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ON SUBMANIFOLDS OF AN ALMOST CONTACT MANIFOLD

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Abstract. The present paper deals with the study of submanifolds of an almost contact manifold. We have studied certain results for a submanifold of an almost contact manifold equipped with quarter-symmetric metric connection.

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1. INTRODUCTION

In the field of differential geometry, submanifolds theory is one of the most interesting topic for geometers. The submanifolds theory has the root in the study of the geometry of the plane curves started by Fermat. Since then it has been involving in the different directions of mechanics and dynamics. In 1958, Boothby and Wang initiated the study of odd dimensional contact and almost contact manifolds. In 1976, I. Sato [15] has introduced the notion of an almost para-contact structure on differentiable manifolds. These structures are analogues to almost contact structures [3]. Since para-contact structures and almost contact structures on differentiable manifold are odd dimensional analogue to almost product structures and almost complex structures respectively.

Golab [8] introduced and studied the properties of quarter- symmetric connection on a differentiable manifold and further it was studied by Mishra and Pandey [9] and many others. Sengupta and Biswas have introduced a quarter-symmetric non-metric connection on a Sasakian manifold [16]. In 1982 and 1991, Yano and Imai [17] studied quarter-symmetric metric connection in Hermitian and Kaehlerian manifold, whereas Mukhopadhyay [11] studied this connection on Riemannian manifold with an almost complex structure ϕ.

In 2010 R. H. Ojha and S. k. Chaubey [4] studied a quarter-symmetric non-metric connection on an almost Hermite manifold and further it is developed by many authors. S. Ali and R. Nivas [1] introduced quarter-symmetric connection on submanifolds of a manifold.

Motivated by the above works, we study certain topological properties of the induced submanifold of an almost contact manifold equipped with quarter-symmetric metric connection. The present paper is organized as follows: Section 2 concern with some preliminaries of almost contact structure and its submanifold. In section 3, we defined quarter-symmetric metric connection. Section 4 contains some results on submanifolds of an almost contact manifold equipped with quarter-symmetric metric connection.

2. ALMOST CONTACT MANIFOLD AND ITS SUBMANIFOLD

An m-dimensional differentiable manifold $\tilde{M}$ is called an almost contact manifold [7], if it admits an almost contact structure (ϕ, ξ, η) consisting of a (1,1) tensor field ϕ, vector field ξ and 1-form η satisfying

$$\phi^2 X = -X + \eta(X)\xi \tag{2.1}$$

$$\phi(\xi) = 0, \ \eta o \phi = 0, \tag{2.2}$$

$$\mathrm{Rank}(\phi) = m-1, \tag{2.3}$$

for all $X \in T\tilde{M}$.

Let g be Riemannian metric satisfying

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.4}$$

$$g(X, \xi) = \eta(X), \tag{2.5}$$

for all $X, Y \in T\tilde{M}$. Then $\tilde{M}$ becomes an almost contact Riemannian manifold equipped with the almost contact Riemannian structure (ϕ, ξ, η, g).

An almost contact Riemannian manifold is called a Sasakian manifold [12], if it satisfies

$$(D_X\phi)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.6}$$

$$D_X \xi = - \phi X, \tag{2.7}$$

for all $X,Y \in T(\tilde{M})$, where D is Riemannian connection on $\tilde{M}$.

Let $\tilde{N}$ be an n-dimensional submanifold of an almost contact manifold $\tilde{M}$ with a positive definite metric g. Let g is also induced metric on $\tilde{N}$. The usual Gauss and Weingarten formulae are given by

$$D_X Y = \tilde{D}_X Y + h(X,Y), \ X,Y \in T\tilde{N}, \tag{2.8}$$

$$D_X N = \tilde{D}_X^{\perp} N - A_N X, \ N \in T^{\perp}\tilde{N}, \tag{2.9}$$

where D is Riemannian connection on $\tilde{M}$ and $\tilde{D}$ is the induced Riemannian connection on $\tilde{N}$, h is the second fundamental form of $\tilde{N}$ and $-A_N X$ and $\tilde{D}_X^{\perp} N$ are the tangential and normal parts of $D_X N$. A_N is the Weingarten operator with respect to N and $\tilde{D}^{\perp}$ is the normal connection in the normal bundle $T^{\perp}\tilde{N}$. Furthermore, h and A_N are related as [18]

$$g(h(X,Y),N) = g(A_N X,Y). \tag{2.10}$$

The submanifold $\tilde{N}$ of an almost contact manifold $\tilde{M}$ is called an invariant submanifold [13] if ϕ preserves the tangent space of $\tilde{N}$, i.e., $\phi(T_p\tilde{N}), \subset T_p\tilde{N}$, for each point $p \in \tilde{N}$.

