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THE SOLUTION OF DISPLACEMENT EQUATION OF CONICAL SHELL

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ABSTRACT

In this paper, the displacement differential equations of conical shell can be changed into the eight-order soluble differential equation by introducing a displacement function. The general solutions of the equations are given by using the generalized hvpergeometric function and applying the Bessel-Function for the axisymmtric problems.

Keywords: conical shells, displacement, exact solution

1. INTRODUCTION

Conical shell is widely used in engineering construction. For example ,designing vessel deals with the calculated theory of conical shell. Because of the complexity of the equation of conical shell, however, the research of the axisymmtric problems is more than that of the general bending problems of conical shell up to date. For the soluble differential equations of conical shell, the extant documents are expressed by the deflection function and stress function of shell (called the mixed solution) .Analyzing the combined structure, the author finds it difficulty to treat the boundary condition of elasticity with the extant mixed solution of conical shell. Gained by contrast, taking the displacement method has some advantages. Moreover, for the characteristic problem of shell as welle as the problem of shell on elastic foundation, it is more effective to use displacement method. But, the systemic research about the displacement solution of conical shell is less at present. This paper makes some attempt in the displacement solution of conical shell.

Starting from the general displacement differential equations of conical shell, introducing the displacement function, the displacement equation system can be changed into a eight-order differential equation. Meantime, the entire stress components and displacement components in shell are expressed by displacement function. The general solutions of the partial differential equations are given by the hypergeometric function and expressed by the Bessel function for the axisymmtric bending problems.

1. The Basic Relations of Conical Shell

Conical shell acted the surface loads satisfies the basic relations as follows:

in which, each component and its direction is shown in Fig. 1.

the equilibrium defferential:

$$\frac{\partial(sT_1)}{\partial s}+\frac{\partial T}{\partial\theta}\sec\phi-T_2+sq_1=0$$

$$\frac{\partial(sT)}{\partial s}+\frac{\partial T_2}{\partial\theta}\sec\phi+T+sq_2=0$$

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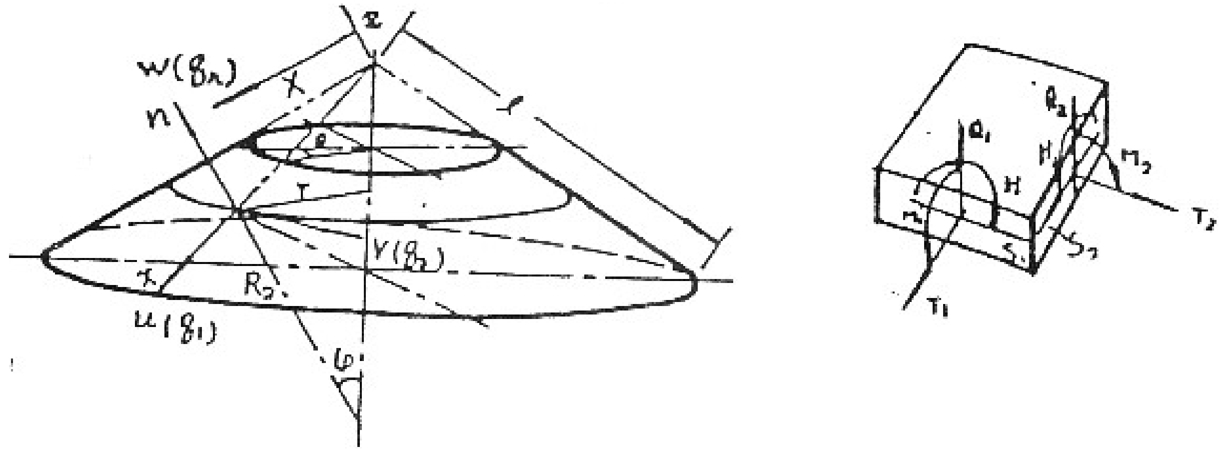

Figure 1, a b

$$\frac{\partial (sN_1)}{\partial s}+\frac{\partial N_2}{\partial \theta}\sec\phi - T_2 tg\phi + sq_n = 0 \qquad \text{(1-1a,b,c,d,e)}$$

$$\frac{\partial (sM_1)}{\partial s}+\frac{\partial H}{\partial \theta}\sec\phi - M_2 - sN_1 = 0$$

$$\frac{\partial (sM)}{\partial s}+\frac{\partial M_2}{\partial \theta}\sec\phi + H - sN_2 = 0$$

the geometric relations:

$$\varepsilon_1 = \frac{\partial u}{\partial s}$$

$$\varepsilon_2 = \frac{u}{s}+\frac{w}{s}tg\phi+\frac{1}{s}\frac{\partial v}{\partial \theta}\sec\phi$$

$$w = \frac{1}{s}+\frac{\partial u}{\partial \theta}\sec\phi+\frac{\partial v}{\partial\ s}-\frac{v}{s} \qquad \text{(1-2a,b,c,d,e,f)}$$

$$x_1 = -\frac{\partial^2 w}{\partial s^2}$$

$$x_2 = -\frac{1}{x^2}\frac{\partial^2 w}{\partial \theta^2}\sec^2\phi - \frac{1}{s}\frac{\partial w}{\partial s}$$

$$\tau = -\frac{1}{s}\frac{\partial^2 w}{\partial s \partial \theta}\sec\phi + \frac{1}{s^2}\frac{\partial w}{\partial \theta}\sec\phi$$

the physical relations:

$$T_1 = D(\varepsilon_1 + \mu\varepsilon_2)$$

$$T_2 = D(\varepsilon_2 + \mu\varepsilon_1)$$

$$T = \frac{1-\mu}{2}Dw$$

$$M_1 = K(x_1 + \mu x_2) \qquad \text{(1-3a,b,c,d,e,f)}$$

$$M_2 = K(x_2 + \mu x_1)$$

$$H = (1 - \mu)K\tau$$

$$D = Eh\big/(1-\mu^2),\ K = Eh^3\big/12(1-\mu^2)$$

2. THE DISPLACEMENT FUNCTION AND THE BASIC EQUATIONS

Using Mushtari-Donnell simplification, we get the following displacement equations:

$$\begin{aligned} &L^{11}(u) + L^{12}(v) + L^{13}(w) + q_1 / D = 0 \\ &L^{21}(u) + L^{22}(v) + L^{23}(w) + q_2 / D = 0 \\ &L^{31}(u) + L^{32}(v) + L^{33}(w) + q_n / K = 0 \end{aligned} \tag{2.1}$$

in which

$$L^{11} = \frac{\partial^2}{\partial s^2} + \frac{1}{x}\frac{\partial}{\partial s} - \frac{1}{s^2} + \frac{1-\mu}{2}\frac{1}{s^2\cos^2\phi}\frac{\partial^2}{\partial\theta^2};$$

$$L^{12} = \frac{1+\mu}{2} + \frac{1}{s\cos\phi}\frac{\partial^2}{\partial s\partial\theta} - \frac{3-\mu}{2}\frac{1}{s^2\cos\phi}\frac{\partial}{\partial\theta}$$

$$L^{13} = \mu\frac{\sin\phi}{s\cos\phi}\frac{\partial}{\partial s} - \frac{\sin\phi}{s^2\cos\phi}; L^{21} = \frac{1+\mu}{2}\frac{1}{s\cos\phi}\frac{\partial^2}{\partial s\partial\theta} + \frac{3-\mu}{2}\frac{1}{s^2\cos\phi}\frac{\partial}{\partial\theta};$$

$$L^{22} = \frac{1-\mu}{2}\left(\frac{\partial^2}{\partial s^2} + \frac{1}{s}\frac{\partial}{\partial s} - \frac{1}{s^2}\right) + \frac{1}{s^2\cos\phi}\frac{\partial^2}{\partial\theta^2}; L^{23} = \frac{\sin\phi}{s^2\cos\phi}\frac{\partial}{\partial\theta};$$

$$L^{31} = \frac{D}{K}tg\phi\left(\mu\frac{1}{s^3}\frac{\partial}{\partial s} + \frac{1}{s^4}\right); L^{32} = \frac{D}{K}tg\phi\frac{1}{s^4\cos\phi}\frac{\partial}{\partial\theta};$$

$$L^{33} = \left(\frac{\partial^2}{\partial s^2} + \frac{1}{s}\frac{\partial}{\partial s} + \frac{1}{s^2\cos^2\phi}\frac{\partial^2}{\partial\theta^2}\right)^2 + \frac{D}{K}tg^2\phi\frac{1}{s^4}.$$

Introducing the conversion, we assume:

$$s = e^{t\cos\phi} \tag{2-2}$$

in which, t is a new independent variable. Thus,the equations (2-1)becomes:

$$\begin{gathered} \frac{\partial^2 u}{\partial t^2} + \frac{1-\mu}{2}\frac{\partial^2 u}{\partial\theta^2} - u\cos^2\phi + \frac{1+\mu}{2} + \frac{\partial^2 v}{\partial t\partial\theta} - \frac{3-\mu}{2}\frac{\partial v}{\partial\theta}\cos\phi \\ + \left(\mu\frac{\partial w}{\partial t} - w\cos\phi\right)\sin\phi = -\frac{q_1 r^2}{D} \end{gathered} \tag{2-3}$$

$$\begin{gathered} \frac{1+\mu}{2}\frac{\partial^2 u}{\partial t\partial\theta} + \frac{3-\mu}{2}\frac{\partial u}{\partial\theta}\cos\phi + \frac{1-\mu}{2}\frac{\partial^2 v}{\partial t^2} + \frac{\partial^2 v}{\partial\theta^2} - \frac{1-\mu}{2}v\cos^2\phi \\ + \frac{\partial w}{\partial\theta}\sin\phi = -\frac{q_2 r^2}{D} \end{gathered}$$

$$\left[\left(\frac{\partial}{\partial t} - 2\cos\phi\right)^2 + \frac{\partial^2}{\partial\theta^2}\right]\left(\frac{\partial^2 w}{\partial t^2} + \frac{\partial^2 w}{\partial\theta^2}\right) + \frac{D}{K}r^2\sin\phi\left[\mu\frac{\partial u}{\partial t} + \mu\cos\phi + \frac{\partial v}{\partial\theta} + w\sin\phi\right] = \frac{q_n v^4}{K}$$

where r is the radius of parallel circle (Fig. 1). By means of Eq. (2-2),we yield:

$$r = s\cos\phi = e^{t\cos\phi}\cos\phi$$

For the coordinate (t, ϕ), Eqs. (2-3) a,b.are the constant coefficient equations. As a result, we can introduce the displacement functions $U(t, \theta)$ when the relations between the displacement components of conical shell and the function $U(t, \theta)$, exist, i.e.,

$$\begin{aligned} u &= -\sin\phi L_u^t(U) - L_{up_1}^t(\psi_1) + L_{up_2}^t(\psi_2) \\ v &= -\sin\phi L_v^t(U) + L_{2p_1}^t(\psi_1) - L_{vp_2}^t(\psi_2) \\ w &= L_w^t(U) \end{aligned} \tag{2-4}$$

The Eqs.(2-3,a,b) satisfy naturally. Here, each differential operator is ,respectively,

$$L_u^t = \mu\frac{\partial^3}{\partial t^3} - \frac{\partial^3}{\partial t\partial\theta^2} - \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial\theta^2}\right)\cos\phi - \mu\frac{\partial}{\partial t}\cos^2\phi + \cos^8\phi$$

$$L_v^t = (2+\mu)\frac{\partial^3}{\partial t^2\partial\theta} + \frac{\partial^3}{\partial\theta^3} + (1-\mu)\frac{\partial^2}{\partial t\partial\theta}\cos\theta + \frac{\partial}{\partial\theta}\cos^2\phi$$

$$L_v^t = \left(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial\theta^2}\right)^2 - 2\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial\theta^2}\right)\cos^2\phi + \cos^4\phi$$

$$L_v^t = \left(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial\theta^2}\right)^2 - 2\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial\theta^2}\right)\cos^2\phi + \cos^4\phi$$

$$L_{up_1}^t = \frac{1-\mu}{2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial\theta^2} - \frac{1-\mu}{2}\cos^2\phi \tag{2-5}$$

$$L_{up_2}^t = \frac{1+\mu}{2}\frac{\partial^2}{\partial t\partial\theta} + \frac{3-\mu}{2}\frac{\partial}{\partial\theta}\cos\phi$$

$$L_{vp_1}^t = \frac{1+\mu}{2}\frac{\partial^2}{\partial t\partial\theta} - \frac{3-\mu}{2}\frac{\partial}{\partial\theta}\cos\phi$$

$$L_{vp_2}^t = \frac{\partial^2}{\partial t^2} + \frac{1-\mu}{2}\frac{\partial^2}{\partial\theta^2} - \cos^2\phi$$

and $\psi_1(t, \theta)$, $\psi_2(t, \theta)$ are the functions concerned with loads q_1, q_2, respectively. they are the solutions of equations as follows:

$$L_w^t(\psi_i) = -\frac{2}{1-\mu}\frac{q_i r^2}{D} \quad (i = 1, 2) \tag{2-6}$$

Generally, there is no use getting the general solution of Eq. (2-6).We only obtain a arbitrary particular integral, because the two former formulas of Eqs. (2-3)are satisfied naturally even this case. The particular solution of Eqs. (2-6) is given behind.

