***IJMS**, Vol. 19, Nos. 1, (January-March 2020), pp. 39-53*

ISSN: 0972-754X

CLAW DECOMPOSITION OF PRODUCT GRAPHS

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Abstract: A decomposition of a graph G is a family of edge-disjoint subgraphs $\{G_1, G_2,\ldots ,G_k\}$ such that $E(G) = E(G_1) \cup E(G_2) \cup \ldots \cup E(G_k)$. If each G_i is isomorphic to H for some subgraph H of G, then the decomposition is called a H-decomposition of G. A star with three edges is called a claw. In this paper, we give necessary and sufficient condition for the decomposition of cartesian product of standard graphs into claws. Also, we give a sufficient condition for the claw decomposition of lexicographic product of standard graphs.

AMS Subject Classification: 05C70

Keywords: Decomposition, claw decomposition, cartesian product, lexicographic product

1. INTRODUCTION

Let $G = (V, E)$ be a simple undirected graph without loops or multiple edges. A path on n vertices is denoted by P_n, cycle on n vertices is denoted by C_n and complete graph on n vertices is denoted by K_n. The *neighbourhood* of a vertex v in G is the set $N(v)$ consisting of all vertices that are adjacent to v. $|N(v)|$ is called the degree of v and is denoted by $d(v)$. A complete bipartite graph with partite sets V_1 and V_2, where $|V_1| = r$ and $|V_2| = s$, is denoted by $K_{r, s}$. The graph $K_{1, r}$ is called a star and is denoted by S_r. The vertex of degree r in the star S_r is called the central vertex of the star. Claw is a star with three edges. The complement of a graph G is denoted by $\overline{G}$. kG denotes the union of k copies of G. The join $G + H$ of two graphs G and H consists of $G \cup H$ and all edges joining each vertex of G to all the vertices of H. Terms not defined here are used in the sense of [5].

A decomposition of a graph G is a family of edge-disjoint subgraphs $\{G_1, G_2,\ldots ,G_k\}$ such that $E(G) = E(G_1) \cup E(G_2) \cup\ldots\cup E(G_k)$. If each G_i is isomorphic to H for some subgraph H of G, then the decomposition is called a H-decomposition of G. If H has at least three edges, then the problem of deciding if a graph G has a H-decomposition is NP-complete [2]. In 1975, Sumiyasu Yamamoto et al., [6] gave necessary and sufficient condition for the S_k-decomposition of complete graphs and complete bipartite graphs. In 1996, C. Lin and T. W. Shyu [4] presented a necessary and sufficient condition for decomposing K_n into stars $S_{k_1}, S_{k_2},\ldots, S_{k_t}$. In 2004, H. L. Fu et al., [3] decomposed a complete graph into cartesian product of two complete graphs K_r and

K_c. In 2012, Darryn E. Bryant et al., [1] gave necessary and sufficient condition for the existence of k-star factorizations of any power K_q^s where q is prime and the products $C_{r_1} \times C_{r_2} \times ... \times C_{r_k}$ of k cycles of arbitrary length. In 2013, Tay-Woei Shyu [7] gave necessary and sufficient condition for the decomposition of complete graph into C_l's and S_k's. In this paper, we give necessary and sufficient condition for the decomposition of cartesian product of standard graphs into claws. Also, we give a sufficient condition for the claw decomposition of lexicographic product of standard graphs.

2. BUILDING BLOCKS

In this section, we collect certain lemmas and results which are used in the subsequent sections. These are the building blocks in the construction of the main theorems.

Definition 2.1: The corona of two graphs G and H, is the graph $G \circ H$ formed from one copy of G and $|V(G)|$ copies of H where the i^{th} vertex of G is adjacent to every vertex in the i^{th} copy of H.

Definition 2.2: The Cartesian product of two graphs G and H is a graph, denoted by $G \times H$, whose vertex set is $V(G) \times V(H)$. Two vertices (g, h) and (g', h') are adjacent precisely if $g = g'$ and $hh'' \in E(H)$, or $gg'' \in E(G)$ and $h = h'$. Thus,

$$V(G \times H) = \{(g, h)/g \in V(G) \text{ and } h \in V(H)\},$$

$$E(V \times H) = \{(g, h)(g', h')/g = g' \text{ and } hh' \in E(H), \text{ or } gg' \in E(G) \text{ and } h = h'\}.$$

Theorem 2.3: [6] A complete graph, K_l with l points and $\binom{l}{2}$ lines can be decomposed into a union of line disjoint $\binom{l}{2}/c$ claws, $K_{1,c}$, with c lines each if and only if

(1) $\binom{l}{c}$ is an integral multiple of c, and

(2) $l \geq 2c$.

Theorem 2.4: [6] A complete bigraph, $K_{m,n}$, with m and n points and mn lines can be decomposed into union of mn/c line disjoint $\binom{l}{2}/c$ claws, $K_{1,c}$, with c lines each if and only if m and n satisfy one of the following three conditions:

(1) $n \equiv 0 \ (mod\ c)$ when $m < c$

(2) $m \equiv 0$ $(mod\ c)$ when $n < c$

(3) $mn \equiv 0$ $(mod\ c)$ when $m \geq c$ and $n \geq c$.

Lemma 2.5: The graph $C_n \text{ o } \overline{K}_2$ is claw decomposable for all n.

Proof: Let $V(C_n) = \{v_1, v_2,...v_n\}$ and let u_i and w_i be the pendant vertices at v_i.

