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ABSOLUTE BANACH SUMMABILITY OF FOURIER SERIES

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Abstract

A result on Banach Summability is established

1. DEFINITION

Let $\{s_n\}$ be the sequence of partial sums of the series $\sum u_n$. Then the sequence $\{t_k(n)\}_{k=1}^{\infty}$ defined by

$$t_k(n) = \frac{1}{k}\sum_{\nu=0}^{k-1} s_{n+\nu}\ ,\ k \in N \tag{1.1}$$

is said to be the k-th element of the Banach transformed sequence. If

$$\lim_{k\to\infty} t_k(n) = s, \text{ a finite number,} \tag{1.2}$$

uniformly for all n $\in$ N, then $\sum u_n$ is said to be Banach summable to s [1].

Further, if

$$\sum_{k=1}^{\infty}\left|t_k(n) - t_{k+1}(n)\right| < \infty, \tag{1.3}$$

uniformly for all n $\in$ N, then the series $\sum u_n$ is said to be absolutely Banach summable or simply $|B|$-summable.

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2. INTRODUCTION

Let $\sum_{n=0}^{\infty} A_n(x)$ be the Fourier series of a 2π-periodic function f(t) which is L-integrable on (–π, π). Then

$$A_n(x) = \frac{2}{\pi}\int_0^{\pi} \phi(t)\cos nt\ dt, \text{ n = 0,1,2,….} \tag{2.1}$$

Dealing with Cesàro summability Bosanquet[2] established the following theorem:

Theorem. A: If $\phi(t) \in BV(0\ ,\ \pi)$, then the Fourier series of f(t)is summable $|C\ ,\ \delta|$ at the point t = x for $\delta > 0$.

Later, in 1961, Pati [4] showed that $\phi(t)\ \log\frac{k}{t} \in BV$ does not ensure absolute harmonic summability of Fourier series. He proved

Theorem.B: There exists a function f(t) of class-L such that $\phi(t)\ \log\frac{k}{t}$ is a function of bounded variation, but its Fourier series, at t = x, is not summable $\left|N\ ,\ \frac{1}{n+1}\right|$.

In 1997, Misra and Misra [3] proved the following theorem.

Theorem. C: If $\phi(t) \in BV(0\ ,\ \pi)$, then the Fourier series $\sum A_n(x)$ of f(t) is $|B|$-summable.

In the present paper we prove an analogue theorem for $|B|$-summability of Fourier series.

3. MAIN RESULT

Theorem. If $\phi(t)\ \log\frac{k}{t} \in BV(0\ ,\pi)$, then the Fourier series $\sum A_n(x)$ of *f*(*t*) is $|B|$-summables.

4. REQUIRED LEMMAS

***Lemma-1[3]*:** The series $\sum u_n$ is $|B|$-summable if and only if

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} \left| \sum_{\nu=1}^{k} \nu \, u_{n+\nu} \right| < \infty, \text{ uniformaly for all } n \in N.$$

***Lemma-2[5]*:** $$\int_0^t \frac{\cos nu}{log \frac{k}{u}} \, du = \left(\log \frac{k}{t} \right)^{-1} + 0 \left(\frac{1}{n(\log n)^2} \right).$$

5. PROOF OF THE THEOREM

For the series $\sum A_n(x)$, by Lemma-1, we have

$$\sum_{r=1}^{\infty} \left| t_r(n) - t_{r+1}(n) \right| = \sum_{r=1}^{\infty} \frac{1}{r(r+1)} \left| \sum_{\nu=1}^{r} \nu \, A_{n+\nu}(x) \right|$$

$$= \frac{2}{\pi} \sum_{r=1}^{\infty} \frac{1}{r(r+1)} \left| \sum_{\nu=1}^{r} \nu \int_0^{\pi} \phi(t) \cos(n+\nu) t \, dt \right|$$

Now

$$\int_0^{\pi} \phi(t) \cos(n+\nu) t \, dt = \int_0^{\pi} h(t) \frac{\cos(n+\nu)t}{\log \frac{k}{t}} \, dt,$$

where $h(t) = \phi(t) \, log \frac{k}{t}$

$$= \left[h(t) \int_0^t \frac{\cos(n+\nu)u}{\log \frac{k}{u}} \, du \right]_o^{\pi} - \int_0^{\pi} \left\{ dh(t) \int_0^t \frac{\cos(n+\nu)}{\log \frac{k}{u}} \cdot du \right\}$$

$$= 0\left(\frac{1}{(n+\nu)\left(\log(n+\nu)\right)^2}\right)$$

$$-\int_0^\pi dh(t)\left[\left(\log\frac{k}{t}\right)^{-1}\frac{\sin(n+\nu)t}{n+\nu}+0\left(\frac{1}{(n+\nu)\left(\log(n+\nu)\right)^2}\right)\right]$$

$$= 0\left(\frac{1}{(n+\nu)\left(\log(n+\nu)\right)^2}\right)-\int_0^\pi dh(t)\left\{\left(\log\frac{k}{t}\right)^{-1}\frac{\sin(n+\nu)t}{n+\nu}\right\}$$