Substituting the solutions (2-4) into Eq. (2-3c), we obtain the equation which is satisfied by the displacement function $U(t, \theta)$, i.e.:

$$L_1^t L_2^t(U) + (1-\mu^2)(D/K)r^2\sin^2\phi L_3^t(U) = Q(t,\theta) \tag{2-7}$$

in which ,each differential operator:

$$L_1^t=\left[\left(\frac{\partial}{\partial t}-2\cos\phi\right)^2+\frac{\partial^2}{\partial\theta^2}\right]\left(\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial\theta^2}\right),\ L_2^t=L_w^t,\ \ L_3^t=\frac{\partial^4}{\partial t^4}-\frac{\partial^2}{\partial t^2}\cos^2\phi \tag{2-8}$$

free term:

$$Q(t,\theta)=\frac{q_n r^4}{K}+\frac{1-\mu}{2}\frac{D}{K}r^2\sin\phi\left\{\left[\mu\frac{\partial^3}{\partial t^3}-\frac{\partial^3}{\partial t\partial\theta^2}+\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial\theta^2}\right)\cos\phi\right.\right.$$

$$\left.\left.-\mu\frac{\partial}{\partial t}\cos^2\phi-\cos^3\phi\right]\psi_1+\left[(2+\mu)\frac{\partial^3}{\partial t^2\partial\theta}+\frac{\partial^3}{\partial\theta^3}-(1-\mu)\frac{\partial^2}{\partial t\partial\theta}\cos\phi+\frac{\partial}{\partial\theta}\cos^2\phi\right]\psi_2\right\} \tag{2-9}$$

The formulas (2-4) and Eqs. (2-7) are the basic equations expressed by the displacement function U(t, θ) for the general problem of bending conical shell. From these general relations , we can obtain, the two result in the limif:

1. if $\phi\to\frac{\pi}{2}$, then cos $\phi\to 0$, sin $\phi\to 1$; $r\to a$, i.e. it coincides with cylindric shell. Noting the differential operator of Eq. (2-5), we obtain from formulas (2-4)

$$u=-\mu\frac{\partial^3 U}{\partial t^3}+\frac{\partial^3 U}{\partial t\partial\theta^2}$$

$$v=-(2+\mu)\frac{\partial^3 U}{\partial t^2\partial\theta}-\frac{\partial^3 U}{\partial\theta^3} \tag{a}$$

$$w=\left(\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial\theta^2}\right)^2 U$$

the basic equation (2-7) can be reduced to :

$$\nabla^4\nabla^4 U+\frac{1-\mu^2}{c^2}\frac{\partial^4 U}{\partial t^4}=Q(t,\theta) \tag{b}$$

When $D/K = 12/h$, The formular(a) and Eq.(b) are given by V. Z. Vlacov for cylindric shell. In which,a is the radius of cylindric shell, $c = h/12$a.

2. if $\phi\to 0$, then cos $\phi\to 1$, sin $\phi\to 0$; i.e. it coincides with circular plate. In this case, we obtain from (2-4) and (2-7):

$$u=-L_{up_1}^t(\psi_1)+L_{up_2}^t(\psi_2)$$

$$v=L_{vp_1}^t(\psi_1)-L_{vp_2}^t(\psi_2) \tag{c}$$

$$L_w^t(\psi_i)=-\frac{q_i r^4}{D}$$

$$L_1^t(w)=q_n r^4/K \tag{d}$$

We can prove that the formular (c) is the basic equation of circular plate in plane stress and the formular (d) is the basic equation of circular plate in transverse bending.

Obviously, the displacement function and the basic equation of cylindric shell or circular plate are all directly derived from the general relations in this paper.

Substituting the original independent variable s, formular (2-4) and Eq. (2-7) become into:

$$u = -\sin\phi\cos^3\phi L_u^s(U) - \cos^2\phi L_{uP_1}^s(\psi_1) + \cos^2\phi L_{uP_2}^s(\psi_2)$$

$$v = -\sin\phi\cos^3\phi L_v^s(U) + \cos^2\phi L_{vP_1}^s(\psi_1) - \cos^2\phi L_{vP_2}^s(\psi_2) \tag{2-10}$$

$$w = \cos^4\phi L_w^s(U), \quad \frac{\partial w}{\partial s} = \cos^4\phi \frac{\partial}{\partial s} L_w^s(U)$$

$$L_1^s L_2^s(U) + \frac{12(1-\mu^2)tg^2\phi}{h^2} s^2 L_3^s(U) = Q(s,\theta) \tag{2-11}$$

$$L_2^s(\psi_i) = -\frac{2q_i s^2 \sec^2\phi}{(1-\mu)D} \quad (i=1,2) \tag{2-12}$$

here, we have used relation r = s cos ϕ. Each differential operator in above formulas is

$$L_u^s = \mu\left(s^3\frac{\partial^3}{\partial s^3} + 3s^2\frac{\partial^2}{\partial s^2}\right) - s\frac{\partial^3}{\partial s\partial\theta^2}\sec^2\phi - \Delta_2 + 1$$

$$L_v^s = \left[(2+\mu)\frac{\partial}{\partial\theta}\left(s^2\frac{\partial^2}{\partial s^2} + s\frac{\partial}{\partial s}\right) + \frac{\partial^3}{\partial\theta^3}\sec^2\phi + (1-\mu)s\frac{\partial^2}{\partial s\partial\theta} + \frac{\partial}{\partial\theta}\right]\sec\phi$$

$$L_w^s = \Delta_1^2 - 2\Delta_2 + 1$$

$$L_{up_1}^s = \frac{1-\mu}{2}\Delta_1 + \frac{1+\mu}{2}\frac{\partial^2}{\partial\theta^2}\sec^2\phi - \frac{1-\mu}{2}$$

$$L_{up_2}^s = \left(\frac{1+\mu}{2}s\frac{\partial^2}{\partial s\partial\theta} - \frac{3-\mu}{2}\frac{\partial^2}{\partial\theta}\right)\sec\phi$$

$$L_{vp_1}^s = \left(\frac{1+\mu}{2}s\frac{\partial^2}{\partial\sin\theta} - \frac{3-\mu}{2}\frac{\partial}{\partial\theta}\right)\sec\phi \tag{2-13}$$

$$L_{vp_2}^s = \Delta_1 - \frac{1+\mu}{2}\frac{\partial^2}{\partial\theta^2}\sec^2\phi - 1$$

$$L_1^s = \left[\left(s\frac{\partial}{\partial s} - 2\right)^2 + \frac{\partial^2}{\partial\theta^2}\right]\Delta_1, \quad L_2^s = L_w^s$$

$$L_3^s = \Delta_0(\Delta_0 - 1), \quad \Delta_0 = s^2\frac{\partial^2}{\partial s^2} + s\frac{\partial}{\partial s}$$

$$\Delta_1 = \Delta_0 + \frac{\partial^2}{\partial\theta^2}\sec^2\phi$$

$$\Delta_2 = \Delta_0 - \frac{\partial^2}{\partial\theta^2}\sec^2\phi$$

free term is

$$Q(s,\theta)=\frac{q_n s^4\sec^4\phi}{K}+\frac{1-\mu}{2}\frac{D}{K}s^2\sec^3\phi\sin\phi$$
$$\left[\mu\left(s^3\frac{\partial^3}{\partial s^3}+3s^2\frac{\partial^2}{\partial s^2}\right)-s\frac{\partial^3}{\partial s\partial\theta^2}+\Delta_2-1\right]\psi_1+\frac{1-\mu}{2}\frac{D}{K}\cdot s^2\sec^4\phi\sin\phi \tag{2-14}$$
$$\left[\frac{\partial^2}{\partial\theta^2}+(2+\mu)\left(s^2\frac{\partial^2}{\partial s^2}+s\frac{\partial}{\partial s}\right)-(1-\mu)s\frac{\partial}{\partial s}+1\right]\frac{\partial\psi_2}{\partial\theta}$$

We make and vanished in formular (2-14) only when act normal load q ($q_1 = q_2 = 0$) on shell.

It is necessary to consider the boundary condition when solving the basic equation(2-11).The boundary conditions of conical shell, generally speaking , are several types as follows , i.e. An the boundary $s = s_1$;

$$u = u^b(\theta) \text{ or } T_1 = T_1^b(\theta)$$

$$v = v^b(\theta) \text{ or } T = T^b(\theta)\, T = T^b(\theta) \tag{2-16a,b}$$

$$w = w^b(\theta) \text{ or } V_1 = V_1^b(\theta) \tag{2-17a,b}$$

$$\frac{\partial w}{\partial s}=\left(\frac{\partial w}{\partial s}\right)^b(\theta) \text{ or } M_1 = M_1^b(\theta) \tag{2-18a,b}$$

where the right top footnote “b”indicate the corresponding boundary value given. The combination of Eq . (2-15a), (2-16a), (2-17a)and (2-18a) is called as (generalized) fixed boundary, and the combination of Eq. (2-15b), (2-16b), (2-17b) and (2-18b) is called as (generalized) free boundary, and the combination of Eq. (2-16a), (2-17a), (2-15b) and (2-18b) is called as (generalized) simply supported boundary. On the special condition, if $u^b = v^b = w^b = \left(\partial w/\partial s\right)^b = 0$, and $T_1^b = T^b = V_1^b = M_1^b = 0$. the preceding several combinations can be reduced to the following expressions, i.e.

$$\text{clamped: } u = 0,\ v = 0,\ w = 0,\ \partial w/\partial s = 0;$$

$$\text{free: } T_1 = 0,\ T = 0,\ V_1 = 0,\ M_1 = 0; \tag{2-19}$$

$$\text{simply supported: } v = 0,\ w = 0,\ T_1 = 0,\ M_1 = 0.$$

The boundary conditions of shell are also other combinations. Generally, they are not named specially.

By means of the geometric relations, the physical relations and formula (2-10) ,each component of internal force and moment of conical shell is expressed by the displacement function U(s, θ) as follows:

$$T_1=(1-\mu^2)D\sin\phi\cos^3\phi\ L^S_{T_1}(U)-\frac{1-\mu}{2}D\cos^2\phi\left[L^S_{T_1p_1}(\psi_1)+L^S_{T_1p_2}(\psi_2)\right]$$

$$T_2=(1-\mu^2)D\sin\phi\cos^3\phi\ L^S_{T_2}(U)-\frac{1-\mu}{2}D\cos^2\phi\left[L^S_{T_2p_1}(\psi_1)+L^S_{T_2p_2}(\psi_2)\right]$$

$$T=(1-\mu^2)D\sin\phi\cos^3\phi\ L^S_{T}(U)-\frac{1-\mu}{2}D\cos^2\phi\left[L^S_{Tp_1}(\psi_1)+L^S_{Tp_2}(\psi_2)\right]$$

$$M_1=-K\cos^4\phi\ L^S_{M_1}(U) \tag{2-20}$$

$$H=-(1-\mu)K\cos^4\phi\ L^S_H(U)$$

$$V_1=-K\cos^4\phi\ L^S_{V_1}(U)$$

in which, each differential operator is

$$L_{T_1}^s = \frac{1}{s}\left[s\frac{\partial}{\partial s}(\Delta_2 - 1) + \frac{\partial^2}{\partial\theta^2}\Delta_0 \sec^2\phi \right]$$

$$L_{T_1 p_1}^s = \frac{1}{s}\left[s\frac{\partial}{\partial s}(\Delta_1 - 1) + \mu(\Delta_2 - 1) + (1+\mu)s\frac{\partial^3}{\partial s\partial\theta^2}\sec^2\phi \right]$$

$$L_{T_1 p_2}^s = \frac{1}{s}\left[\mu(\Delta_1 + 1) - (1+\mu)\Delta_0 + (3+\mu)s\frac{\partial}{\partial s} \right]\frac{\partial}{\partial\theta}\sec\phi$$

$$L_{T_2}^s = \frac{1}{s}(\Delta_0^2 + \Delta_0)$$

$$L_{T_2 p_1}^s = \frac{1}{S}\left[\mu s\frac{\partial}{\partial s}(\Delta_1 - 1) + (\Delta_2 - 1) - (1-\mu)s\frac{\partial^3}{\partial s\partial\theta^2}\sec^2\phi \right]$$

$$L_{T_2 p_2}^s = \frac{1}{s}\left[(\Delta_1 + 1) + (1+\mu)\Delta_1 - (1-\mu)s\frac{\partial}{\partial s} \right]\frac{\partial}{\partial\theta}\sec\phi \tag{2-21}$$

$$L_T^s = \frac{1}{S}(s\frac{\partial}{\partial s}\Delta_0 - \Delta_0)\frac{\partial}{\partial\theta}\sec\phi$$

$$L_{Tp_1}^s = \frac{1}{s}\left[(\Delta_1 + 1) + (1+\mu)\Delta_0 - (1-\mu)s\frac{\partial}{\partial s} \right]\frac{\partial}{\partial\theta}\sec\phi$$

$$L_{Tp_2}^s = \frac{1}{s}\left[s\frac{\partial}{\partial s}(\Delta_1 - 1) - (\Delta_2 - 1) - (1+\mu)s\frac{\partial^3}{\partial s\partial\theta^2}\sec^2\phi \right]$$

$$L_{M_1}^s = \frac{1}{s^2}\left[\Delta_1 - (1-\mu)\left(s\frac{\partial}{\partial s} + \frac{\partial^2}{\partial\theta^2}\sec^2\phi \right) \right] L_w^s$$

$$L_{M_2}^s = \frac{1}{s^2}\left[\mu\Delta_1 + (1-\mu)\left(s\frac{\partial}{\partial s} + \frac{\partial^2}{\partial\theta^2}\sec^2\phi \right) \right] L_w^s$$

$$L_H^s = \frac{1}{s^2}\left[\left(s\frac{\partial}{\partial s} - 1 \right)\frac{\partial}{\partial\theta}\sec\phi \right] L_w^s$$

$$L_{V_1}^s = \frac{1}{s^3}\left\{ \left(s\frac{\partial}{\partial s} - 1 \right)\left[\Delta_1 + (1-\mu)\frac{\partial^2}{\partial\theta^2}\sec^2\phi \right] - \Delta_1 \right\} L_w^s$$