Then $< \{u_i, w_i, v_i, v_{i+1}\} > \cong K_{1,3}$ for all $1 \leq i \leq n - 1$

and $< \{u_n, w_n, v_n, v_1\} > \cong K_{1,3}$.

Thus $$E(C_n \text{ o } \overline{K}_2) = \underbrace{E(K_{1,3}) \cup ... \cup E(K_{1,3})}_{n\ times}.$$

Hence $C_n \text{ o } \overline{K}_2$ is claw decomposable. □

Lemma 2.6: *If* n is even and $n \equiv 0(mod\ 3)$, then $K_2 \times C_n$ is claw decomposable.

Proof: Let $V(K_2) = \{x_1, x_2\}$ and let $V(C_n) = \{y_1, y_2,..., y_n\}$.

Then $V(K_2 \times C_n) = \{(x_i, y_j)/i = 1, 2 \text{ and } 1 \leq j \leq n\}$.

Rename $(x_1, y_j) = v_j$ and $(x_2, y_j) = u_j$ for all $1 \leq j \leq n$.

Now, $< \{v_1, v_2, v_n, u_1\} > \cong K_{1,3}$,

$< \{u_1, u_{n-1}, u_n, v_n\} > \cong K_{1,3}$,

$< \{u_{i+1}, v_i, v_{i+1}, v_{i+2}\} > \cong K_{1,3}$ for all $i \in \{2, 4,..., n-2\}$ and

$< \{u_i, u_{i+1}, u_{i+2}, v_{i+1}\} > \cong K_{1,3}$ for all $i \in \{1, 3, ..., n - 3\}$.

Thus $$\text{E}(K_2 \times C_n) = \underbrace{E(K_{1,3}) \cup ... \cup E(K_{1,3})}_{n\ times}.$$

Hence $K_2 \times C_n$ is claw decomposable. □

Lemma 2.7: $K_n \text{ o } K_1$ is claw decomposable if and only if $n > 3$ and $n \neq 1(mod\ 3)$.

Proof: Let $V(K_n) = \{v_1, v_2,..., v_n\}$ and let u_i be the pendant vertex at v_i for all $1 \leq i \leq n$.

Suppose that $n > 3$ and $n \neq 1(mod\ 3)$.

Case (i): $n \equiv 2(\text{mod } 3)$.

Now, $< \{v_5, v_6,..., v_n\} > \cong K_{n-4}$,

$< \{v_3, v_4, v_i, u_i\} > \cong K_{1,3}$ for all $5 \leq i \leq n$,

$< \{v_1, v_2, v_4, u_4\} > -\{v_1v_2\} \cong K_{1,3}$,

$< \{v_1, v_3, v_4, u_3\} > -\{v_1v_4\} \cong K_{1,3}$,

$< \{u_1, v_1, v_2, v_5, v_6,\ldots, v_n\} > -E\,(< \{v_5, v_6,\ldots, v_n\} >) \cong K_{1,n-2}$ and

$< \{u_2, v_2, v_3, v_5, v_6,\ldots, v_n\} > -E(< \{v_5, v_6,\ldots, v_n\} >) \cong K_{1,n-2}$.

Thus $E(K_2 \text{ o } C_1) = E(K_{n-4}) \cup \underbrace{E(K_{1,3}) \cup \ldots \cup E(K_{1,3})}_{(n-2)\ times} \cup E(K_{1,n-2}) \cup E(K_{1,n-2})$.

Since $n \equiv 2(mod\ 3)$, $n - 4 \equiv 1(mod\ 3)$. Hence by Theorem 2.3, K_{n-4} is claw decomposable. Also, $K_{1,n-2}$ is claw decomposable.

Hence K_n o K_1 is claw decomposable.

Case (ii): $n \equiv 0(mod\ 3)$.

Then $< \{v_1, v_2,\ldots, v_{n-1}, u_1, u_2,\ldots, u_{n-1}\} > \cong K_{n-1}$ o K_1 and

$< \{v_1, v_2,\ldots, v_n\} > -E(< \{v_1, v_2,\ldots, v_{n-1}\} >) + \{u_nv_n\} \cong K_{1,n}$.

Thus $E(K_n \text{ o } K_1) = E(K_{n-1} \text{ o } K_1) \equiv E(K_{1,n})$.

Since $n \equiv 0(mod\ 3)$, $n - 1 \equiv 2(mod\ 3)$. Hence by Case (*i*), K_{n-1} o K_1 is claw decomposable. Also, $K_{1,n}$ is claw decomposable.

Hence K_n o K_1 is claw decomposable.

Conversely, suppose that K_n o K_1 is claw decomposable.

Then $|E(K_n \text{ o } K_1)| \equiv 0\ (mod\ 3)$. That is, $\frac{n(n-1)}{2} + n \equiv 0\ (mod\ 3)$ which implies $\frac{n(n+1)}{2} \equiv 0\ (mod\ 3)$ and thus $n \equiv 0(mod\ 3)$ or $n \equiv 2\ (mod\ 3)$. Hence $n \neq 1(mod\ 3)$. Also, K_3 o K_1 is not claw decomposable. Thus $n > 3$.

Hence $n > 3$ and $n \neq 1(mod\ 3)$. □

Lemma 2.8: The graph $K_2 \times K_n$ is claw decomposable if and only if $n > 3$ and $n \equiv 0\ (mod\ 3)$.

Proof: Let $V(K_2) = \{x_1, x_2\}$ and let $V(C_n) = \{y_1, y_2,\ldots, y_n\}$.

Then $V(K_2 \times C_n) = \{(x_i, y_j)/i = 1, 2 \text{ and } 1 \leq j \leq n\}$.