Thus

$$\sum_{r=1}^{\infty}\left|t_r(n)-t_{r+1}(n)\right| = \frac{2}{\pi}\sum_{r=1}^{\infty}\frac{1}{r(r+1)}\left|\sum_{\nu=1}^{r}\nu\left[0\left(\frac{1}{(n+\nu)\left(\log(n+\nu)\right)^2}\right)\right.\right.$$

$$\left.\left.-\int_0^\pi dh(t)\left\{\left(\log\frac{k}{t}\right)^{-1}\frac{\sin(n+\nu)t}{n+\nu}\right\}\right]\right|$$

$$\leq \frac{A}{\pi}\left[\sum_{r=1}^{\infty}\frac{1}{r(r+1)}\left|\sum_{\nu=1}^{r}\frac{\nu}{(n+\nu)\left(\log(n+\nu)\right)^2}\right|\right.$$

$$\left.+\sum_{r=1}^{\infty}\frac{1}{r(r+1)}\left|\sum_{\nu=1}^{r}\int_0^\pi dh(t)\left\{\left(\log\frac{k}{t}\right)^{-1}\frac{\sin(n+\nu)t}{n+\nu}\right\}\right|\right]$$

$$= \frac{A}{\pi}\left[S_1+S_2\right], \text{ say.}$$

Now

$$S_1 = \sum_{r=1}^{\infty}\frac{1}{r(r+1)}\sum_{\nu=1}^{r}\frac{\nu}{(n+\nu)\left(\log(n+\nu)\right)^2}$$

$$= \sum_{\nu=1}^{\infty}\frac{\nu}{(n+\nu)\left(\log(n+\nu)\right)^2}\sum_{r=\nu}^{\infty}\frac{1}{r(r+1)}$$

$$= \sum_{\nu=1}^{\infty} \frac{\nu}{(n+\nu)\left(\log(n+\nu)\right)^2} . 0\left(\frac{1}{\nu}\right)$$

$$= 0(1) \sum_{\nu=1}^{\infty} \frac{1}{(n+\nu)(\log(n+\nu))^2}$$

$< \infty$, uniformly in n.

Next

$$S_2 = \sum_{r=1}^{\infty} \frac{1}{r(r+1)} \left| \sum_{\nu=1}^{r} \nu \int_0^{\pi} dh(t) \left(\log \frac{k}{t}\right)^{-1} \frac{\sin(n+\nu)t}{n+\nu} \right|$$

$$= \sum_{r=1}^{\infty} \frac{1}{r(r+1)} \left| \sum_{\nu=1}^{r} \nu \left(\log \frac{k}{t}\right)^{-1} \frac{\sin(n+\nu)t}{n+\nu} \right|,$$

as $h(t) \in B.V(0, \pi)$

$$= \left(\sum_{r=1}^{\tau} + \sum_{r>\tau}^{\infty} \right) \frac{1}{r(r+1)} \left| \sum_{\nu=1}^{r} \nu \log\left(\frac{k}{t}\right)^{-1} \frac{\sin(n+\nu)\ t}{n+\nu} \right|,$$

where $\tau = \left[\frac{1}{t}\right]$

$= S_{21} + S_{22}$, say.

Now

$$S_{21} = \sum_{\nu=1}^{\tau} \frac{1}{r(r+1)} \left| \sum_{\nu=1}^{r} \nu \left(\log \frac{k}{t}\right)^{-1} \frac{\sin(n+\nu)t}{n+\nu} \right|$$

$$= (\log \tau)^{-1} \sum_{r=1}^{\tau} \frac{1}{r(r+1)} \sum_{\nu=1}^{r} \left(\frac{\nu}{n+\nu}\right)$$

$$= (\log \tau)^{-1} \log \tau$$

$$= 0(1)$$

Finally,

$$S_{22} = \sum_{r > \tau} \frac{1}{r(r+1)} \left| \sum_{\nu=1}^{r} \nu \left(\log \frac{k}{t} \right)^{-1} \frac{\sin(n+\nu)t}{(n+\nu)} \right|$$

$$= (\log \tau)^{-1} \sum_{r > \tau} \frac{1}{r(r+1)} \left| \sum_{\nu=1}^{r} \left(\frac{\nu}{n+\nu} \right) \sin(n+\nu)t \right|$$

$$= (\log \tau)^{-1} \sum_{r > \tau} \frac{1}{r(r+1)} \left(\frac{\nu}{n+\nu} \right) \left| \sum_{\nu=1}^{r} \sin(n+\nu)t \right|$$

$$= (\log \tau)^{-1} \sum_{r > \tau} \frac{1}{(r+1)(n+\nu)} \; 0(\tau)$$

$$= 0(\tau)(\log \tau)^{-1} \sum_{r > \tau} \frac{1}{(r+1)(n+\nu)}$$

$$= 0(\tau)(\log \tau)^{-1} \; 0(\tau)^{-1}$$

$$= 0(1)$$

Thus

$$\sum_{r=1}^{\infty} \left| t_r(n) - t_{r+1}(n) \right| < \infty,$$

uniformly in n.

This proves the theorem.

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