Now, we reduce the basic equations and boundary conditions to nondimensional form. Introducing the dimensionless amount as follows:

$$\tilde{U} = U/\ell, \ \tilde{u} = u/\ell, \ \tilde{v} = v/\ell, \ \tilde{w} = w/\ell,$$

$$\tilde{T}_1 = \frac{s_0^3\ell^2 tg^2\phi}{K}T_1, \tilde{M}_1 = \frac{s_0^2\ell M_1}{K}, \tilde{T}_2 = \frac{s_0^3\ell^2 tg^2\phi}{K}T_2,$$

$$\tilde{M}_2 = \frac{s_0^2 \ell M_2}{K},\ \tilde{T} = \frac{s_0^3 \ell^2 tg^2\phi}{K}T,\ \tilde{H} = \frac{s_0^2 \ell H}{K}, \tag{2-22}$$

$$\tilde{V}_1 = \frac{s_0^3 \ell^2 V_1}{K},\ \tilde{q}_n = \frac{s_0^4 \ell^3}{K} q_n,\ \tilde{\psi}_i = \frac{\psi_i}{\ell},$$

$$\tilde{q}_i = \frac{s_0^4 \ell^3 tg^2\phi}{K} q_i\ \ (i=1,2),\ \ \alpha = \sqrt{12(1-\mu^2)}\ \ tg\phi \cdot \frac{s}{h}$$

then ,the formulas (2-10), (2-20) and (2-12) become

$$\tilde{u} = -\sin\phi\cos^3\phi\ \ L_u^\alpha(\tilde{U}) - \cos^2\phi\ \ L_{up_1}^\alpha(\tilde{\psi}_1) + \cos^2\phi\ \ L_{up_2}^\alpha(\tilde{\psi}_2)$$

$$\tilde{v} = -\sin\phi\cos^3\phi\ \ L_u^\alpha(\tilde{U}) - \cos^2\phi\ \ L_{up_1}^\alpha(\tilde{\psi}_1) + \cos^2\phi\ \ L_{vp_2}^\alpha(\tilde{\psi}_2)$$

$$\tilde{w} = \cos^4\phi\ \ L_w^\alpha(\tilde{U})$$

$$\tilde{T}_1 = \sin\phi\cos^3\phi L_{T_1}^\alpha(\tilde{U}) - \frac{1}{2(1+\mu)}\cos^2\phi\left[L_{T_1p_1}^\alpha(\tilde{\psi}_1) + L_{T_1p_2}^\alpha(\tilde{\psi}_2)\right]$$

$$\tilde{T}_2 = \sin\phi\cos^3\phi L_{T_2}^\alpha(\tilde{U}) - \frac{1}{2(1+\mu)}\cos^2\phi\left[L_{T_2p_1}^\alpha(\tilde{\psi}_1) + L_{T_2p_2}^\alpha(\tilde{\psi}_2)\right]$$

$$\tilde{T} = -\sin\phi\ \cos^3\phi\, L_T^\alpha(\tilde{U}) - \frac{1}{2(1+\mu)}\cos^2\ \phi\left[L_{T_{p_1}}^\alpha(\tilde{\psi}_1) + L_{T_{p_2}}^\alpha(\tilde{\psi})\right] \tag{2-23}$$

$$\tilde{M}_1 = -\cos^4\phi\ \ L_{M_1}^\alpha(\tilde{U})$$

$$\tilde{M}_2 = -\cos^4\phi\ \ L_{M_2}^\alpha(\tilde{U})$$

$$\tilde{H} = -(1-\mu)\cos^4\phi\ \ L_H^\alpha(\tilde{U})$$

$$\tilde{V}_1 = -\cos^4\phi\ \ L_{V_1}^\alpha(\tilde{U})$$

$$L_2^\alpha(\tilde{\psi}_i) = -2(1+\mu)\tilde{g}_i\alpha^2\sec^2\phi\ \ (i=1,2) \tag{2-24}$$

where the symbol "α" denotes nondimension. In this case, the basic equations (2-11) reduce the simple form:

$$L_1^\alpha L_2^\alpha(\tilde{U}) + \alpha^2 L_3^\alpha(\tilde{U}) = \tilde{Q}(\alpha,\theta) \tag{2-25}$$

and from Eq.(2-14), the free term is

$$\tilde{Q}(\alpha,\theta) = \tilde{q}_n\alpha^4\sin^4\phi + \frac{1}{2(1+\mu)}ctg^2\phi\sec^3\phi\sin\phi\Big\{\alpha^2\Big[\mu\Big(\alpha^3\frac{\partial^3}{\partial\alpha^3} + 3\alpha^2\frac{\partial^2}{\partial\alpha^2}\Big)$$

$$-\alpha\frac{\partial^3}{\partial\alpha\partial\theta^2} + \Delta_2 - 1\Big]\tilde{\psi}_1 + \alpha^2\Big[\frac{\partial^2}{\partial\theta^2} + (2+\mu)\Big(\alpha^2\frac{\partial^2}{\partial\alpha^2} + \alpha\frac{\partial}{\partial\alpha}\Big)$$

$$-(1-\mu)\alpha\frac{\partial}{\partial\alpha} + 1\Big]\frac{\partial\tilde{\psi}_2}{\partial\theta}\sec\phi\Big\} \tag{2-26}$$

In preceding formulas, the linear differential operators signal the top footnote "α" indicate that the independent variable s is substituted by the dimensionless independent variable α for the corresponding operators in Eq.(2-13) and (2-21).

Now the dimensionless boundary conditions can be (written as) when $\alpha = \alpha_1$

the fixed (generalized) boundary:

$$\tilde{u} = \tilde{u}^b \quad \tilde{v} = \tilde{v}^b \quad \tilde{w} = \tilde{w}^b \quad \frac{\partial \tilde{w}}{\partial \alpha} = s_0 \left(\frac{\partial w}{\partial s} \right)^b$$

the free (generalized) boundary:

$$\tilde{T}_1 = \tilde{T}_1^b \quad \tilde{T} = \tilde{T}^b \quad \tilde{v}_1 = \tilde{v}_1^b \quad \tilde{M}_1 = \tilde{M}_1^b \tag{2-27}$$

the simply supported boundary:

$$\tilde{w} = \tilde{w}^b \quad \tilde{v} = \tilde{v}^b \quad \tilde{T}_1 = \tilde{T}_1^b \quad \tilde{M}_1 = \tilde{M}_1^b$$

and the formulas (2-19) become naturally:

the fixed boundary: $\left\{ \tilde{u}, \tilde{v}, \tilde{w}, \partial \tilde{w} / \partial \alpha \right\} = 0$

the free boundary: $\left\{ \tilde{T}_1, \tilde{T}, \tilde{V}_1, \tilde{M}_1 \right\} = 0$ (2-28)

the simply supported boundary: $\left\{ \tilde{w}, \tilde{v}, \tilde{T}_1, \tilde{M}_1 \right\} = 0$

3. TWO KINDS OF BOUNDARY VALUE PROBLEM

The general bending problem of conical shell acted arbitrary loads can be reduced to solving Eq. (2-25) under the boundary conditions (2-26) or (2-27).

Now, we can write the Fourier's series for $q_1(\alpha, \theta)$, $q_2(\alpha, \theta)$, $q_3(\alpha, \theta)$ and the given boundary value $\tilde{u}^b(\theta)$, $\tilde{v}^b(\theta)$, $\tilde{w}^b(\theta)$, $\left(\partial \tilde{w} / \partial \alpha\right)^b$, $\tilde{T}_1^b(\theta)$, $\tilde{T}^b(\theta)$, $\tilde{v}_1^b(\theta)$, $\tilde{M}_1^b(\theta)$.

$$\tilde{q}_i(\alpha,\theta) = \tilde{q}_{io}(\alpha) + \sum_{k=1}^{\infty} \tilde{q}_{ikc}(\alpha)\cos k\theta + \sum_{k=1}^{\infty} \tilde{q}_{iks}(\alpha)\sin k\theta \quad (i = 1,2,3) \tag{3-1a,b,c}$$

$$\tilde{u}^b(\theta) = \tilde{u}_0^b + \sum_{k=1}^{\infty} \tilde{u}_{kc}^b \cos k\theta + \sum_{k=1}^{\infty} \tilde{u}_{ks}^b \sin k\theta \tag{3-2a}$$

$$\tilde{v}^b(\theta) = \tilde{v}_0^b + \sum_{k=1}^{\infty} \tilde{v}_{kc}^b \cos k\theta + \sum_{k=1}^{\infty} \tilde{v}_{ks}^b \sin k\theta \tag{3-2b}$$

$$\tilde{w}^b(\theta) = \tilde{w}_0^b + \sum_{k=1}^{\infty} \tilde{w}_{kc}^b \cos k\theta + \sum_{k=1}^{\infty} \tilde{w}_{ks}^b \sin k\theta \tag{3-2c}$$

$$\left(\frac{\partial \tilde{w}}{\partial \alpha} \right)^b = \left(\frac{\partial \tilde{w}}{\partial \alpha} \right)_0^b + \sum_{k=1}^{\infty} \left(\frac{\partial \tilde{w}}{\partial \alpha} \right)_{kc}^b \cos k\theta + \sum_{k=1}^{\infty} \left(\frac{\partial \tilde{w}}{\partial \alpha} \right)_{ks}^b \sin k\theta \tag{3-2d}$$

$$\tilde{T}_1^b(\theta) = \tilde{T}_{10}^b + \sum_{k=1}^{\infty} \tilde{T}_{1kc}^b \cos k\theta + \sum_{k=1}^{\infty} \tilde{T}_{1ks}^b \sin k\theta \tag{3-2e}$$

$$\tilde{T}^b(\theta) = \tilde{T}_0^b + \sum_{k=1}^{\infty} \tilde{T}_{kc}^b \cos k\theta + \sum_{k=1}^{\infty} \tilde{T}_{ks}^b \sin k\theta \tag{3-2f}$$

$$\tilde{V}_1^b(\theta) = \tilde{V}_{10}^b + \sum_{k=1}^{\infty} \tilde{V}_{1kc}^b \cos k\theta + \sum_{k=1}^{\infty} \tilde{V}_{1ks}^b \sin k\theta \tag{3-2g}$$

$$\tilde{M}_1^{\,b}(\theta)=\tilde{M}_1^{\,b}(\theta)+\sum_{k=1}^{\infty}\tilde{M}_{1kc}^{\,b}\cos k\theta+\sum_{k=1}^{\infty}\tilde{M}_{1ks}^{\,b}\sin k\theta \tag{3-2h}$$

$$\tilde{q}_{i0}^{(\alpha)}=\frac{1}{2\pi}\int_0^{2\pi}\tilde{q}_i(\alpha,\theta)d\theta\quad(i=1,2,3)$$

$$\tilde{q}_{ikc}^{(\alpha)}=\frac{1}{\pi}\int_0^{2\pi}\tilde{q}_i(\alpha,\theta)\cos k\theta d\theta\quad(i=1,2,3) \tag{3-3}$$

$$\tilde{q}_{iks}^{(\alpha)}=\frac{1}{\pi}\int_0^{2\pi}\tilde{q}_i(\alpha,\theta)\sin k\theta d\theta\quad(i=1,2,3)$$

$$\left[\tilde{u}_0^b,\tilde{v}_0^b,\tilde{w}_0^b,\left(\frac{\partial\tilde{w}}{\partial\alpha}\right)_0^b\right]=\frac{1}{2\pi}\int_0^{2\pi}\left[\tilde{u}^b,\tilde{v}^b,\tilde{w}^b,\left(\frac{\partial\tilde{w}}{\partial\alpha}\right)^b\right]d\theta$$

$$\left[\tilde{u}_{kc}^b,\tilde{v}_{kc}^b,\tilde{w}_{kc}^b,\left(\frac{\partial\tilde{w}}{\partial\alpha}\right)_{kc}^b\right]=\frac{1}{\pi}\int_0^{2\pi}\left[\tilde{u}^b,\tilde{v}^b,\tilde{w}^b,\left(\frac{\partial\tilde{w}}{\partial\alpha}\right)^b\right]\cos k\theta d\theta$$

$$\left[\tilde{u}_{ks}^b,\tilde{v}_{ks}^b,\tilde{w}_{ks}^b,\left(\frac{\partial\tilde{w}}{\partial\alpha}\right)_{ks}^b\right]=\frac{1}{\pi}\int_0^{2\pi}\left[\tilde{u}^b,\tilde{v}^b,\tilde{w}^b,\left(\frac{\partial\tilde{w}}{\partial\alpha}\right)^b\right]\sin k\theta d\theta \tag{3-4}$$

$$\left[\tilde{T}_{10}^b,\tilde{T}_0^{\,b},\tilde{V}_{10}^b,\tilde{M}_{10}^{\,b}\right]=\frac{1}{2\pi}\int_0^{2\pi}\left[\tilde{T}_1^b,\tilde{T}^b,\tilde{V}_1^b,\tilde{M}_1^{\,b}\right]d\theta$$

$$\left[\tilde{T}_{1kc}^b,\tilde{T}_{kc}^{\,b},\tilde{V}_{1kc}^b,\tilde{M}_{1kc}^{\,b}\right]=\frac{1}{\pi}\int_0^{2\pi}\left[\tilde{T}_1^b,\tilde{T}^b,\tilde{V}_1^b,\tilde{M}_1^{\,b}\right]\cos k\theta d\theta$$

$$\left[\tilde{T}_{1ks}^b,\tilde{T}_{ks}^{\,b},\tilde{V}_{1ks}^b,\tilde{M}_{1ks}^{\,b}\right]=\frac{1}{\pi}\int_0^{2\pi}\left[\tilde{T}_1^b,\tilde{T}^b,\tilde{V}_1^b,\tilde{M}_1^{\,b}\right]\sin k\theta d\theta \tag{3-5}$$