Rename $(x_1, y_j) = v_j$ and $(x_2, y_j) = u_j$ for all $1 \leq j \leq n$.

Now, $< \{v_1, v_2,..., v_n, u_1, u_2,..., u_n\} > -E(< \{u_1, u_2,..., u_n\} >) \cong K_n \text{ o } K_1$

and $< \{u_1, u_2,..., u_n\} > \cong K_n$.

Thus $E(G) = E(K_n \text{ o } K_1) \cup E(K_n)$.

Suppose that $n > 3$ and $n \equiv 0$ (*mod* 3).

Then by Lemma 2.7, K_n o K_1 is claw decomposable. Also, by Theorem 2.3, K_n is claw decomposable.

Hence K_2 o K_n is claw decomposable.

Conversely, suppose that $K_2 \times K_n$ is claw decomposable.

Then $|E(K_2 \times K_n)| \equiv 0$ (*mod* 3). That is, $2.\frac{n(n-1)}{2} + 1.n \equiv 0$ (*mod* 3) which implies $n^2 \equiv 0$ (*mod* 3) and hence $n \equiv 0$ (*mod* 3). Also, $K_2 \times K_3$ is not claw decomposable. Thus $n > 3$. Hence $n > 3$ and $n \equiv 0$ (*mod* 3).

Lemma 2.9: The graph $K_2 \times K_n$ together with a pendant vertex attached to each vertex of one copy of K_n is claw decomposable if and only if $n \neq 1$(*mod* 3).

Proof: Let G be the graph $K_2 \times K_n$ together with a pendant vertex attached to the each vertex of one copy of K_n.

Let $V(K_2) = \{x_1, x_2\}$ and let $V(K_n) = \{y_1, y_2,..., y_n\}$.

Then $V(K_2 \times K_n) = \{(x_i, y_j)/i = 1, 2 \text{ and } 1 \leq j \leq n\}$.

Rename $(x_1, y_j) = v_j$ and $(x_2, y_j) = u_j$ for all $1 \leq j \leq n$.

Let w_j be the pendant vertex at v_j in G for all $1 \leq j \leq n$.

Now, $< \{u_1, u_2,..., u_n, v_1, v_2,..., v_n\} > -E(< \{v_1, v_2,..., v_n\} >) \cong K_n \text{ o } K_1$

and $< \{v_1, v_2,..., v_n, w_1, w_2,..., w_n\} > \cong K_n \text{ o } K_1$.

Thus $E(G) = E(K_n \text{ o } K_1) \cup E(K_n \text{ o } K_1)$.

Suppose that $n \neq 1$ (*mod* 3).

Then by Lemma 2.7, K_n o K_1 is claw decomposable.

Hence G is claw decomposable.

Conversely, suppose that G is claw decomposable.

Then $|E(G)| \equiv 0$ (*mod* 3). That is, $2.\frac{n(n-1)}{2} + 1.n + n \equiv 0$ (*mod* 3) which implies $n(n + 1) \equiv 0$ (*mod* 3) and thus $n \equiv 0$ (*mod* 3) or $n \equiv 2$(*mod* 3).

Hence $n \neq 1$(*mod* 3). □

3. CLAW DECOMPOSITION OF CARTESIAN PRODUCT OF GRAPHS

In this section, we give necessary and sufficient condition for the decomposition of cartesian product of some standard graphs into claws.

Theorem 3.1: If G_1 and G_2 are H-decomposable, then $G_1 \times G_2$ is H-decomposable.

Proof: Let $V(G_1) = \{v_1, v_2,..., v_k\}$ and $V(G_2) = \{u_1, u_2,..., u_n\}$.

Then $V(G_1 \times G_2) = \{(v_i, u_j)/1 \leq i \leq k, 1 \leq j \leq n\}$.

Rename $(v_i, u_j) = v_{ij}$; $1 \leq i \leq k, 1 \leq j \leq n$.

Now, $< \{v_{1j}, v_{2j},..., v_{kj}\} > \cong G_1$ for all $1 \leq j \leq n$ and

$< \{u_{i1}, u_{i2},..., u_{in}\} > \cong G_2$ for all $1 \leq i \leq k$.

Thus, $E(G_1 \times G_2) = \underbrace{E(G_1) \cup ... \cup E(G_1)}_{n\ times} \cup \underbrace{E(G_2) \cup ... \cup E(G_2)}_{k\ times}$.

Since G_1 and G_2 are H-decomposable, $G_1 \times G_2$ is H-decomposable. □

Corollary 3.2: If $m, n \equiv 0\ (mod\ 3)$, then $K_{1,m} \times K_{1,n}$ is claw decomposable.

Corollary 3.3: If $m \equiv 0\ (mod\ 3)$ and $n \neq 2\ (mod\ 3)$ then $K_{1,m} \times K_n$ is claw decomposable.

Proof: It follows from Theorems 2.3 and 3.1.

Corollary 3.4: *If* $rs \equiv 0\ (mod\ 3)$ and $n \equiv 2\ (mod\ 3)$, then $K_{r,s} \times K_n$ *is* claw decomposable.

Proof: It follows from Theorems 2.3, 2.4 and 3.1.

Corollary 3.5: If $rs \equiv 0\ (mod\ 3)$ and $n \equiv 0\ (mod\ 3)$, then $K_{r,s} \times K_{1,n}$ is $K_{1,3}$ -decomposable.

Proof: It follows from Theorems 2.4 and 3.1. □

Remark 3.6: P_n o K_1 and C_n o K_1 are not claw decomposable for any values of n.

