IJMS, Vol. 15, Nos. 1-2, (January-June 2016), pp. 9-15

ISSN: 0972-754X

PROPER ROUGH FUZZY SETS

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Abstract

Under rough fuzzy approximation, the boundary of the given fuzzy set represents the region of ambiguity. In general, an Information system is updated based on several reasons. Clearly, it gives changes and modifications in the existing knowledge base. In this paper, a mathematical model is derived to reduce the region of ambiguity by considering other knowledge base, which is a refinement of the existing one. The resultant rough fuzzy set is called the *proper rough fuzzy set.*

1. INTRODUCTION

In 1982, the theory of rough sets [10] was introduced by Pawlak. This theory provides several applications in data mining; bioinformatics etc. In 1990, by hybridizing fuzzy and rough sets, Dubois and Prade developed the theory of rough fuzzy and fuzzy rough sets. This paper concentrates on rough fuzzy sets. In the theories of rough sets and rough fuzzy sets, the universe of discourse is partitioned into granules. In rough fuzzy sets, for each granule, the lower and upper membership values are defined. The interval estimate of membership values between the lower and upper membership values gives the ambiguity. In this paper, we derive a model to reduce the ambiguity by considering a refinement of the existing partition, which satisfies the rules given in section 4.

First, we shall discuss the basic definitions and properties of rough fuzzy sets.

2. ROUGH FUZZY SETS

In this section, we begin with the relationship between the topological spaces obtained from equivalence classes and Boolean algebra [8].

Let $U=\{x_1, x_2,\ldots, x_n\}$ be the universe of discourse. Let $Z = \{X_1, X_2,\ldots, X_t\}$ be the partition of U. Here, the elements of Z are called as granules. The topology obtained by taking Z as the open base is called Pawlak Topology.

Keywords: Rough fuzzy approximation, boundary, rough fuzzy set, proper rough fuzzy set, information system, knowledge base.

Now, we prove that Pawlak topology is the smallest Boolean algebra or Algebra of Sets.

Theorem 2.1: Let Z be a partition of a finite set U. If T(Z) is the Pawlak Topology defined on U, then T (Z) is the unique smallest Boolean Algebra containing Z.

Proof: Let M = {T/ T is the Boolean algebra containing Z}. As the power set of U, P(U)$\in$M, M is nonempty. Define $\Im = \cap T$ for all T$\in$M.

Let A,B$\in\Im$, then A,B$\in$T for all T$\in$M. As T is the Boolean algebra, A$\cup$B$\in$T and $A^c \in T$ for all T$\in$M. Therefore, A$\cup$B$\in\Im$ and $A^c\in\Im$. Hence, $\Im$ is a Boolean algebra. Moreover, as each T$\in$M contains Z, $\Im$ contains Z.

Let $\Re$ be the smallest Boolean algebra containing Z $\Rightarrow \Re \subseteq \Im$. But, as $\Re$ is the Boolean algebra containing Z, $\Re \in$M. Therefore, $\Im \subseteq \Re$. Hence, $\Im = \Re$, which shows the uniqueness.

Now, by the choice of $\Im$, $\Im \subseteq$T(Z). Suppose that R$\in$T(Z), then R is the union of the elements of Z $\Rightarrow$ R lies in every Boolean algebra containing Z $\Rightarrow$ R$\in\Im$. Therefore, T(Z) is the smallest Boolean algebra containing Z.

As the topology T(Z) is induced by some indiscernibility relation R, we denote T(Z) by $\sigma\left(U/R\right)$. Now, we discuss the definition and properties of rough fuzzy sets. In 1990, Dubois and Prade introduced rough fuzzy and fuzzy rough sets, which found wide applications in decision-making in fuzzy environment, which is same as that of rough sets in crisp environment. This paper discusses rough fuzzy sets. It is defined as follows:

For any fuzzy subset F of U with μ_F as membership function, the lower and upper rough approximations of F are defined as

Let U={x_1, x_2,…,x_n} be the universe of discourse. Let Z={X_1, X_2,…., X_t} be the partition of U. Here, the elements of Z are called as granules. Then for any fuzzy subset F of U with μ_F as membership function, the lower and upper rough approximations of F are defined as

$$\mu_{\underline{F}}(X_i) = \inf_{x_j}\{\mu_F(x_j) / x_j \in X_i\} \text{ and}$$

$$\mu_{\overline{F}}(X_i) = \sup_{x_j}\{\mu_F(x_j) / x_j \in X_i\} \text{ respectively.}$$