Meantime, we can expand the functions

$$\tilde{u}(\alpha,\theta),\tilde{v}(\alpha,\theta),\tilde{w}(\alpha,\theta),\tilde{T}_1(\alpha,\theta),\tilde{T}(\alpha,\theta),\tilde{M}_1(\alpha,\theta),\tilde{M}_2(\alpha,\theta),\tilde{H}(\alpha,\theta),\tilde{V}_1(\alpha,\theta)$$

in the following form:

$$\tilde{U}=\tilde{U}_0+\sum_{K=1}^{\infty}\tilde{U}_{kc}\cos k\theta+\sum_{K=1}^{\infty}\tilde{U}_{ks}\sin k\theta \tag{3-6}$$

$$\tilde{u}=\tilde{u}_0+\sum_{K=1}^{\infty}\tilde{u}_{kc}\cos k\theta+\sum_{K=1}^{\infty}\tilde{u}_{ks}\sin k\theta \tag{3-7}$$

$$\tilde{v}=\tilde{v}_0+\sum_{K=1}^{\infty}\tilde{v}_{kc}\cos k\theta+\sum_{K=1}^{\infty}\tilde{v}_{ks}\sin k\theta \tag{3-8}$$

$$\tilde{w}=\tilde{w}_0+\sum_{K=1}^{\infty}\tilde{w}_{kc}\cos k\theta+\sum_{K=1}^{\infty}\tilde{w}_{ks}\sin k\theta \tag{3-9}$$

$$\tilde{T}_1=\tilde{T}_{10}+\sum_{K=1}^{\infty}\tilde{T}_{1kc}\cos k\theta+\sum_{K=1}^{\infty}\tilde{T}_{1ks}\sin k\theta \tag{3-10}$$

$$\tilde{T}_2=\tilde{T}_{20}+\sum_{K=1}^{\infty}\tilde{T}_{2kc}\cos k\theta+\sum_{K=1}^{\infty}\tilde{T}_{2ks}\sin k\theta \tag{3-11}$$

$$\tilde{T}=\tilde{T}_0+\sum_{K=1}^{\infty}\tilde{T}_{kc}\cos k\theta+\sum_{K=1}^{\infty}\tilde{T}_{ks}\sin k\theta \tag{3-12}$$

$$\tilde{M}_1 = \tilde{M}_{10} + \sum_{K=1}^{\infty} \tilde{M}_{1kc} \cos k\theta + \sum_{K=1}^{\infty} \tilde{M}_{1ks} \sin k\theta \tag{3-13}$$

$$\tilde{M}_2 = \tilde{M}_{20} + \sum_{K=1}^{\infty} \tilde{M}_{2kc} \cos k\theta + \sum_{K=1}^{\infty} \tilde{M}_{2ks} \sin k\theta \tag{3-14}$$

$$\tilde{H} = \tilde{H}_0 + \sum_{K=1}^{\infty} \tilde{H}_{kc} \cos k\theta + \sum_{K=1}^{\infty} \tilde{H}_{ks} \sin k\theta \tag{3-15}$$

$$\tilde{V}_1 = \tilde{V}_{10} + \sum_{K=1}^{\infty} \tilde{V}_{1kc} \cos k\theta + \sum_{K=1}^{\infty} \tilde{V}_{1ks} \sin k\theta \tag{3-16}$$

$$\tilde{\psi}_i = \tilde{\psi}_{i0} + \sum_{K=1}^{\infty} \tilde{\psi}_{ikc} \cos k\theta + \sum_{K=1}^{\infty} \tilde{\psi}_i \sin k\theta \tag{3-17}$$

Where, $\tilde{U}_0(\alpha), \tilde{U}_{kc}(\alpha)\ (k=1,2,3,\ldots\ldots), \tilde{U}_{ks}(\alpha)(k=1,2,3,\ldots\ldots)$, are unknown function to be determined. Functions $\tilde{u}_0^{(\alpha)}, \tilde{u}_{kc}^{(\alpha)}, \tilde{u}_{ks}^{(\alpha)}, \ldots, \tilde{V}_{1ks}(\alpha)$ are the derivable amount which can be obtained from functions $\tilde{U}_0(\alpha), \tilde{U}_{kc}(\alpha)$ and $\tilde{U}_{ks}(\alpha)$. Functions $\tilde{\psi}_{i0}(\alpha), \tilde{\psi}_{ikc}(\alpha), \tilde{\psi}_{iks}(\alpha)$ are connected with $\tilde{q}_i(\alpha)$ by Eq.(2-24). Substituting formulas (4-6) –(4-17) into (2-23) and introducing the operator

$$\delta_\alpha(:) = \alpha \frac{d(:)}{d\alpha} \tag{3-18}$$

we get relations as follows, compared with the coefficient of each term of angular series:

$$\begin{gathered} \begin{matrix}\tilde{u}_{kc}\\ \tilde{u}_{ks}\end{matrix} = -\sin\phi\cos^3\phi\left[\mu\delta_\alpha^3 - \delta_\alpha^2 + (m^2-\mu)\delta_\alpha - (m^2-1)\right]\begin{matrix}\tilde{U}_{kc}\\ \tilde{U}_{ks}\end{matrix} \\ -\cos^2\phi\left[\frac{1-\mu}{2}\delta_\alpha^2 - (m^2 + \frac{1-\mu}{2})\right]\begin{matrix}\tilde{\psi}_{1kc}\\ \tilde{\psi}_{1ks}\end{matrix} \pm \cos^2\phi\left[\frac{1+\mu}{2}\delta_\alpha - \frac{3-\mu}{2}\right]m\begin{matrix}\tilde{\psi}_{2ks}\\ \tilde{\psi}_{2kc}\end{matrix} \begin{cases}k=0,1,2,\ldots\\ k=1,2\ldots\end{cases} \end{gathered} \tag{3-19a,b}$$

$$\begin{gathered} \begin{matrix}\tilde{v}_{kc}\\ \tilde{v}_{ks}\end{matrix} = \mp\sin\phi\cos\phi\left[(2+\mu)\delta_\alpha^2 + (1-\mu)\delta_\alpha - (m^2-1)\right]m\begin{matrix}\tilde{U}_{ks}\\ \tilde{U}_{kc}\end{matrix} \\ \pm\cos^2\phi\left[\frac{1+\mu}{2}\delta_\alpha + \frac{3-\mu}{2}\right]m\begin{matrix}\tilde{\psi}_{1ks}\\ \tilde{\psi}_{1kc}\end{matrix} - \cos^2\phi\left[\delta_\alpha^2 - \left(\frac{1-\mu}{2}m^2+1\right)\right]\begin{matrix}\tilde{\psi}_{2kc}\\ \tilde{\psi}_{2ks}\end{matrix} \begin{cases}k=0,1,2,\ldots\\ k=1,2\ldots\end{cases} \end{gathered} \tag{3-20a,b}$$

$$\begin{matrix}\tilde{w}_{kc}\\ \tilde{w}_{ks}\end{matrix} = \cos^4\phi\left[\delta_\alpha^4 - 2\left(m^2+1\right)\delta_\alpha^2 + \left(m^2-1\right)^2\right]\begin{matrix}\tilde{U}_{kc} & k=0,1,2,\cdots\\ \tilde{U}_{ks} & k=1,2,\cdots\end{matrix} \tag{3-21 a,b}$$

$$\begin{gathered} \begin{matrix}\tilde{T}_{1kc}\\ \tilde{T}_{1ks}\end{matrix} = \sin\phi\cos^3\phi\frac{1}{\alpha}\left[\delta_\alpha^3 - m^2\delta_\alpha^2 + \left(m^2-1\right)\delta_\alpha\right]\begin{matrix}\tilde{U}_{kc}\\ \tilde{U}_{ks}\end{matrix} \\ -\frac{1}{2(1+\mu)}\cos^2\phi\frac{1}{\alpha}\left\{\left[\delta_\alpha^3 + \mu\delta_\alpha^2 - \left(2m^2+\mu m^2+1\right)\delta_\alpha + \mu\left(m^2-1\right)\right]\begin{matrix}\tilde{\psi}_{1kc}\\ \tilde{\psi}_{1ks}\end{matrix}\right. \\ \left.\pm\left[-\delta_\alpha^2 + (3+\mu)\delta_\alpha - \mu\left(m^2-1\right)\right]\ m\begin{matrix}\tilde{\psi}_{2ks}\\ \tilde{\psi}_{2kc}\end{matrix}\right\} \begin{matrix}k=0,1,2,\cdots\\ k=1,2,\cdots\end{matrix} \end{gathered} \tag{3-22 a,b}$$

$$\begin{matrix}\tilde{T}_{2kc}\\ \tilde{T}_{2ks}\end{matrix} = \sin\phi\cos^3\phi\frac{1}{\alpha}\left[\delta_\alpha^4 - \delta_\alpha^2\right]\begin{matrix}\tilde{U}_{kc}\\ \tilde{U}_{ks}\end{matrix} - \frac{1}{2(1+\mu)}\cos^2\phi\frac{1}{\alpha}$$

$$\left\{\left[\mu\delta_\alpha^3+\delta_\alpha^2+\left(m^2-\mu\right)\delta_\alpha+\left(m^2-1\right)\right]\begin{matrix}\tilde{\psi}_{1kc}\\ \tilde{\psi}_{1ks}\end{matrix}\pm\left[\left(2+\mu\right)\delta_\alpha^2\right.\right.$$

$$\left.\left.-\left(1-\mu\right)\delta_\alpha-\left(m^2-1\right)\right]\ m\begin{matrix}\tilde{\psi}_{2ks}\\ \tilde{\psi}_{2kc}\end{matrix}\right\}\begin{matrix}k=0,1,2,\cdots\\ k=1,2,\cdots\end{matrix} \tag{3-23 a,b}$$

$$\begin{matrix}\tilde{T}_{kc}\\ \tilde{T}_{ks}\end{matrix}=-\sin\phi\cos^3\phi\frac{1}{\alpha}\left[\delta_\alpha^3-\delta_\alpha^2\right]\ m\begin{matrix}\tilde{U}_{ks}\\ \tilde{U}_{kc}\end{matrix}-\frac{1}{2\left(1+\mu\right)}\cos^2\phi\frac{1}{\alpha}$$

$$\left\{\pm\left[-\mu\delta_\alpha^2-\left(1-\mu\right)\delta_\alpha-\left(m^2-1\right)\right]\ m\begin{matrix}\tilde{\psi}_{1ks}\\ \tilde{\psi}_{1kc}\end{matrix}+\left[\delta_\alpha^3-\delta_\alpha^2-\left(2m^2+\mu m^2+1\right)\delta_\alpha-\left(m^2-1\right)\right]\begin{matrix}\tilde{\psi}_{2kc}\\ \tilde{\psi}_{2ks}\end{matrix}\right\} \tag{3-24 a,b}$$

$$\begin{matrix}\tilde{M}_{1kc}\\ \tilde{M}_{1ks}\end{matrix}=-\cos^4\phi\frac{1}{\alpha}\left[\delta_\alpha^2-\left(1-\mu\right)\delta_\alpha-\mu m^2\right]\ \left[\delta_\alpha^4-2\left(m^2+1\right)\delta_\alpha^2+\left(m^2-1\right)^2\right]\begin{matrix}\tilde{U}_{kc}\\ \tilde{U}_{ks}\end{matrix}\begin{matrix}k=0,1,2,\cdots\\ k=1,2,\cdots\end{matrix} \tag{3-25 a,b}$$

$$\begin{matrix}\tilde{M}_{2kc}\\ \tilde{M}_{2ks}\end{matrix}=-\cos^4\phi\frac{1}{\alpha^2}\left[\mu\delta_\alpha^2+\left(1-\mu\right)\delta_\alpha-m^2\right]\ \left[\delta_\alpha^4-2\left(m^2+1\right)\delta_\alpha^2+\left(m^2-1\right)^2\right]\begin{matrix}\tilde{U}_{kc}\\ \tilde{U}_{ks}\end{matrix}\begin{matrix}k=0,1,2,\cdots\\ k=1,2,\cdots\end{matrix} \tag{3-26 a,b}$$

$$\begin{matrix}\tilde{H}_{kc}\\ \tilde{H}_{ks}\end{matrix}=-\left(1-\mu\right)\cos^4\phi\frac{1}{\alpha^2}\left[\pm\left(\delta_\alpha-1\right)m\right]\ \left[\delta_\alpha^4-2\left(m^2+1\right)\delta_\alpha+\left(m^2-1\right)^2\right]\begin{matrix}\tilde{U}_{kc}\\ \tilde{U}_{ks}\end{matrix}\begin{matrix}k=0,1,2,\cdots\\ k=1,2,\cdots\end{matrix} \tag{3-27 a,b}$$

$$\begin{matrix}\tilde{V}_{1kc}\\ \tilde{V}_{1ks}\end{matrix}=-\cos^4\phi\frac{1}{\alpha^3}\left[\delta_\alpha^3-2\delta_\alpha^2-\left(2-\mu\right)m^2\delta_\alpha+m^2\left(3-\mu\right)\right]$$

$$\left[\delta_\alpha^4-2\left(m^2+1\right)\delta_\alpha^2+\left(m^2-1\right)^2\right]\begin{matrix}\tilde{U}_{kc}\\ \tilde{U}_{ks}\end{matrix}\begin{matrix}k=0,1,2,\cdots\\ k=1,2,\cdots\end{matrix} \tag{3-28 a,b}$$