Remark 3.7: If $G = P_m$ o K_1, then G o C_n is not claw decomposable.

Theorem 3.8: Let $G_1 = P_m$ o K_1. If G_2 and G_2 o K_1 are claw decomposable, then $G_1 \times G_2$ is claw decomposable.

Proof: Let $V(G_1) = \{u_1, u_2,..., u_m, w_1, w_2,...,w_m\}$ where w_i is the pendant edge at u_i for all $1 \leq i \leq m$, $u_1u_2... u_m$ is the m-path in G and $V(G_2) = \{v_1, v_2,..., v_n\}$.

Then $V(G_1 \times G_2) = \{(u_i, v_j), (w_i, v_j)/1 \leq i \leq m, 1 \leq j \leq n\}$.

Rename $(u_i, v_j) = u_{ji}$ and $(w_i, v_j) = w_{ji}$ for all $1 \le i \le m,$ and $1 \le j \le n.$

Now, $< \{u_{1j}, u_{2j},..., u_{nj}, w_{1j}, w_{2j},...,w_{nj}\} > - E(< \{u_{1j}, u_{2j},..., u_{nj}\} >)$

$\cong G_2 \text{ o } K_1$ for all $1 \le j \le m$,

$< \{u_{1j}, u_{2j},..., u_{nj}, u_1(j + 1), u_2(j + 1),..., u_n(j + 1)\}$

$> - E(< \{u_{1(j+1)}, u_{2(j+1)},..., u_{n(j+1)}\} >) \cong G_2 \text{ o } K_1$ for all $1 \le j \le m - 1$

and $< \{u_{1m}, u_{2m},..., u_{nm}\} > \cong G_2$.

Thus $E(G_1 \times G_2) = \underbrace{E(G_2 o K_1 \cup ... \cup E(G_2 o K_1)}_{(2m-1)\ times} \cup E(G_2)$.

By assumption, G_2 and G_2 o K_1 are claw decomposable.

Hence $G_1 \times G_2$ is claw decomposable. □

Corollary 3.9: If $G = P_m \text{ o } K_1$ and $n \equiv 0$ (*mod* 3), then $G \times K_n$ is claw decomposable.

Proof: Since $n \equiv 0$ (*mod* 3), by Theorem 2.3, K_n is claw decomposable. Also, by Lemma 2.7, K_n o K_1 is claw decomposable. Hence the result follows from above theorem. □

Remark 3.10: If $G = P_m \text{ o } K_1$, then $G \times K_{1,n}$ is not claw decomposable.

Proof: Suppose not. Then let $S = \{S_1, S_2,..., S_k\}$ be a claw decomposition of $G \times K_{1,n}$. Let $V(G) = \{u_1, u_2,..., u_m, w_1, w_2,...,w_m\}$ where w_i is the pendant edge at u_i for all $1 \le i \le m$ and $u_1u_2....u_m$ is the m-path in G.

Let $V(K_{1,n}) = \{v_0, v_1,...,v_n\}$ where $d(v_0) = n$.

Then $V(G \times K_{1,n}) = \{(u_i, v_j), (w_i, v_j)/1 \le i \le m, 0 \le j \le n\}$.

Rename $(u_i, v_j) = u_{ji}$ and $(w_i, v_j) = w_{ji}$ for all $1 \le i \le m, 0 \le j \le n$.

Now, $w_{11}u_{11} \in E(G \times K_{1,n})$ and hence must be in some member of S, say S_1. Since $d(u_{11}) = 3$ and $d(w_{11}) = 2$, $u_{11}u_{12} \in S_1$. Similarly, $w_{1i}\,u_{1i}$ and $u_{1i}\,u_{1(i+1)}$ will be in the same member of S, say S_i for all $1 \le i \le m - 1$.

Then in $G \times K_{1,n} - U_{i=1}^{n} E(S_i)$, $d(u_{1n}) = 2$ and $d(w_{1n}) = 2$. Thus $w_{1n}\,u_{1n} \notin S$, a contradiction.

Hence $G \times K_{1,n}$ is not claw decomposable. □

Theorem 3.11: If $n \equiv 0$ (*mod* 3), then $P_k \times K_n$ is claw decomposable for all values of k.

Proof: Let $V(K_n) = \{v_1, v_2,..., v_n\}$ and $V(P_k) = \{u_1, u_2,..., u_k\}$ where $P_k = u_1u_2...u_k$.

Then $V(P_k \times K_n) = \{(u_i, v_j)/1 \leq i \leq k, 1 \leq j \leq n\}$.

Rename $(u_i, v_j) = v_{ji}$ for all $1 \leq i \leq k$, $1 \leq j \leq n$.

Assume that $n \equiv 0$ $(mod\ 3)$.

Now, $< \{v_{1j}, v_{2j},..., v_{nj}, v_{1(j+1)}, v_{2(j+1)},..., v_{n(j+1)}\}$

$> - E(< \{v_{1(j+1)}, v_{2(j+1)},..., v_{n(j+1)}\} >) \cong K_n \text{ o } K_1$ for all $1 \leq j \leq k-1$

and $< \{v_{1k}, v_{2k},..., v_{nk}\} > \cong K_n$.

Thus $E(G) = \underbrace{E(K_n o K_1 \cup ... \cup E(K_n o K_1)}_{(k-1)\ times} \cup E(K_n)$.

Since $n \equiv 0$ $(mod\ 3)$, by Lemma 2.7, K_n o K_1 is claw decomposable. Also, by Theorem 2.3, K_n is claw decomposable.

