n-4)$ in number. Again, consider the collection of all topologies which is the intersection of ultraspaces of the form $\{\tau(x_1, \Upsilon(x_i)) \wedge \tau(x_1, \Upsilon(x_j)) : i \neq j = 3, 4, \ldots, n\}$ and $\{\tau(x_2, \Upsilon(x_i)) \wedge \tau(x_2, \Upsilon(x_i)) : i = 3, 4, \ldots, n\}$. Each member of the above collection are complements of τ.

Proceeding, similarly we can see that topologies contained in the intersection of 3 ultraspaces, 4 ultraspaces and so on, $(n-2)$ ultraspaces are all complements of τ. Thus, we get $2^{2n-4} - 1$ complements for τ.

Corollary 2.2. The topology $\tau_c = \{\phi, X \setminus A, X\}$ has the same number of complements as τ in T(X).

Theorem 2.3. Let $\tau = \{\phi, A, X\}$ be an arbitrary infraspace in T(X) such that $|X| = n$. Then τ has largest number of complements in T(X) if $|A| = \frac{n-1}{2}$ *or* $\frac{n+1}{2}$ *or* $\frac{n}{2}$ according as n is odd or even.

Proof. Let $A = \{x_1, x_2, \ldots, x_k\} \subset X$ where $1 < k < n$. Consider the collection of ultraspaces $\{\tau(x_i, \Upsilon(x_j)) : i = 1, 2, \ldots, k; j = k+1, k+2, \ldots, n\}$ in $T(X)$. The above collection contains $k(n-k)$ ultraspaces in which for each $x \in X$ if $x \neq x_i$ for all *i*. The set $\{x\}$ is open and open sets containing x_i are the set containing x_i in $\Upsilon(x_j)$ and are complements of τ.

Again, consider the collection of all ultraspaces of the form $\{\tau(x_i, \Upsilon(x_j)) \wedge \tau(x_i, \Upsilon(x_{j'})) : i = 1, 2, \ldots, k; j \neq j' = k+1, k+2, \ldots, n\}$ and $\{\tau(x_i, \Upsilon(x_j)) \wedge \tau(x_{i'}, \Upsilon(x_j)) : i \neq j' = 1, 2, \ldots, k; j = k+1, k+2, \ldots, n\}$. Each member of the above collection are topologies contained in the intersection of two ultraspaces and are complements of τ.

Proceeding, similarly we can see that the collection of topologies contained in the intersection of 3 ultraspaces, 4 ultraspaces and so on, $(n-k)$ ultraspaces are complements of $\mathcal{T}$. $\mathcal{T}$ and $\tau_c = \{\phi, X \setminus A, X\}$ have equal number of complements in T(X). Hence if n is even $k = n - k$, i.e, $k = \frac{n}{2}$. If n is odd, $k = (n-1) - k$ or $k = (n+1) - k$, i.e, $k = \frac{n-1}{2}$ or $\frac{n+1}{2}$.

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