Substituting formula (3-6) into Eq. (2-25). formula (3-7)~(3-16), (3-2a)~(3-2h) into the boundary conditions (2-26), formula (3-17) into Eq.(2-24), we can yield two kinds of boundary value problem through comparing with the coefficient of each term of trigonometric series:

The basic equations about first kind of problem are

$$\begin{aligned}&\left(\delta_\alpha^2-m^2\right)\ \left[\left(\delta_\alpha-2\right)^2-m^2\right]\ \left[\delta_\alpha^4-2\left(m^2+1\right)\delta_\alpha^2+\left(m^2-1\right)^2\right]\tilde{U}_{kc}\\&+\alpha^2\left(\delta_\alpha^4-\delta_\alpha^2\right)\tilde{U}_{kc}=\tilde{Q}_{kc}\quad k=0,1,2,\cdots\end{aligned} \tag{3-29}$$

the right side term is

$$\tilde{Q}_{kc}^{(\alpha)}=\sec^4\phi\alpha^4\tilde{q}_{3kc}^{(\alpha)}+\frac{ctg^2\phi}{2\left(1+\mu\right)}\sec^2\phi\sin\phi\alpha^2\left\{\left[\mu\delta_\alpha^3+\delta_\alpha^2+\left(m^2-\mu\right)\delta_\alpha+\left(m^2-1\right)\right]\right.$$

$$\left.\tilde{\psi}_{1kc}+m\left[\left(2+\mu\right)\delta_\alpha^2-\left(1-\mu\right)\delta_\alpha-\left(m^2-1\right)\right]\ \tilde{\psi}_{2ks}\right\} \tag{3-30}$$

the corresponding boundary conditions are:

when $\alpha=\alpha_1$:

fixed (generalized) side:

$$\begin{aligned}&\tilde{u}_{kc}=\tilde{u}_{kc}^b,\ \tilde{w}_{kc}=\tilde{w}_{kc}^b,\ \frac{\partial\tilde{w}_{kc}}{\partial\alpha}=\left(\frac{\partial\tilde{w}}{\partial\alpha}\right)_{kc}^b\quad k=0,1,\cdots\\&\tilde{v}_{ks}=\tilde{v}_{ks}^b\quad k=1,2,\cdots\end{aligned} \tag{3-31}$$

free (generalized) side:

$$\begin{aligned}&\tilde{T}_{1kc}=\tilde{T}_{1kc}^{b},\ \tilde{V}_{1kc}=\tilde{V}_{1kc}^{b},\ \tilde{M}_{1kc}=\tilde{M}_{1kc}^{b}\quad k=0,1,\cdots\\&\tilde{T}_{ks}=\tilde{T}_{ks}^{b}\quad k=1,2,\cdots\end{aligned}\tag{3-32}$$

simply supported (generalized) side:

$$\begin{aligned}&\tilde{w}_{kc}=\tilde{w}_{kc}^{b},\ \tilde{T}_{1kc}=\tilde{T}_{1kc}^{b},\ \tilde{M}_{1kc}=\tilde{M}_{1kc}^{b}\quad k=0,1,2\cdots\\&\tilde{v}_{ks}=\tilde{v}_{ks}^{b}\quad k=1,2,\cdots\end{aligned}\tag{3-33}$$

The basic equations about second kind of problem are:

$$\begin{aligned}&\left(\delta_\alpha^2-m^2\right)\left[\left(\delta_\alpha-2\right)^2-m^2\right]\left[\delta_\alpha^4-2\left(m^2+1\right)\delta_\alpha^2+\left(m^2-1\right)^2\right]\tilde{U}_{ks}\\&+\alpha^2\left(\delta_\alpha^4-\delta_\alpha^2\right)\tilde{U}_{ks}=\tilde{Q}_{ks}\quad k=1,2,\cdots\end{aligned}\tag{3-34}$$

the right side term is

$$\begin{aligned}\tilde{Q}_{ks}=\sec^4\phi\alpha^4\tilde{q}_{3ks}^{(\alpha)}+\frac{\cot^2\phi}{2\left(1+\mu\right)}\sec^3\phi\sin\phi\alpha^2\Big\{&\left[\mu\delta_\alpha^3+\delta_\alpha^2+\left(m^2-\mu\right)\delta_\alpha+\left(m^2-1\right)\right]\\&\tilde{\psi}_{1ks}-m\left[\left(2+\mu\right)\delta_\alpha^2-\left(1-\mu\right)\delta_\alpha-\left(m^2-1\right)\right]\ \tilde{\psi}_{2kc}\Big\}\end{aligned}\tag{3-35}$$

the corresponding boundary conditions are
when $\alpha=\alpha_1$:

fixed (generalized) side:

$$\begin{aligned}&\tilde{u}_{ks}=\tilde{u}_{ks}^{b},\ \tilde{v}_{ks}=\tilde{v}_{ks}^{b},\ \tilde{w}_{ks}=\tilde{w}_{ks}^{b},\\&\left(\partial\tilde{\omega}_{ks}/\partial\alpha\right)=\left(\partial\tilde{\omega}/\partial\alpha\right)_{ks}^{b}\quad k=1,2,\cdots\end{aligned}\tag{3-36}$$

free (generalized) side:

$$\begin{aligned}&\tilde{T}_{1ks}=\tilde{T}_{1ks}^{b},\ \tilde{V}_{1ks}=\tilde{V}_{1ks}^{b},\ \tilde{M}_{1ks}=\tilde{M}_{1ks}^{b}\\&\tilde{T}_{kc}=\tilde{T}_{kc}^{b}\quad k=1,2,\cdots\end{aligned}\tag{3-37}$$

simply supported side:

$$\begin{aligned}&w_{ks}=w_{ks}^{b},\ \tilde{T}_{1ks}=\tilde{T}_{1ks}^{b},\ \tilde{M}_{1ks}=\tilde{M}_{1ks}^{b}\\&\tilde{v}_{kc}=\tilde{v}_{kc}^{b}\quad k=1,2,\cdots\end{aligned}\tag{3-38}$$

Moreover, the functions $\tilde{\psi}_{ikc}$ (i = 1, 2), $\tilde{\psi}_{iks}$ (i = 1, 2) show the solution of equation

$$\left[\delta_\alpha^4-2\left(m^2+1\right)\delta_\alpha^2+\left(m^2-1\right)^2\right]\begin{matrix}\tilde{\psi}_{ikc}\\\tilde{\psi}_{iks}\end{matrix}=-2\left(1+\mu\right)\sec^2\phi\alpha^2\begin{matrix}\tilde{q}_{ikc}\\\tilde{q}_{iks}\end{matrix}\quad(i=1,2)\tag{3-39}$$

Generally, we only obtain the particular solution of Eq.(3-39). A arbitrary solution of Eq.(3-39) can be written as integral form as follows

$$\begin{aligned}&\left.\begin{matrix}\tilde{\psi}_{ikc}\\\tilde{\psi}_{iks}\end{matrix}\right\}=-2\left(1+\mu\right)\sec^2\phi\alpha^{-(m-1)}\int\alpha^{2m-3}d\alpha\int\alpha^{-(2m+1)}d\alpha\\&\int\alpha^{2m+1}d\alpha\int\alpha^{-m}\begin{matrix}\tilde{q}_{ikc}\\\tilde{q}_{iks}\end{matrix}d\alpha\quad(i=1.2)\end{aligned}\tag{3-40}$$

Thus, the general bending problems of conical shell are reduced to solving the basic equation (3-29.), under the boundary condition (3-31) ~ (3-33) and the basic equation (3-39); under the boundary condition (3-36) ~ (3-38) and equation (3-34), i.e. two kinds of boundary value problem.

4. THE GENERAL SOLUTION OF BASIC EQUATION

At first, we discussed the solution about first kind of basic equation which is a variable coefficient eight-order differential equation. Applying interchangeability of operator δ(:), Eq. (3-29) can be rewritten as more compact form:

$$\prod_{i=1}^{8}\left(\delta_\alpha-\bar{\lambda}_i\right)\tilde{U}_{kc}+\alpha^2\prod_{i=1}^{4}\left(\delta_\alpha+\bar{a}_i\right)\tilde{U}_{kc}=\tilde{Q}_{kc}\quad (k=0,1,2,\cdots) \tag{4-1}$$

Where the coefficient $\bar{\lambda}_i$, $\bar{a}_i$ is, respectively

$$\begin{aligned}&\bar{\lambda}_1=m+2,\ \bar{\lambda}_2=-m+2,\ \bar{\lambda}_3=m+1,\ \bar{\lambda}_4=-m+1\\&\bar{\lambda}_5=m,\ \bar{\lambda}_6=-m,\ \bar{\lambda}_7=m-1,\ \bar{\lambda}_8=-m-1\\&\bar{a}_1=0,\ \bar{a}_2=0,\ \bar{a}_3=1,\ \bar{a}_4=-1\end{aligned} \tag{4-2}$$

The solution of Eq. (4-1) can be expressed:

$$\tilde{U}_{kc}=\tilde{U}^h_{kc}+\tilde{U}^p_{kc} \tag{4-3a}$$

in which $\tilde{U}^h_{kc}$ is the homogeneous solution and $\tilde{U}^p_{kc}$ is the particular solution, they satisfy the equation as follows, respectively

$$\begin{aligned}&\prod_{i=1}^{8}\left(\delta_\alpha-\bar{\lambda}_i\right)\tilde{U}^h_{kc}+\alpha^2\prod_{i=1}^{4}\left(\delta_\alpha+\bar{a}_i\right)\tilde{U}^h_{kc}=0\\&\prod_{i=1}^{8}\left(\delta_\alpha-\bar{\lambda}_i\right)\tilde{U}^p_{kc}+\alpha^2\prod_{i=1}^{4}\left(\delta_\alpha+\bar{a}_i\right)\tilde{U}^p_{kc}=\tilde{Q}_{kc}\end{aligned} \tag{4-3}$$

Eq. (4-3) can also reduced to the generalized hypergeometric equation. For this, we assume

$$\alpha=4i\xi^{\frac{1}{2}}\quad \tilde{U}^h_{kc}=\xi^\lambda\eta^h \tag{4-4}$$

Where, ξ is a new independent variable and η^h is a unknown function, λ is a waiting determination constant and $i=\sqrt{-1}$. Now, we define a new operator

$$\delta_\xi(:)=\xi\frac{d(:)}{d\xi} \tag{4-5}$$

and pay attention to the relations:

$$\delta^r_\alpha(:)=2^r\delta^r_\xi(:)\quad \delta_\xi\left(\xi^\lambda\eta^h\right)=\xi^\lambda\left(\delta_\xi+\lambda\right)\eta^h$$

because r is a positive integer, Eq. (4-3a) can be rewritten

$$\prod_{i=1}^{8}\left(\delta_\xi+\lambda-\lambda_i\right)\eta^h-\xi\prod_{i=1}^{4}\left(\delta_\xi+\lambda+a_i\right)\eta^h=0 \tag{a}$$

in which, each parameter is

$$\lambda_i=\frac{1}{2}\bar{\lambda}_i\quad a_i=\frac{1}{2}\bar{a}_i \tag{4-2}$$

Thus, we can determine the constant which satisfies the equation as follows

$$\prod_{i=1}^{8}(\lambda-\lambda_i)=0 \tag{4-6}$$

From this, we obtain eight values of the parameter

$$\lambda=\lambda_i \quad (i=1,2,\cdots,8) \tag{4-7}$$

it, can be obtained from the formula (4-2). In Eq.(a), we let the parameter be a arbitrary value given by formula (4-7). If $\lambda = \lambda_1$, for example, then the equation (a) reduce to the generalized hypergeometric equation

$$\prod_{j=0}^{7}\left(\delta_\xi+\beta_j-1\right)\eta^h-\xi\prod_{i=1}^{4}\left(\delta_\xi+a_i'\right)\eta^h=0 \tag{4-8a}$$

in which the coefficients β_i and a'_i are concerned with the coefficient

$$\begin{aligned}&\beta_j=1+\lambda_1-\lambda_{j+1} \quad (j=0,1,2,\cdots,7)\\&a_j'=a_i+\lambda_1 \quad (i=1,2,\cdots,4)\end{aligned} \tag{4-9}$$

Similarly, the equation (4-3a) reduces to nonhomogeneous generalized hypergeometric equation letting $\tilde{U}_{kc}^{p}$ be equal to $\xi^{\lambda_1}\eta^p$:

$$\prod_{j=0}^{7}\left(\delta_\xi+\beta_j-1\right)\eta^p-\xi\prod_{i=1}^{4}\left(\delta_\xi+a_i'\right)\eta^p=\left(\frac{1}{16}\right)^2\xi^{-\lambda_1}\tilde{Q}_{kc} \tag{4-8b}$$

We discuss the solution of the homogeneous equation (4-8a) at first. The solution of Eq. (4-8a) are directly concerned with the property of the coefficient β_j. According to the theory of the generalized hypergeometric equation, if the difference between two arbitrary parameters, $\beta_1,\cdots,\beta_7\,(i.e.\lambda_1,\lambda_2,\lambda_3,\cdots\lambda_8)$ is not integer or zero, the solution of Eq. (4-8a) is

$$\eta^h={}_4F_7\left(a_1',\cdots,a_4';\ \beta_1,\beta_2,\cdots\beta_7;\ \xi\right) \tag{4-10}$$