Hence $P_k \times K_n$ is claw decomposable. □

Conjecture 3.12: The graph $P_k \times K_n$ is claw decomposable if and only if $n \equiv 0$ $(mod\ 3)$.

Theorem 3.13: If $n \neq 1(mod\ 3)$, then $C_k \times K_n$ is claw decomposable.

Proof: Let $V(K_n) = \{v_1, v_2,..., v_n\}$ and $V(C_k) = \{u_1, u_2,..., u_k\}$.

Then $V(C_k \times K_n) = \{(u_i, v_j)/1 \leq i \leq k, 1 \leq j \leq n\}$.

Rename $(u_i, v_j) = v_{ji}$ for all $1 \leq i \leq k$, $1 \leq j \leq n$.

Assume that $n \neq 1(mod\ 3)$.

Now, $< \{v_{1i}, v_{2i},...,v_{ni}, v_{1(i+1)}, v_{2(i+1)},..., v_{n(i+1)}\}$

$> -E(< \{v_{1(i+1)}, v_{2(i+1)},..., v_n(i+1)\} >) \cong K_n \text{ o } K_1$ for all $1 \leq i \leq k-1$

and $< \{v_{1k}, v_{2k},..., v_{nk}, v_{11}, v_{21},..., v_{n1}\} > - E(< \{v_{11}, v_{21},..., v_{n1}\} >) \cong K_n \text{ o } K_1$.

Thus $E(G) = \underbrace{E(K_n o K_1 \cup ... \cup E(K_n o K_1)}_{k\ times}$

Since $n \neq 1(mod\ 3)$, by Lemma 2.7, K_n o K_1 is claw decomposable.

Hence $C_k \times K_n$ is claw decomposable. □

Conjecture 3.14: The graph $C_k \times K_n$ is claw decomposable if and only if $n \neq 1(mod\ 3)$.

Theorem 3.15: The graph $K_{1,m} \times K_{1,n}$ is claw decomposable if and only if $2mn + m + n \equiv 0 (mod\ 3)$.

Proof: Let $V(K_{1,m}) = \{u_0, u_1,...,u_m\}$ and $V(K_{1,n}) = \{v_0, v_1,...,v_n\}$ where $d(u_0) = m$ and $d(v_0) = n$.

Then $V(K_{1,m} \times K_{1,n}) = \{(u_i, v_j)/0 \le i \le m, 0 \le j \le n\}$.

Rename $(u_i, v_j) = v_{ji}$ for all $0 \le i \le m$ and $0 \le j \le n$.

Suppose that $2mn + m + n \equiv 0\ (mod\ 3)$.

Case (i): $m \equiv 0\ (mod\ 3)$.

Since $2mn + m + n \equiv 0\ (mod\ 3)$, $n \equiv 0\ (mod\ 3)$. Thus both $K_{1,m}$ and $K_{1,n}$ are claw decomposable. Hence by Theorem 3.1, $K_{1,m} \times K_{1,n}$ is claw decomposable.

Case (ii): $m \equiv 1\ (mod\ 3)$.

Then $2mn + m + n \equiv 1 (mod\ 3)$ for all values of n, a contradiction.

Hence this case does not arise.

Case (iii): $m \equiv 2\ (mod\ 3)$.

If $n \equiv 0\ (mod\ 3)$, then $2mn + m + n \equiv 2(\ mod\ 3)$, a contradiction.

If $n \equiv 1\ (mod\ 3)$, then $2mn + m + n \equiv 1 (mod\ 3)$, a contradiction.

Thus $n \equiv 2\ (mod\ 3)$.

Now, $< \{v_{0j}, v_{1j},...,v_{nj}\} > + \{v_{0j}\, v_{00}\} \cong K_{1,n+1}$ for all $1 \le j \le m$,

$< \{v_{i0}, v_{i1},...,v_{i(m-2)}\} > \cong K_{1,(m-2)}$ for all $1 \le i \le n$ and

$< \{v_{00}, v_{10},...,v_{n0}, v_{1(m-1)}, v_{2(m-1)},...,v_{n(m-1)}, v_{1m}, v_{2m},..., v_{nm}\} > \cong G'$

where G' is the graph obtained by identifying one pendant vertex of each copy of $K_{1,3}$ in $nK_{1,3}$.

Thus $$E(K_{1,m} \times K_{1,n}) = \underbrace{E(K_{1,(n+1)}) \cup ... \cup E(K_{1,(n+1)})}_{m\ times} \cup$$

$$\underbrace{E(K_{1,(m-2)}) \cup ... \cup E(K_{1,(m-2)})}_{m\ times} \cup E(G').$$

Since $n, m \equiv 2(mod\ 3)$, $K_{1,(n+1)}$ and $K_{1,(m-2)}$ areclaw decomposable.

Also, G' is claw decomposable.

Hence $K_{1,m} - K_{1,n}$ is claw decomposable.

Conversely, suppose that $K_{1,m} \times K_{1,n}$ is claw decomposable.

Then $|E(K_{1,m} \times K_{1,n})| \equiv 0 \ (mod\ 3)$.

That is, $(m + 1)\ n + (n + 1)\ m \equiv 0 \ (mod\ 3)$.

That is, $2mn + m + n \equiv 0 \ (mod\ 3)$. □

Remark 3.16. $K_2 \times C_5$ is not claw decomposable.

Theorem 3.17: Let n be even and $n \equiv 0\ (mod\ 3)$ and $m \equiv 1(mod\ 3)$. Then $K_{1,m} \times C_n$ is claw decomposable.