The submanifold $\tilde{N}$ of an almost contact manifold $\tilde{M}$ is called an anti-invariant submanifold [18] if ϕ maps the tangent space of $\tilde{N}$ into the normal space, i.e., $\phi(T_p\tilde{N}), \subset T_p^{\perp}\tilde{N}$ for each point $p \in \tilde{N}$.

The submanifold $\tilde{N}$ is called totally umbilical if $h(X,Y) = g(X,Y)H$, for all $X, Y \in T\tilde{N}$, where H is the mean curvature vector defined by $H = (1/n)\Sigma\{h(e_i, e_i)\}$, where $\{e_i\}$ is an orthonormal basis of $T\tilde{N}$. The submanifold is called totally geodesic if $h(X, Y) = 0$ for all $X, Y \in T\tilde{N}$.

3. QUARTER-SYMMETRIC METRIC CONNECTION

Let $(\tilde{M}, g)$ be a Riemannian manifold with a linear connection B. Then B is said to be quarter-symmetric connection [5], [6] if its torsion tensor T defined by

$$T(X, Y) = B_X Y - B_Y X - [X,Y], \tag{3.1}$$

satisfies

$$T(X, Y) = \eta(Y)\phi X - \eta(X)\phi Y, \tag{3.2}$$

where η is a 1-form and ϕ is a tensor of type (1, 1).

In addition, if a quarter-symmetric linear connection B satisfies the condition $(B_X g)(Y,Z)=0$ for all vector fields X, Y, Z on $\tilde{M}$, then B is said to be quarter-symmetric metric connection, otherwise it is called a quarter-symmetric non-metric connection.

In particular, if $\phi(X) = X$, then the quarter-symmetric metric connection reduces to semi-symmetric metric connection [[2], [14]].

Relation between the Riemannian connection D and the quarter-symmetric

metric connection B is given by [10]

$$B_X Y = D_X Y - \eta(X)\phi Y. \tag{3.3}$$

4. QUARTER-SYMMETRIC METRIC CONNECTION ON SUBMANIFOLDS OF AN ALMOST CONTACT MANIFOLD

In (3.3) we define quarter-symmetric metric connection B on an almost contact manifold. Now let $\tilde{B}$ is the induced connection on submanifold of an almost contact manifold from the connection B, then we have

$$B_X Y = \tilde{B}_X Y + \psi(X, Y), \tag{4.1}$$

where ψ is the second fundamental form of $\tilde{N}$ in $\tilde{M}$.

For $X \in T\tilde{N}$ and $N \in T^{\perp}\tilde{N}$, we put

$$\phi X = KX + LX, \quad KX \in T\tilde{N}, LX \in T^{\perp}\tilde{N}, \tag{4.2}$$

$$\phi N = tN + rN, tN \in T\tilde{N}, rN \in T^{\perp}\tilde{N}. \tag{4.3}$$

Using (3.3) in (4.1), we have

$$\tilde{B}_X Y + \psi(X, Y) = D_X Y - \eta(X)\phi Y. \tag{4.4}$$

Now using (2.8) and (4.2) in (4.4), we have

$$\tilde{B}_X Y + \psi(X, Y) = \tilde{D}_X Y + h(X, Y) - \eta(X)KY - \eta(X)LY. \tag{4.5}$$

Now in (4.5) equating tangential and normal parts, we have

$$\tilde{B}_X Y = \tilde{D}_X Y - \eta(X)KY, \tag{4.6}$$

$$\psi(X, Y) = h(X, Y) - \eta(X)LY. \tag{4.7}$$

From (4.1) and (4.7), we have

$$B_X Y = \tilde{B}_X Y + h(X, Y) - \eta(X)LY. \tag{4.8}$$

Torsion tensor with respect to the induced quarter-symmetric metric connection $\tilde{B}$ is given by

$$T(X,Y) = \tilde{B}_X Y - \tilde{B}_Y X - [X,Y]. \tag{4.9}$$

Using (4.6) in above equation, we have

$$T(X, Y) = \eta(Y)KX - \eta(X)KY, \tag{4.10}$$

Also using (4.8), we have

$$(\tilde{B}_X g)(Y, Z) = (B_X g)(Y, Z). \tag{4.11}$$

Hence we have the following theorem:

Theorem 1. If $\tilde{M}$ be an almost contact manifold equipped with a quarter-symmetric metric connection and $\tilde{N}$ be its submanifold then induced connection on $\tilde{N}$ is also quarter-symmetric metric connection.

If the submanifold $\tilde{N}$ is anti-invariant i.e. for any Y tangent to $\tilde{N}$, ϕY is normal to $\tilde{N}$, then from (4.2) we have KY = 0.