Here, ${}_4F_7\left(a_1',\cdots,a_4';\ \beta_1,\beta_2,\cdots\beta_7;\ \xi\right)$ is a generalized hypergeometric function and can be expressed by infinite series as follows:

$${}_4F_7\left(a_1',\cdots,a_4';\ \beta_1,\beta_2,\cdots\beta_7;\ \xi\right)=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{4}[a_i']}{n!\prod_{j=1}^{7}\left[\beta_j\right]_n}\xi^n \tag{4-11}$$

This series is convergent for a arbitrary value of ξ. In which, $[a]_n$ is:

$$[a]_n=a(a+1)(a+2)\cdots(a+n-1)=\frac{\Gamma(a+n)}{\Gamma(a)}$$

Here, $\Gamma(a)$ is a Gamma function. If noting $\beta_0 = 1$ and $n! = [1]_n$, the solution (4-10) can be directly rewritten by means of the formula (4-11)

$$\eta_v^h=\xi^{1-\beta_v}\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{4}\left[a_i'+1-\beta_v\right]_n}{\prod_{j=0}^{7}\left[\beta_j+1-\beta_v\right]_n}\xi^n \quad (v=1,2,\cdots,7) \tag{4-12}$$

Using the formula (4-9) and noting the relation of subscript between the coefficients β and

$$i = j + 1$$

we assume $v + 1 = e$

From the formula, we yield

$$\beta_j = 1 + \lambda_1 - \lambda_i \quad 1 - \beta_v = \lambda_e - \lambda_1 \tag{b}$$

In the exchange formula (4-4), we let λ to be λ_1. Considering the formula (b), we can obtain from the formula (4-12)

$$\tilde{U}^h_{kce} = \alpha^{2\lambda_e} \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{16}\right)^n \frac{\prod_{i=1}^{4} [a_i + \lambda_e]_n}{\prod_{i=1}^{8} [1 + \lambda_e - \lambda_i]_n} \alpha^{2n} \quad (e = 1,2,\cdots,8) \tag{4-13}$$

In which, we let successively λ_e be equal to λ_1, λ_2, ..., λ_8. Appling the formula (4-2), we know that only $\lambda_e = \lambda_1, \lambda_2, \lambda_3, \lambda_4$ four particular solutions in the eight particular solutions in formula (4-14) are independent each other when m≠ integer or m/2≥2. In this case, the four solutions of $\lambda_e = \lambda_5, \lambda_6, \lambda_7, \lambda_8$ are not significance when difference between λ_1 and λ_5 (λ_2 and λ_6; λ_3 and λ_7; λ_4 and λ_8) is one, therefore the solutions of $\lambda_e = \lambda_5, \lambda_6, \cdots, \lambda_8$ include logarithm terms[5], which can be known from the theory of series solving for differential equation. So, we can give the other four particular solutions as follows

$$\begin{aligned} \tilde{U}^h_{kce} = \alpha^{2\lambda_e} \Bigg\{ & \left(2\sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{16}\right)^n \frac{\prod_{i=1}^{4} [a_i + \lambda_e]_n}{\prod_{\lambda=1}^{8} [1 + \lambda_e - \lambda_i]_n} \alpha^{2n} \right) \ln\alpha \\ & + \sum_{n=1}^{1} \left(-\frac{1}{16}\right)^{-n} (-1)^{4n-1} \frac{(n-1)\prod_{i=1}^{8}{}^{*} [1-(1+\lambda_e-\lambda_i)]_n}{\prod_{i=1}^{4} [1-(a_i+\lambda_e)]_n} \alpha^{-2} + \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{16}\right)^n \frac{\prod_{i=1}^{4} [a_i + \lambda_e]_n}{\prod_{i=1}^{8} [1 + \lambda_e - \lambda_i]_n} \\ & \cdot \sum_{n=1}^{n} \left(\sum_{i=1}^{4} \frac{1}{a_i + \lambda_e + r - 1} - \sum_{i=1}^{8} \frac{1}{\lambda_e - \lambda_i + r} \right) \alpha^{2n} \Bigg\} \quad (e = 1,2,3,4) \end{aligned} \tag{4-14}$$

in which symbol "*" indicates that the multiplication does not include the term of $e = i$. If we still use the symbol expressed by hypergeometric function, we have

$$\begin{aligned} & {}_4F_7 \left(a_1 + \lambda_e, \cdots, a_4 + \lambda_e;\ 1 + \lambda_e - \lambda_1, \cdots, 1 + \lambda_e - \lambda_8{}^*;\ \alpha \right) \\ & = \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{16}\right)^n \frac{\prod_{i=1}^{4} [a_i + \lambda_e]_n}{n! \prod_{i=1}^{8}{}^{*} [1 + \lambda_e - \lambda_i]_n} \alpha^{2n} \end{aligned} \tag{4-15a}$$

$$\begin{aligned} & {}_4\Phi_7 \left(a_1 + \lambda_e, \cdots, a_4 + \lambda_e;\ 1 + \lambda_e - \lambda_1, \cdots, 1 + \lambda_e - \lambda_8{}^*;\ \alpha \right) \\ & = {}_4F_7 \left(a_1 + \lambda_e, \cdots, a_4 + \lambda_e;\ 1 + \lambda_e - \lambda_1, \cdots, 1 + \lambda_e - \lambda_8{}^*;\ \alpha \right) \ln\alpha \end{aligned}$$

$$+\sum_{n=1}^{1}\left(-\frac{1}{16}\right)^{-n}(-1)^{4n-1}\frac{(n-1)!\prod_{i=1}^{8}{}^{*}\left[1-(1+\lambda_e-\lambda_i)\right]n}{\prod_{i=1}^{4}\left[1-(a_i+\lambda_e)\right]_n}\alpha^{-2n}+\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{16}\right)^n$$

$$\frac{\prod_{i=1}^{4}[a_i+\lambda_e]_n}{n!\prod_{i=1}^{8}{}^{*}[1+\lambda_e-\lambda_i]_n}\sum_{r=1}^{n}\left(\sum_{i=1}^{4}\frac{1}{a_i+\lambda_e+r-1}-\sum_{i=1}^{8}\frac{1}{\lambda_e-\lambda_i+r}\right)\alpha^{2n} \tag{4-15b}$$

When $m \neq$ integer or m/2 $\geq$ 2, the solution of Eq.(4-3a) becomes

$$\begin{aligned}\tilde{U}_{kc}^{h}=&\sum_{e=1}^{4}C_e\alpha^{2\lambda_e}\ {}_4F_7(a_1+\lambda_e,\cdots,a_4+\lambda_e;\ 1+\lambda_e-\lambda_1,1+\lambda_e-\lambda_2,\cdots,1+\lambda_e-\lambda_g^{*};\alpha)\\&+\sum_{e=1}^{4}C'_e\alpha^{2\lambda_e}\ {}_4\Phi_7(a_1+\lambda_e,\cdots,a_4+\lambda_e;\ 1+\lambda_e-\lambda_1,1+\lambda_e-\lambda_2,\cdots,1+\lambda_e-\lambda_g^{*};\alpha)\end{aligned} \tag{4-16}$$

where C_e and C'_e are all arbitrary constant which can be determined by boundary conditions.

In some case, it is possible to appear $m/2$ = 1. For this, we can determine the corresponding coefficient λ_i by getting $m/2$=1 in the formula (4-2)′. In order to writing compactly, we rearrange the footnote order of coefficient. Here, we yield

$$\begin{aligned}&\lambda_1=2,\ \lambda_2=0,\ \lambda_3=-\tfrac{1}{2},\\&\lambda_4=\tfrac{3}{2},\ \lambda_5=\tfrac{1}{2},\ \lambda_6=1,\\&\lambda_7=-1,\ \lambda_8=-\tfrac{3}{2}\end{aligned} \tag{4-17}$$

In the formula (4-13), if we let successively λ_e be equal to the value above and then have relation of a fraction. Thus, we known that the two particular solutions when $\lambda_e = \lambda_1$, λ_4 are naturally independent. For the parameters λ_2, λ_3 and λ_5, although the difference between them and λ_4 or λ_1, is integer. Because

$$F(\lambda_e)=\lim_{\varepsilon\to 0}\prod_{i=1}^{4}[a_i+\lambda_e+\varepsilon]_n=0\quad(e=2,3,5) \tag{4-18}$$

we can define that the particular solutions of $\lambda_e = \lambda_2$, λ_3, λ_5 don't include logarithm terms So we can obtain the five particular solutions from the formula (4-13)

$$\tilde{U}_{kce}^{h}=\alpha^{2\lambda_e}\ {}_1F_4\left(\lambda_e;\ 1+\lambda_e-\lambda_1,1+\lambda_e-\lambda_2,1+\lambda_e-\lambda_3,1+\lambda_e-\lambda_7,1+\lambda_e-\lambda_8{}^{*};\alpha\right)\quad(e=1,2,3) \tag{4-19}$$

$${}_1F_4=\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{16}\right)^n\frac{[\lambda_e]_n}{n!\prod_{i=1}^{3}{}^{*}[1+\lambda_e-\lambda_i]_n\prod_{i=1}^{8}{}^{*}[1+\lambda_e-\lambda_i]_n}\alpha^{2n} \tag{4-20}$$

$$\tilde{U}_{kce}^{h}=\alpha^{2\lambda_e}\ {}_2F_5\left(1,\lambda_e;\ 1+\lambda_e-\lambda_1,1+\lambda_e-\lambda_2,1+\lambda_e-\lambda_3,1+\lambda_e-\lambda_7,1+\lambda_e-\lambda_8{}^{*};\alpha\right)\quad(e=4,5) \tag{4-21}$$

$$ {}_2F_5 = \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{16}\right)^n \frac{[1]_n [\lambda_e]_n}{n! \prod_{i=1}^{3} {}^*[1+\lambda_e-\lambda_i]_n \cdot \prod_{i=7}^{8} {}^*[1+\lambda_e-\lambda_i]_n} \alpha^{2n} \tag{4-22} $$

Because the difference between λ_1 and λ_6(λ_2 and λ_7, λ_3 and λ_8) is equal to one, the particular solutions of λ_6, λ_7 and λ_8 must include logarithm term[5]. Applying the method in bibliography[5] and letting m_1=1, we can obtain the other three particular solutions i.e.

$$ \tilde{U}^h_{kce} = \alpha^{2\lambda_e} {}_1\Phi_4 \left(\lambda_e;\ 1+\lambda_e-\lambda_1, 1+\lambda_e-\lambda_2, 1+\lambda_e-\lambda_3, 1+\lambda_e-\lambda_7, 1+\lambda_e-\lambda_8{}^*;\alpha\right) \\ (e=1,2,3) \tag{4-23} $$

$$ \begin{aligned} {}_1\Phi_4(\cdots) &= 2\ {}_1F_4(\cdots)\ln\alpha + {}^1\!/_{16} \frac{\prod_{i=1}^{3} {}^*[1-(1+\lambda_e-\lambda_i)] \ \prod_{i=7}^{8} {}^*[1-(1+\lambda_e-\lambda_i)]}{[1-\lambda_e]} \alpha^{-2} \\ &+ \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{16}\right)^n \frac{[\lambda_e]_n}{n! \prod_{i=1}^{3} {}^*[1+\lambda_e-\lambda_i]_n \ \prod_{i=7}^{8} {}^*[1+\lambda_e-\lambda_i]_n} \\ &\times \sum_{r=1}^{\infty} \left(\frac{1}{\lambda_e+r-1} - \sum_{i=1}^{3} \frac{1}{\lambda_e-\lambda_i+r} - \sum_{i=7}^{8} \frac{1}{\lambda_e-\lambda_i+r} \right) \alpha^{2n} \end{aligned} \tag{4-24} $$

Combining the formula (4-21) and (4-19), (4-23), we can obtain the general solution of Eq.(4-3a) when m/2=1:

$$ \begin{aligned} \tilde{U}^h_{kc} &= \sum_{e=1}^{3} C_e \alpha^{2\lambda_e} {}_1F_4\left(\lambda_e;\ 1+\lambda_e-\lambda_1, 1+\lambda_e-\lambda_2, 1+\lambda_e-\lambda_3, 1+\lambda_e-\lambda_7, 1+\lambda_e-\lambda_8{}^*;\alpha\right) \\ &+ \sum_{e=4}^{5} C_e \alpha^{2\lambda_e} {}_2F_5\left(1,\lambda_e;\ 1+\lambda_e-\lambda_1, 1+\lambda_e-\lambda_2, 1+\lambda_e-\lambda_3, 1+\lambda_e-\lambda_7, 1+\lambda_e-\lambda_8{}^*;\alpha\right) \\ &+ \sum_{e=1}^{3} C'_e \alpha^{2\lambda_e} {}_1\Phi_4\left(\lambda_e;\ 1+\lambda_e-\lambda_1, 1+\lambda_e-\lambda_2, 1+\lambda_e-\lambda_3, 1+\lambda_e-\lambda_7, 1+\lambda_e-\lambda_8{}^*;\alpha\right) \end{aligned} \tag{4-25} $$

in which C_{e1} and C'_{e1} are arbitrary constants which are determined by boundary conditions.