Proof: Let $V(K_{1,m}) = \{u_0, u_1,..., u_m\}$ where $d(u_0) = m$ and

$$V(C_n) = \{v_1, v_2,..., v_n\}.$$

Then $V(K_{1,m} \times C_n) = \{(u_i, v_j)/0 \le i \le m, 1 \le j \le n\}$.

Rename $(u_i, v_j) = v_{ji};\ 0 \le i \le m,\ 1 \le j \le n.$

Assume that n is even, $n \equiv 0\ (mod\ 3)$ and $m \equiv 1(mod\ 3)$.

Claim: $G_2 = K_{1,3} \times C_n - E(C_n)$ where $E(C_n)$ denotes the edges of the cycle C_n corresponding to the central vertex is claw decomposable if n is even and $n \equiv 0(mod\ 3)$.

Then $G' = K_{1,3} \times C_n - \{v_{i0}v_{(i+1)0}, v_{10}v_{n0}/1 \le i \le n \le 1\}$.

Now, $< \{v_{ni}, v_{1i}, v_{2i}, v_{10}\} > \cong K_{1,3}$ for all $1 \le i \le 3$,

$< \{v_{i0}, v_{i1}, v_{i2}, v_{i3}\} >$ <"= $K_{1,3};\ i \in \{2, 4,..., n\}$,

$< \{v_{ij}, v_{(i+1)j}, v_{(i+2)j}, v_{(i+1)0}\} > \cong K_{1,3}$ for all $1 \le j \le 3$ and $i \in \{2, 4,..., n-2\}$.

Thus $E(G') = E(K_{1,3}) \cup E(K_{1,3}) \cup E(K_{1,3}) \cup \underbrace{E(K_{1,3}) \cup ... \cup E(K_{1,3})}_{\left(\frac{n}{2}\right) times} \cup$

$$\underbrace{E(K_{1,3}) \cup ... \cup E(K_{1,3})}_{3\left(\frac{n-2}{2}\right) times}.$$

Hence G' is claw decomposable if n is even and $n \equiv 0\ (mod\ 3)$.

Since $m \equiv 1(mod\ 3)$, $m = 3t + 1;\ t \in Z.$

Thus $E(K_{1,m} \times C_n) = E(K_2 \times C_n) \cup \underbrace{E(G') \cup ... \cup E(G')}_{t\ times}.$

By the Claim and Lemma 2.6, G'^2 and $K_2 \times C_n$ are claw decomposable.

Hence $K_{1,m} \times C_n$ is claw decomposable. □

Theorem 3.18: $K_1, \times K_n$ is claw decomposable if and only if $n \equiv 0$ $(mod\ 3)$ or $mn + m + n \equiv 1(mod\ 3)$.

Proof: Let $V(K_{1,\mathrm{m}}) = \{u_0, u_1, ..., u_m\}$ where $d(u_0) = m$ and $V(K_n) = \{v_1, v_2, ..., v_n\}$.

Then $V(K_1{,}m \times K_n) = \{(u_i, v_j)/0 \le i \le m,\ 1 \le j \le n\}$.

Rename $(u_i, v_j) = v_{ji}$ for all $0 \le i \le m$, $1 \le j \le n$.

Suppose that $n \equiv 0$ $(mod\ 3)$ or $mn + m + n \equiv 1(mod\ 3)$.

Case (i): $n \equiv 0$ $(mod\ 3)$

Subcase 1: $m \equiv 0$ $(mod\ 3)$

Then $K_{1,m}$ is claw decomposable. Also, since $n \equiv 0$ $(mod\ 3)$, by Theorem 2.3, K_n is claw decomposable.

Hence by Theorem 3.1, $K_{1,m} \times K_n$ is claw decomposable.

Subcase 2: $m \equiv 1(mod\ 3)$

Now, < {v1j , v2j , . . . , vnj} > <“= Kn for all 0 d” j d” m “ 1,

< {vi0, vi1, . . . , vi(m”1)} > <“= K1,m”1 for all 1 d” i d” n and

$< \{v_{10}, v_{20}, ..., v_{n0}, v_{1m}, v_{2m}, ..., v_{nm}\} > - E(< \{v_{10}, v_{20}, ..., v_{n0}\} >) \cong K_n \text{ o } K_1$.

Thus $E(K_{1,m} \times K_n) = \underbrace{E(K_n) \cup ... \cup E(K_n)}_{n\ times} \cup$

$$\underbrace{E(K_{1,m-1}) \cup ... \cup E(K_{1,m-1})}_{n\ times} \cup E(K_n o K_1).$$

By Lemma 2.7, K_n o K_1 is claw decomposable. Since $n \equiv 0$ $(mod\ 3)$, by Theorem 2.3, K_n is claw decomposable. Since $m \equiv 1(mod\ 3)$, $K_{1,m-1}$ is claw decomposable.

Hence $K_{1,m} \times K_n$ is claw decomposable.

Subcase 3: $m \equiv 2(mod\ 3)$

Now, $< \{v_{1j}, v_{2j}, ..., v_{nj}\} > \cong K_n$ for all $0 \le j \le m - 2$,

$< \{v_{i0}, v_{i1}, ..., v_{i(m-2)}\} > \cong K_{1,m-2}$ for all $1 \le i \le n$,

$< \{v_{10}, v_{20},\ldots, v_{n0}, v_{1(m-1)}, v_{2(m-1)},\ldots, v_{n(m-1)}\}$

$> - E(< \{v_{10}, v_{20},\ldots, v_{n0}\} >) \cong K_n \text{ o } K_1$ and

$< \{v_{10}, v_{20},\ldots, v_{n0}, v_{1m}, v_{2m},\ldots, v_{nm}\} > - E(< \{v_{10}, v_{20},\ldots,v_{n0}\} >) \cong K_n \text{ o } K_1$.