Therefore from (4.6), we have the following theorem:

Theorem 2. For an anti-invariant submanifold $\tilde{N}$ of an almost contact manifold $\tilde{M}$ with a quarter-symmetric metric connection, the induced quarter-symmetric metric connection and the induced Riemannian connection on submanifold are equivalent.

Let $\{e_1, e_2, e_3, ..., e_n\}$ be an orthonormal basis of $T\tilde{N}$, where $e_n = \xi$.

From (4.7), we obtain

$$\psi(e_i, e_i) = h(e_i, e_i) - \eta(e_i)L(e_i), \tag{4.12}$$

Since $L(e_n) = 0$ and $\eta(e_i) = 0$, i = 1, 2, ..., n-1, hence (4.12) become

$$\psi(e_i, e_i) = h(e_i, e_i),$$

Summing up for i = 1, 2,..., n and dividing by n, we obtain

$$H = (1/n)\Sigma\{h(e_i, e_i)\} = (1/n)\Sigma\{\psi(e_i, e_i)\}. \tag{4.13}$$

Hence we have the following theorem:

Theorem 3. The mean curvature of the submanifold $\tilde{N}$ with respect to the Riemannian connection coincides with the mean curvature of $\tilde{N}$ with respect to the quarter-symmetric metric connection.

From (4.7), we have

$$h(X, Y) - \psi(X, Y) = \eta(X)LY \tag{4.14}$$

If $\tilde{N}$ is totally umbilical with respect to the Riemannian connection and the quarter-symmetric metric connection, then from (4.13) and (4.14), we have

$$\psi(X, Y) = g(X, Y)H = h(X, Y), \tag{4.15}$$

Therefore, from (4.14), for all $X, Y \in T\tilde{N}$ we have

$$\eta(X)LY = 0. \tag{4.16}$$

Putting $X = \xi$ in above and using (2.1), we obtain $LY = 0$, for all $Y \in T\tilde{N}$, which implies that $\tilde{N}$ is an invariant submanifold.

Hence we have the following theorem:

Theorem 4. If submanifold $\tilde{N}$ of an almost contact manifold $\tilde{M}$ is totally umbilical with respect to the Riemannian connection and the quarter-symmetric metric connection, then $\tilde{N}$ is an invariant submanifold.

Also from (4.2), if submanifold $\tilde{N}$ is invariant then $LY = 0$, for all $Y \in T\tilde{N}$. Then from (4.14), we have following theorem:

Theorem 5. If $\tilde{N}$ is invariant, then $\tilde{N}$ is totally umbilical (resp., totally geodesic) with respect to the quarter-symmetric metric connection if and only if $\tilde{N}$ is totally umbilical (resp., totally geodesic) with respect to the Riemannian connection.

Let us consider that $\tilde{M}$ is a Sasakian manifold. Using (2.8), we have

$$\begin{aligned}(D_X\phi)Y &= D_X(\phi Y) - \phi(D_XY)\\ &= \tilde{D}_X(\phi Y) + h(X, \phi Y) - \phi(\tilde{D}_XY + h(X, Y))\\ &= (\tilde{D}_X\phi)Y + h(X, \phi Y) - \phi h(X, Y)\end{aligned} \tag{4.17}$$

Hence we have the following theorem:

Theorem 6. If $\tilde{M}$ is a Sasakian manifold with a Riemannian connection, then the submanifold $\tilde{N}$ is also a Sasakian manifold with respect to the induced Riemannian connection if $\phi h(X, Y) = h(X, \phi Y)$.

Also the induced quarter-symmetric metric connection on the submanifold $\tilde{N}$ is given by (4.8) using this equation, we have

$$\begin{aligned}(B_X\phi)Y &= B_X(\phi Y) - \phi(B_XY)\\ &= \tilde{B}_X(\phi Y) + h(X, \phi Y) - \eta(X)L(\phi Y) - \phi(\tilde{B}_XY + h(X, Y) - \eta(X)LY)\\ &= (\tilde{B}_X\phi)Y + h(X, \phi Y) - \phi h(X, Y) - \eta(X)L(\phi Y) + \eta(X)\phi(LY)\\ &= (\tilde{B}_X\phi)Y + h(X, \phi Y) - \phi h(X, Y).\end{aligned} \tag{4.18}$$

Hence we have the following theorem:

Theorem 7. If $\tilde{M}$ is a Sasakian manifold with a quarter-symmetric metric connection, then the submanifold $\tilde{N}$ is also a Sasakian manifold with respect to the induced quarter-symmetric metric connection if $\phi h(X, Y) = h(X, \phi Y)$.

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