Now, we derive the particular solution $\tilde{U}^p_{kc}$. For distributed load, function $\tilde{Q}_{kc}$ is expressed by the formula, i.e

$$ \tilde{Q}_{kc} = \sum_{n'=0}^{N'} \tilde{Q}_{kcn'} \alpha^{2\rho_{n'}} \tag{4-26} $$

here, $\tilde{Q}_{kcn'}$ is a known constant and N' is a arbitrary big integer. For the each term $\tilde{Q}_{kcn'}\alpha^{2\rho_{n'}}$ in formula (4-27), Eq.(4-3b) becomes

$$ \prod_{i=1}^{8} \left(\delta_\alpha - \bar{\lambda}_i\right) \tilde{U}^p_{kcn'} + \alpha^2 \prod_{i=1}^{4} \left(\delta_\alpha + \bar{a}_i\right) \tilde{U}^p_{kcn'} = \tilde{Q}_{kcn'} \alpha^{2\rho_{n'}} \quad (n' = 0,1,2,\cdots,N') \tag{4-27} $$

Making exchange, let

$$ \alpha = 4\xi^{1/2} \tag{4-28} $$

noting $\delta^r_\alpha(:) = 2^r \delta^r_\xi(:)$, Eq. (4-28) reduces to

$$\prod_{i=1}^{8}\left(\delta_{\xi}-\lambda_{i}\right)\tilde{U}_{kcn'}^{p}+\xi^{2}\prod_{i=1}^{4}\left(\delta_{\xi}+a_{i}\right)\tilde{U}_{kcn'}^{p}=\left(\frac{1}{16}\right)^{2}\tilde{Q}_{kcn'}\xi^{\rho_{n'}}\quad(n'=0,1,2,\cdots,N') \tag{4-29}$$

The solution of Eq. (4-29) is

$$\tilde{U}_{kcn'}^{p}=\left(\frac{1}{16}\right)^{2}\tilde{Q}_{kcn'}\xi^{\rho_{n'}}\sum_{n=0}^{\infty}b_{n}\xi^{n} \tag{4-30}$$

Substituting the solution (4-30) into Eq.(4-29) and comparing the same power of in both ends of Eq. (4-30), we can obtain the relations as follows:

$$b_{0}=\frac{1}{\prod_{i=1}^{8}\left(\rho_{n'}-\lambda_{i}\right)},\quad b_{n}=-\frac{\prod_{i=1}^{4}\left(\rho_{n'}+a_{i}+n-1\right)}{\prod_{i=1}^{8}\left(\rho_{n'}-\lambda_{i}+n\right)}b_{n-1} \tag{4-31}$$

From the recurrence formula (4-31) and the formula (4-30) we yield

$$\tilde{U}_{kcn'}^{p}=\frac{\left(\frac{1}{16}\right)^{2}\tilde{Q}_{kcn'}}{\prod_{i=1}^{8}\left(\rho_{n'}-\lambda_{i}\right)}\xi^{\rho_{n'}}\sum_{n=0}^{\infty}(-1)^{n}\frac{\prod_{i=1}^{4}\left[a_{i}+\rho_{n'}\right]_{n}}{\prod_{i=1}^{8}\left[1+\rho_{n'}-\lambda_{i}\right]_{n}}\xi^{\rho_{n'}} \tag{4-32}$$
$$(n'=0,1,\cdots,N')$$

using the expression (4-28), the solutions of Eq. (4-29) are

$$\tilde{U}_{kcn'}^{p}=\frac{\left(\frac{1}{16}\right)^{\rho_{n'}+2}\tilde{Q}_{kcn'}}{\prod_{i=1}^{8}\left(\rho_{n'}-\lambda_{i}\right)}\xi^{\rho_{n'}}\,{}_{5}F_{8}\left(a_{1}+\rho_{n'},\cdots,a_{4}+\rho_{n'},\ 1;\ 1+\rho_{n'}-\lambda_{1},\cdots,1+\rho_{n'}-\lambda_{8};\alpha\right) \tag{4-33}$$
$$(n'=0,1,2\cdots,N')$$

where

$$\begin{aligned}&{}_{5}F_{8}\left(a_{1}+\rho_{n'},\cdots,a_{4}+\rho_{n'},\ 1;\ 1+\rho_{n'}-\lambda_{1},\cdots,1+\rho_{n'}-\lambda_{8};\alpha\right)\\&=\sum_{n=0}^{\infty}(-1)^{n}\left(\frac{1}{16}\right)^{n}\frac{\prod_{i=1}^{4}\left[a_{i}+\rho_{n'}\right]_{n}\cdot[1]_{n}}{n!\prod_{i=1}^{8}\left[1+\rho_{n'}-\lambda_{i}\right]_{n}}\alpha^{2n}\end{aligned} \tag{4-34}$$

The particular solutions (4-33), generally, are all significant when m = integer. But, below two cases, the solutions (4-33) is not suitable. Here, we have:

If $\rho_{n'}=\lambda_{e'}$ ($\lambda_{e'}$ is a arbitrary value in λ_1, λ_2, ... λ_8), the particular solutions (4-33) are not significant. Thus, the corresponding particular solutions are derived[5] Letting $\rho_{n'}=\lambda_{e'}+\varepsilon$, and making the linear combination, the particular solutions are:

$$\tilde{U}_{kcn'}^{p} = \lim_{\varepsilon \to 0} \frac{\left(\frac{1}{16}\right)^{\lambda_{e'}+2} \tilde{Q}_{kcn'}}{\varepsilon \prod_{i=1}^{8}{}^{*}(\lambda_{e'} - \lambda_i + \varepsilon)}$$
$$\times\left[\alpha^{2(\lambda_{e'}+\varepsilon)} {}_5F_8\left(a_1 + \lambda_{e'} + \varepsilon, \cdots, a_4 + \lambda_{e'} + \varepsilon,\ 1;\ 1 + \lambda_{e'} - \lambda_1, \cdots, 1 + \lambda_{e'} - \lambda_8; \alpha\right)\right.$$
$$\left. -\alpha^{2\lambda_{e'}} {}_5F_8\left(a_1 + \lambda_{e'}, \cdots, a_4 + \lambda_{e'},\ 1;\ 1 + \lambda_{e'} - \lambda_1, \cdots, 1 + \lambda_{e'} - \lambda_8; \alpha\right)\right]$$

from the I'Hospital's principle, we obtain

$$\tilde{U}_{kcn'}^{p} = \frac{\left(\frac{1}{16}\right)^{\lambda_{e'}+2} \tilde{Q}_{kcn'}}{\prod_{i=1}^{8}{}^{*}(\lambda_{e'} - \lambda_i)} \alpha^{2\lambda_{e'}} {}_5\Phi_8\left(a_1 + \lambda_{e'}, \cdots, a_4 + \lambda_{e'},\ 1;\ 1 + \lambda_{e'} - \lambda_1, \cdots, 1 + \lambda_{e'} - \lambda_8; \alpha\right) \tag{4-35}$$

in which,

$$\begin{aligned}
&{}_5\Phi_8\left(a_1 + \lambda_{e'}, \cdots, a_4 + \lambda_{e'},\ 1;\ 1 + \lambda_{e'} - \lambda_1, \cdots, 1 + \lambda_{e'} - \lambda_8; \alpha\right) \\
&= 2\ {}_5F_8\left(a_1 + \lambda_{e'}, \cdots, a_4 + \lambda_{e'},\ 1;\ 1 + \lambda_{e'} - \lambda_1, \cdots, 1 + \lambda_{e'} - \lambda_8; \alpha\right) \ln \alpha \\
&+ \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{16}\right)^n \frac{\prod_{i=1}^{4} [a_i + \lambda_{e'}]_n [1]_n}{n! \prod_{i=1}^{8} [1 + \lambda_{e'} - \lambda_i]_n} \\
&\sum_{r=1}^{n} \left(\sum_{i=1}^{4} \frac{1}{a_i + \lambda_{e'} + r - 1} - \sum_{i=1}^{8} \frac{1}{\lambda_{e'} - \lambda_i + r} \right) \alpha^{2n}
\end{aligned} \tag{4-36}$$

If $\rho_{n'} - \lambda_{e'} = -m'(m' = 1, 2, \cdots)$ and $\lambda_{e'}$ is a arbitrary value among $\lambda_1, \lambda_2, \cdots, \lambda_8$, when

$$\prod_{i=1}^{4} [a_i + \lambda_{e'}]_n \neq 0$$

the corresponding particular solutions can be obtained[5],

$$\tilde{U}_{kcn'}^{p} = \frac{\left(\frac{1}{16}\right)^{\rho_{n'}+2} \tilde{Q}_{kcn'}}{\prod_{i=1}^{8}{}^{*}(\rho_{n'} - \lambda_i)} \alpha^{2\lambda_{e'}} {}_5\Phi_8\left(a_1 + \lambda_{e'}, \cdots, a_4 + \lambda_{e'},\ 1;\ 1 + \lambda_{e'} - \lambda_1, \cdots, 1 + \lambda_{e'} - \lambda_8; \alpha\right) \tag{4-37}$$

According to the formulas (4-33), (4-35) and (4-37), if the term $\tilde{Q}_{kc}$ of the right side of Eq. (4-35) is given by the formula (4-27), the particular solution is

$$\tilde{U}_{kc}^{p} = \sum_{n'=0}^{N'} \tilde{U}_{kcn'}^{p} \tag{4-38}$$

Next, we discuss the solutions of basic equation (4-1) when $k = 0(m = 0)$. In this case, the deformation of conical shell is symmetric. Removing the common differential operator $\cos^2 \phi\left(\delta_\alpha^2 - 1\right)$ in the formulas (3-19a)~(3-21a) and Eq. (3-29), the basic equation reduces:

$$\tilde{u}_0 = -\sin\phi\cos\phi\left(\mu\delta_\alpha - 1\right)\tilde{U}_0 - \frac{1-\mu}{2}\tilde{\psi}_{10} \tag{4-39}$$

$$\tilde{v}_0 = -\tilde{\psi}_{20} \tag{4-40}$$

$$\tilde{w}_0 = \cos^2\phi\left(\delta_\alpha^2 - 1\right)\tilde{U}_0 \tag{4-41}$$

$$\delta_\alpha^2\left(\delta_\alpha - 2\right)^2\left(\delta_\alpha^2 - 1\right)\tilde{U}_0 + \alpha^2\delta_\alpha^2\tilde{U}_0 = \tilde{Q}_0 \tag{4-42}$$

the free term is

$$\tilde{Q}_0 = \sec^4\phi\left[\alpha^4\tilde{q}_{30} + \frac{1}{2(1+\mu)}\mathrm{tg}\phi\alpha^2\left(\mu\delta_\alpha + 1\right)\tilde{\psi}_{10}\right] \tag{4-43}$$

Thus, the equation (3-39a) can be reduced

$$\left(\delta_\alpha^2 - 1\right)\tilde{\psi}_{i0} = -2\left(1+\mu\right)\sec^2\phi\alpha^2\tilde{q}_{i0} \quad (i = 1, 2) \tag{4-44a,b}$$

From the formulas (3-22a) ~ (3-28a), the corresponding internal forces are expressed as follows:

$$\tilde{T}_{10} = \sin\phi\cos\phi\frac{1}{\alpha}\delta_\alpha\tilde{U}_0 - \frac{1}{2(1+\mu)}\frac{1}{\alpha}\left(\delta_\alpha + \mu\right)\tilde{\psi}_{10} \tag{4-45}$$

$$\tilde{T}_{20} = \sin\phi\cos\phi\frac{1}{\alpha}\delta_\alpha^2\tilde{U}_0 - \frac{1}{2(1+\mu)}\frac{1}{\alpha}\left(\mu\delta_\alpha + 1\right)\tilde{\psi}_{10} \tag{4-46}$$

$$\tilde{T}_0 = -\frac{1}{2(1+\mu)}\frac{1}{\alpha}\left(\delta_\alpha - 1\right)\tilde{\psi}_{20} \tag{4-47}$$

$$\tilde{M}_{10} = -\cos^2\phi\frac{1}{\alpha^2}\left[\delta_\alpha^2 - \left(1-\mu\right)\delta_\alpha\right]\left(\delta_\alpha^2 - 1\right)\tilde{U}_0 \tag{4-48}$$

$$\tilde{M}_{20} = -\cos^2\phi\frac{1}{\alpha^2}\left[\mu\delta_\alpha^2 + \left(1+\mu\right)\delta_\alpha\right]\left(\delta_\alpha^2 - 1\right)\tilde{U}_0 \tag{4-49}$$

$$\tilde{H}_0 = 0 \tag{4-50}$$

$$\tilde{V}_{10} = -\cos^2\phi\frac{1}{\alpha^2}\left(\delta_\alpha^3 - 2\delta_\alpha^2\right)\left(\delta_\alpha^2 - 1\right)\tilde{U}_0 \tag{4-51}$$

and from (3-31) ~ (3-33), we can obtain the corresponding boundary conditions when $\alpha = \alpha_1$:

fixed (generalized) side:

$$\tilde{u}_0 = \tilde{u}_0^b,\ \tilde{w}_0 = \tilde{w}_0^b,\ \frac{\partial\ \tilde{w}_0}{\partial\ \alpha} = \left(\frac{\partial\ \tilde{w}}{\partial\ \alpha}\right)_0^b \tag{4-52}$$

free (generalized) side:

$$\tilde{T}_{10} = \tilde{T}_{10}^b,\ \tilde{M}_{10} = \tilde{M}_{10}^b,\ \tilde{V}_{10} = \tilde{V}_{10}^b \tag{4-53}$$

simply (generalized) side:

$$\tilde{w}_0 = \tilde{w}_0^b, \quad \tilde{T}_0 = \tilde{T}_{10}^b, \quad \tilde{M}_{10} = \tilde{M}_{10}^b \tag{4-54}$$

From the formulas (4-39)~(4-54), we know that there are two symmetric deformation for k = 0. One is described by the equation (4-52)~(4-54) and the boundary conditions (4-43)~(4-54), which expresses the symmetrical bending of conical shell. In this case, it concerns with the components of load $\tilde{q}_{10}$ and $\tilde{q}_{30}$. The other symmetric deformation is the pure torsion of conical shell which is expensed by Eq. (4-44b) and the boundary conditions

$$\alpha = \alpha_1, \quad \tilde{v}_0 = \tilde{v}_0^b, \quad or \quad \tilde{T}_0 = \tilde{T}_{10}^b \tag{4-55a,b}$$

In this case, it only concerns with load $\tilde{q}_{20}$ and causes the tangent internal force $\tilde{T}_0$ and the annular displacement $\tilde{v}_0$ in shell. The equations and boundary conditions corresponding these two symmetric of deformations are independent each other. Here, these equations and boundary conditions may be determined, respectively.