Thus $E(K_{1,m} \times K_n) = \underbrace{E(K_n) \cup \ldots \cup E(K_n)}_{(m-1)\ times} \cup$

$$\underbrace{E(K_{1,m-2}) \cup \ldots \cup E(K_{1,m-2})}_{(m-1)\ times} \cup E(K_n o K_1) \cup E(K_n o K_1).$$

Since $n \equiv 0\ (mod\ 3)$, by Theorem 2.3, K_n is claw decomposable. Since $m \equiv 2(mod\ 3)$, $K_{1,m-2}$ is claw decomposable. Also by Lemma 2.7, K_n o K_1 is claw decomposable.

Hence $K_{1,m} \times K_n$ is claw decomposable.

Case (ii): $mn + m + n \equiv 1(mod\ 3)$

Subcase 1: $m \equiv 0\ (mod\ 3)$

Since $mn + m + n \equiv 1(mod\ 3)$, $n \equiv 1(mod\ 3)$. Thus by Theorem 2.3, K_n is claw decomposable. Also, $K_{1,m}$ is claw decomposable. Hence by Theorem 3.1, $K_{1,m} \times K_n$ is claw decomposable.

Subcase 2: $m \equiv 1(mod\ 3)$

Since $mn + m + n \equiv 1(mod\ 3)$, $n \equiv 0\ (mod\ 3)$. This case is already dealt in Subcase 2 of Case *(i)*.

Subcase 3: $m \equiv 2(mod\ 3)$

If $m \equiv 2(mod\ 3)$, then $mn + m + n \equiv 2(mod\ 3)$ for all values of n, a contradiction. Hence this case does not arise.

Hence in all the cases, $K_{1,m} \times K_n$ is claw decomposable.

Conversely, suppose that $K_{1,m} \times K_n$ is claw decomposable.

Then $|E(K_{1,m} \times K_n)| \equiv 0\ (mod\ 3)$. Thus, $(m+1)\dfrac{n(n-1)}{2} + mn \equiv 0\ (mod\ 3)$. which implies $\dfrac{n}{2}[mn + m + n - 1] \equiv 0\ (mod\ 3)$ and hence $n \equiv 0\ (mod\ 3)$ or $mn + m + n \equiv 1(mod\ 3)$.

4. CLAW DECOMPOSITION OF LEXICOGRAPHIC PRODUCT OF GRAPHS

In this section, we give sufficient condition for the lexicographic product of any graph G with $\overline{K}_n$, K_n, $K_{m,n}$ and $K_2 \times K_n$ to be claw decomposable.

Definition 4.1: The lexicographic product of two graphs G and H is a graph, denoted by $G * H$, whose vertex set is $V(G) \times V(H)$. Two vertices (g, h) and (g', h') are adjacent precisely if $gg' \in E(G)$, or $g = g'$ and $hh' \in E(H)$.

The other way of viewing $G * H$ is by replacing each vertex in G by a copy of H and two vertices in G are adjacent if and only if there exists a complete bipartite subgraph with the corresponding vertices of H as partite sets in $G * H$.

Theorem 4.2: Let G be any non trivial graph. If $n \equiv 0$ $(mod\ 3)$, then $G * \overline{K}_n$ is claw decomposable.

Proof: Assume that n a" 0(mod 3).

Let $V(G) = \{v_1, v_2,...,v_k\}$ and $V(\overline{K}_n) = \{u_1, u_2,..., u_n\}$.

Then $V(G * \overline{K}_n) = \{(v_i, u_j)/1 \leq i \leq k$ and $1 \leq j \leq n\}$.

Rename $(v_i, u_j) = v_{ji}$; $1 \leq i \leq k$ and $1 \leq j \leq n$.

Now, for each $v_iv_j \in E(G)$, $< \{v_{1i}, v_{2i},..., v_{ni}, v_{1j}, v_{2j},...,v_{nj}\} > \cong K_{n,n}$.

Thus, $E(G * \overline{K}_n) = \underbrace{E(K_{n,n}) \cup ... \cup E(K_{n,n})}_{|E(G)|\ times}$.

Since $n \equiv 0$ $(mod\ 3)$, by Theorem 2.4, $K_{n,n}$ is claw decomposable.

Hence $G * \overline{K}_n$ is claw decomposable. □

Theorem 4.3: Let G be any non trivial graph. If $n > 3$ and $n \equiv 0$ $(mod\ 3)$, then $G * K_n$ is claw decomposable.

Proof: Assume that $n > 3$ and $n \equiv 0$ $(mod\ 3)$.

Let $V(G) = \{v_1, v_2,..., v_k\}$ and $V(K_n) = \{u_1, u_2,..., u_n\}$.

Then $V(G * K_n) = \{(v_i, u_j)/1 \leq i \leq k$ and $1 \leq j \leq n\}$.

Rename $(v_i, u_j) = v_{ji}$; $1 \leq i \leq k$ and $1 \leq j \leq n$.

Now, $< \{v_{1i}, v_{2i},..., v_{ni}\} > \cong K_n$ for all $1 \leq i \leq k$.