For axial symmetric bending, the basic equation (4-42) can be rewritten

$$\delta_\alpha^2 (\delta_\alpha - 2)^2 (\delta_\alpha^2 - 1)\tilde{U}_0 + (\delta_\alpha - 2)^2 \alpha^2 \tilde{U}_0 = \tilde{Q}_0 \tag{4-56}$$

Introducing the variable y, we assume:

$$(\delta_\alpha^2 - 1)\delta_\alpha^2 \tilde{U}_0 + \alpha^2 \tilde{U}_0 = y \tag{4-57}$$

hence, the equation (4-56) is equal to the system of equations as follows

$$(\delta_\alpha - 2)^2 y = \tilde{Q}_0$$

$$(\delta_\alpha^2 - 1)\delta_\alpha^2 \tilde{U}_0 + \alpha^2 \tilde{U}_0 = y \tag{4-58a,b}$$

making fransformation, let

$$y = \alpha^2 y' \tag{c}$$

the equation (4-58a) becomes

$$\delta_\alpha^2 y' = \frac{1}{\alpha^2}\tilde{Q}_0$$

The preceding formula can also be written spread form, i.e:

$$\alpha \frac{d}{d\alpha} \alpha \frac{dy'}{d\alpha} = \frac{1}{\alpha^2}\tilde{Q}_0 \tag{d}$$

Integrating the formula (d) and noting the formula a (c), we yield

$$y = \alpha^2 \int \frac{1}{\alpha} \int \frac{\tilde{Q}_0}{\alpha^3} d\alpha \;\; d\alpha + \alpha^2 f_1(c_1, c_2) \tag{e}$$

where

$$f_1(c_1, c_2) = c_1 \text{ In } \alpha + c_2 \tag{4-59}$$

c_1 and c_2 are integral constants. Appling the formula (e), Eq. (4-58b) becomes:

$$\delta_\alpha^2 (\delta_\alpha^2 - 1)\tilde{U}_0 + \alpha^2 \tilde{U}_0 = \alpha^2 \int \frac{1}{\alpha} \int \frac{\tilde{Q}_0}{\alpha^3} d\alpha \;\; d\alpha + \alpha^2 f_1(c_1, c_2) \tag{4-60}$$

Introducing the differential operator:

$$L_0(:) = \alpha \frac{d^2(:)}{d\alpha^2} + 2\frac{d(:)}{d\alpha} \tag{4-61}$$

then, Eq. (4-60) can be reduced to the formula below

$$L_0 L_0(\tilde{U}_0) + \tilde{U}_0 = \int \frac{1}{\alpha} \int \frac{\tilde{Q}_0}{\alpha^3} d\alpha \quad d\alpha + f_1(c_1, c_2) \tag{4-62}$$

The formula above can also be written

$$(L_0 + i)(L_0 + i)\tilde{U}_0 = p(\alpha) + f_1(c_1, c_2) \tag{4-62$'$}$$

$i = \sqrt{-1}$. and

$$p(\alpha) = \int \frac{1}{\alpha} \int \frac{\tilde{Q}_0}{\alpha^3} d\alpha \quad d\alpha \tag{4-63}$$

The solution of E q. (4-62) or (4-62)2 can be expressed:

$$\tilde{U}_0 = \tilde{U}_{01}^h + \tilde{U}_{02}^h + \tilde{U}_{03}^h + \tilde{U}_0^p \tag{4-64}$$

Here, $\tilde{U}_{01}^h$, $\tilde{U}_{02}^h$ and $\tilde{U}_{03}^h$ are homogeneous solutions, $\tilde{U}_0^p$ is a particular solution. They satisfy the equations below

$$\begin{gathered} (L_0 + i) \quad \tilde{U}_{01}^h = 0 \\ (L_0 - i) \quad \tilde{U}_{02}^h = 0 \\ L_0 L_0(\tilde{U}_{03}^h) + \tilde{U}_{03}^h = f_1(c_1, c_2) \\ (L_0 + i)(L_0 - i) \quad \tilde{U}_0^h = p(\alpha) \end{gathered} \tag{4-65a,b,c,d}$$

Eq. (4-65a) can be reduced to Besse equation. For this, we introduce a new variable

$$x = 2 \quad i^{1/2}\alpha^{1/2}, \quad \tilde{U}_{01}^h = \alpha^{1/2} y_{01} \tag{4-66}$$

noting the formula (4-61), the formula (4-65) becomes

$$x^2 \frac{d^2 y_{01}}{dx^2} + x\frac{dy_{01}}{dx}(x^2 - 1)y_{01} = 0 \tag{4-67}$$

The equation (4-67) is a Bessel's equation and its solution is

$$y_{01} = A_1 J_1(x) + B_1 H_1^{(1)}(x)$$

Considering the formula (4-66), the solution of Eq. (4-67a) is

$$\tilde{U}_{01}^h = A_1 \alpha^{-1/2} J_1(2\alpha^{1/2} i^{1/2}) + B_1 \alpha^{-1/2} H_1^{(1)}(2\alpha^{1/2} i^{1/2}) \tag{4-68a}$$

In which $J_1(2\alpha^{1/2} i^{1/2})$, $H_1^{(1)}(2\alpha^{1/2} i^{1/2})$ is Bessel and Hankel function of one order of first kind. They can be expressed by Thomson's function, i, e

$$\begin{gathered} J_1(2\alpha^{1/2} i^{1/2}) = -ber_1(2\alpha^{1/2}) + ibei_1(2\alpha^{1/2}) \\ H_1^{(1)}(2\alpha^{1/2} i^{1/2}) = (2/\pi)(Kei_1(2\alpha^{1/2}) + iKer_1(2\alpha^{1/2}) \end{gathered} \tag{g}$$

Because the equation (4-65b) conjugetes with the equation (4-65a), its solution is naturally

$$\tilde{U}_{02}^h = A_2 \alpha^{-1/2} J_1(2\alpha^{1/2} i^{3/2}) + B_2 \alpha^{-1/2} H_1^{(1)}(2\alpha^{1/2} i^{3/2}) \tag{4-68b}$$

Where A_1, B_1, A_2 and B_2 are arbitrary imaginary constants. $J_1(2\alpha^{1/2}i^{3/2})$, $H_1^{(1)}(2\alpha^{1/2}i^{3/2})$ and $J_1(2\alpha^{1/2}i^{1/2})$, $H_1^{(1)}(2\alpha^{1/2}i^{1/2})$ are conjugate each other.

With the method of substituting, we can prove that the solution of Eq. (4-65c) is

$$\tilde{U}_{03}^h = f_1(c_1, c_2) \tag{4-68c}$$

Combining (4-68a), (4-68b), (4-68c) and using the formula (g), either A_1 and A_2 or B_1 and B_2 are conjugate for real function $\tilde{U}_0^h$. The homogeneous solution of Eg. (4-56) is

$$\tilde{U}_0^h = f_1(c_1, c_2) + c_3 ber_1(2\alpha^{1/2}) + c_4 bei_1(2\alpha^{1/2}) + c_5 Ker_1(2\alpha^{1/2}) + c_6 Kei_1(2\alpha^{1/2}) \tag{4-69}$$

here, c_1, c_2, $\cdots$, c_6 are arbitrary real constants, which are determined by the boundary condition.

We can determine the particular solution U_0^p The formula (4-65d) can be written

$$\begin{aligned} (L_1 + i) y_0^p &= p(\alpha) \\ (L_1 - i)\tilde{U}_0^p &= y_0^p \end{aligned} \tag{4-70a,b}$$

The homogeneous solutions of Eq. (4-70a) and Eq. (4-70b) are given by the formulas (4-68a) and (4-68b), respectively. Applying the method of altering coefficient and noting the relation

$$J_n(x)(dH_n^{(1)}(x)/dx) - H_n^{(1)}(x)(dJ_n(x)/dx) = 2i/\pi \quad x$$

We have

$$\begin{aligned} y_0^p = i\pi(&\alpha^{-1/2} J_1(2\alpha^{1/2}i^{1/2})\int_\alpha x^{3/2} H_1^{(1)}(2x^{1/2}i^{1/2})p(x)dx \\ &- \alpha^{-1/2} H_1^{(1)}(2\alpha^{1/2}i^{1/2})\int_\alpha x^{3/2} J_1(2x^{1/2}i^{1/2})p(x)dx) \end{aligned} \tag{4-71a}$$

$$\begin{aligned} \tilde{U}_0^p = i\pi(&\alpha^{1/2} J_1(2\alpha^{1/2}i^{3/2})\int_\alpha x^{3/2} H_1^{(1)}(2x^{1/2}i^{3/2})y_0^p dx \\ &- \alpha^{-1/2} H_1^{(1)}(2\alpha^{1/2}i^{3/2})\int_\alpha x^{3/2} J_1(2x^{1/2}i^{3/2})y_0^p dx) \end{aligned} \tag{4-71b}$$

Substituting the formula (4-71a) into (4-71b) and exchanging integral variable, the particular solution of Eq. (4-56) is

$$\begin{aligned} \tilde{U}_0^p =\ &\pi^2\alpha^{-1/2} J_1(2\alpha^{1/2}i^{3/2})\int_\alpha x_1 H_1^{(1)}(2x_1^{1/2}i^{3/2})\int_{x_1} x_2^{3/2} J_1(2x_2^{1/2}i^{1/2})p(x_2)dx_2 dx_1 \\ &-\pi^2\alpha^{-1/2} J_1(2\alpha^{1/2}i^{3/2})\int_\alpha x_1 H_1^{(1)}(2x_1^{1/2}i^{3/2})J_1(2x_1^{1/2}i^{1/2})\int_{x_1} x_2^{3/2} H_1^{(1)}(2x_2^{1/2}i^{1/2})p(x_2)dx_2 dx_1 \\ &+\pi^2\alpha^{-1/2} H_1^{(1)}(2\alpha^{1/2}i^{3/2})\int_\alpha x_1 J_1(2x_1^{1/2}i^{3/2})J_1(2x_1^{1/2}i^{1/2})\int_{x_1} x_2^{3/2} H_1^{(1)}(2x_2^{1/2}i^{1/2})p(x_2)dx_2 dx_1 \\ &+\pi^2\alpha^{-1/2} H_1^{(1)}(2\alpha^{1/2}i^{3/2})\int_\alpha x_1 J_1(2x_1^{1/2}i^{3/2})H_1^{(1)}(2x_1^{1/2}i^{1/2})\int_{x_1} x_2^{3/2} J_1(2x_2^{1/2}i^{1/2})p(x_2)dx_2 dx_1 \end{aligned} \tag{4-72}$$

For the deformation of pure torsion, the corresponding basic equation can to obtained from Eq. (4-44b)

$$(\delta_\alpha^2 - 1)\tilde{\psi}_{20} = -2(1+\mu)\sec^2\phi \quad \tilde{q}_{20}\alpha^2 \tag{4-73}$$

The solution of Eq. (4-73) can be solved easily. If we introduce a unknown variable y_1, the equation (4-73) is equal to equation system as follows, i.e

$$(\delta_\alpha - 1)y_1 = -2(1+\mu)\sec^2\phi \quad \alpha^2\tilde{q}_{20} \tag{4-74a}$$

$$(\delta_\alpha + 1)\tilde{\psi}_{20} = y_1 \tag{4-74b}$$

let

$$y_1 = \alpha^2 y' \quad \tilde{\psi}_{20} = \alpha^{-1}\tilde{\psi}'_{20} \tag{h}$$

Then, Eq. (4-74) becomes integrable ferm

$$dy'_1 / d\alpha = -2(1+\mu)\sec^2\phi \;\; \tilde{q}_{20} \qquad d\tilde{\psi}'_{20} / d\alpha = y_1 \tag{i}$$

Considering the formula (h), we yield

$$y'_1 = -2(1+\mu)\sec^2\phi\;\alpha\int \tilde{q}_{20} d\alpha + c'_1\alpha \tag{4-75a}$$

$$\tilde{\psi}'_{20} = (1/\alpha)\int y_1 d\alpha + (c'_2/\alpha) \tag{4-75b}$$

Substituting the formula (4-75a) into (4-75b) and exchange integral variable properly, we can get the general solution of Eq. (4-73):

$$\tilde{\psi}_{20} = -2(1+\mu)\sec^2\phi \;\; (1/\alpha)\int_\alpha x_1 \int_{x_1} \tilde{q}_{20} dx_2 dx_1 + (c'_1\alpha/2) + (c'_2/\alpha) \tag{4-76}$$

in which, c'_1 and c'_2 are integral constant which can be determined by boundary condition (4-55).

The solutions of second kind of problem, i.e., basic equation (3-34), is similar to the preceding method, which is no need of discussion again.

When determining the displacement functions $\tilde{U}_{kc}$, $\tilde{U}_{ks}$ and $\tilde{U}_0$, we can calculate the displacement and internal force of the shell by the formulas (3-19)-(3-27), (4-39)-(4-41) and (4-45)-(4-50).

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