Also, for each $v_iv_j \in E(G)$,

$< \{v_{1i}, v_{2i}, \ldots, v_{ni}, v_{1j}, v_{2j}, \ldots, v_{nj}\} > -E(< \{v_{1i}, v_{2i}, \ldots, v_{ni}\} >)$

$- E(< \{v_{1j}, v_{2j}, \ldots, v_{nj}\} >) \cong K_{n,n}.$

Thus, $E(G * K_n) = \underbrace{E(K_n) \cup \ldots \cup E(K_n)}_{k\ times} \cup \underbrace{E(K_{n,n}) \cup \ldots \cup E(K_{n,n})}_{|E(G)|\ times}.$

Since $n \equiv 0 (mod\ 3)$, by Theorem 2.3 and 2.4, K_n and $K_{n,n}$ are claw decomposable.

Hence $G * K_n$ is claw decomposable. □

Theorem 4.4: Let G be any non trivial graph. If $m \equiv 0\ (mod\ 3)$ and $n \equiv 0\ (mod\ 3)$, then $G * K_{m,n}$ is claw decomposable.

Proof: Assume that $m \equiv 0\ (mod\ 3)$ and $n \equiv 0\ (mod\ 3)$.

Let $V(G) = \{v_1, v_2, \ldots, v_k\}$ and $V(K_{m,n}) = \{u_1, u_2, \ldots, u_m, w_1, w_2, \ldots, w_n\}$ where $d(u_i) = n$ for all $1 \le i \le m$ and $d(w_j) = m$ for all $1 \le j \le n$.

Then $V(G \times K_{m,n}) = \{(v_i, u_j), (v_i, w_l)/1 \le i \le k, 1 \le j \le m, 1 \le 1 \le n\}$.

Rename $(v_i, u_j) = u_{ji}$ and $(v_i, w_l) = w_{li}$ for all $1 \le i \le k$, $1 \le j \le m$, $1 \le 1 \le n$.

Now for each $v_iv_j \in E(G)$,

$< \{u_{1i}, u_{2i}, \ldots, u_{mi}, w_{1i}, w_{2i}, \ldots, w_{ni}, u_{1j}, u_{2j}, \ldots, u_{mj}, w_{1j}, w_{2j}, \ldots, w_{nj}\}$

$> - E(< \{u_{1i}, u_{2i}, \ldots, u_{mi}, w_{1i}, w_{2i}, \ldots, w_{ni}\} >)$

$- E(< \{u_{1j}, u_{2j}, \ldots, u_{mj}, w_{1j}, w_{2j}, \ldots, w_{nj}\} >) \cong Km + n, m + n$ and

$< \{u_{1i}, u_{2i}, \ldots, u_{mi}, w_{1i}, w_{2i}, \ldots, w_{ni}\} > \cong K_{m,n}$ for all $1 \le i \le k$.

Thus, $E(G * K_{m,n}) = \underbrace{E(K_{m,n}) \cup \ldots \cup E(K_{m,n})}_{k\ times} \cup \underbrace{E(K_{m+n,m+n}) \cup \ldots \cup E(K_{m+n,m+n})}_{|E(G)|\ times}$

Since $m \equiv 0 (mod\ 3)$ and $n \equiv 0\ (mod\ 3)$, by Theorem 2.4, $K_{m,n}$ and $K_{m+n,m+n}$ are claw decomposable.

Hence $G * K_{m,n}$ is claw decomposable.

Theorem 4.5: Let G be any non trivial graph. If $n > 3$ and $n \equiv 0\ (mod\ 3)$, then $G * [K_2 \times K_n]$ is claw decomposable.

Proof: Assume that $n > 3$ and $n \equiv 0\ (mod\ 3)$.

Let $V(G) = \{w_1, w_2, \ldots, w_k\}$, $V(K_2 \times K_n) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$ and $E(K_2 \times K_n) = \{v_iv_j, u_iu_j, u_iv_i/1 \le i, j \le n, i\ 6 = j\}$.

Then $V(G * [K_2 \times K_n]) = \{(w_i, v_j), (w_i, u_j)/1 \le i \le k, 1 \le j \le n\}$.

Rename $(w_i, v_j) = v_{ji}$ and $(w_i, u_j) = u_{ji}$ for all 1 ≤ *i d" k*, $1 \le j \le n$.

Now, $< \{v_{1i}, v_{2i},..., v_{ni}, u_{1i}, u_{2i},..., u_{ni}\} > \cong K_2 \times K_n$ for all $1 \le i \le k$.

Also, for each $w_i w_j \in E(G)$,

$$< \{v_{1i}, v_{2i},..., v_{ni}, u_{1i}, u_{2i},..., u_{ni}, v_{1j}, v_{2j},..., v_{nj}, u_{1j}, u_{2j},..., u_{nj}\} > - E(< \{v_{1i}, v_{2i},...., v_{ni}, u_{1i}, u_{2i},...., u_{ni}\} >) - E(< \{v_{1j}, v_{2j},...,v_{nj}, u_{1j}, u_{2j},..., u_{nj}\} >) \cong K_{2n,2n}.$$

Thus $E(G * [K_2 \times K_n]) = \underbrace{E(K_2 \times K_n) \cup ... \cup E(K_2 \times K_n)}_{k\ times} \cup \underbrace{E(K_{2n,2n}) \cup ... \cup E(K_{2n,2n})}_{|E(G)|\ times}$.

Since $n > 3$ and $n \equiv 0\ (mod\ 3)$, by Lemma 2.8, $K_2 \times K_n$ is claw decomposable. Also, by Theorem 2.4, $K_{2n,2n}$ is claw decomposable.

Hence $G * [K_2 \times K_n]$ is claw decomposable. □

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