The lower approximation of F is given by $\underline{F} = (\mu_{\underline{F}}(X_1), \mu_{\underline{F}}(X_2), \ldots, \mu_{\underline{F}}(X_t))$ and the upper approximation is given by $\overline{F} = (\mu_{\overline{F}}(X_1), \mu_{\overline{F}}(X_2), \ldots \mu_{\overline{F}}(X_t))$

Here $(\underline{F},\overline{F})$ is called a rough fuzzy set [1,2,5]. It can be illustrated by the following example.

Example 2.2: Consider the universe of discourse U={a,b,c,d,e,f} with the partition Z={X_1, X_2, X_3} where X_1={a,c,e}, X_2={b,f} and X_3={d}. Consider the fuzzy subset F = (0.2,0.4,0.3,0.6,0.2,0.7) of U. Then $\mu_{\underline{F}}(X_1)$ = min {0.2,0.3,0.2}= 0.2; $\mu_{\underline{F}}(X_2)$ = min {0.4, 0.7}=0.4 and $\mu_{\underline{F}}(X_3)$=min {0.6}=0.6, which gives $\underline{F}$ =(0.2,0.4,0.6). Similarly, $\mu_{\overline{F}}(X_1)$ = max {0.2,0.3,0.2}=0.3; $\mu_{\overline{F}}(X_2)$=max {0.4, 0.7}=0.7 and $\mu_{\overline{F}}(X_3)$= max {0.6} = 0.6, which gives $\overline{F}$ = (0.3,0.7,0.6).

From the definition, some of the properties of rough fuzzy sets can be observed, which are listed below.

Properties 2.3: (a) $\underline{F}\subseteq\overline{F}$ (b) $\overline{F\cup G}=\overline{F}\cup\overline{G}$ (c) $\overline{F\cap G}\subseteq\overline{F}\cap\overline{G}$

(d) $\underline{F}\cup\underline{G}\subseteq\underline{F\cup G}$ (e) $\underline{F}\cup\underline{G}=\underline{F\cup G}$ (f) $[\overline{F}]^c=\underline{F^c}$

(g) $[\overline{F}]^c=\overline{F^c}$

Now, we use a threshold as defined below for the construction of proper rough fuzzy sets.

2.4 Threshold on Fuzzy Set

Any fuzzy set A of finite universe of discourse U, [9] can be expressed in terms of a membership function. A can be represented as a set (($\mu_A(x_1)$, $\mu_A(x_2)$,…, $\mu_A(x_n)$)).

For any two fuzzy sets A and B, the union and intersection [3,4,7] of them can be obtained by using the max and min operators say t-conorms and t-norms.

$\mu_{A\cup B}(x_i)$ = max ($\mu_A(x_i)$, $\mu_B(x_i)$) and
$\mu_{A\cap B}(x_i)$ = min ($\mu_A(x_i)$, $\mu_B(x_i)$) respectively.

Additionally, the strong α-cut, inclusion and complement of fuzzy sets are defined as follows:

(a) A is said to be a subset of B if and only if $\mu_A(x_i) \le \mu_B(x_i)\ \forall\ x_i \in U$.

(b) For any fuzzy set A, the complement of A is given by its grade of membership of each element of U; i.e., $\mu_{A^c}(x_i) = 1-\mu_A(x_i)\ \forall\ x_i \in U$.

(c) For a given aÎ(0,1), the strong a-cut of a fuzzy set A is defined as {x∈U/ $\mu_A(x)>\alpha$} and is denoted by A[α]

Now, it is necessary to discuss the possible values of thresholds taken in (0,1). Here, the set D is constructed as follows, which is the domain of thresholds.

Consider a set D, called domain, satisfying the following properties:

(a) $D \subset (0,1)$

(b) If a fuzzy set A is under computation, eliminate the values $\mu_A(x)$ and $\mu_A{}^c$ $\forall$ $x \in U$ from the domain D, if they exist.

(c) After the computation using A, the values removed in (b) may be included in D provided A must not involve in further computation

Consider the universe of discourse $U = (x_1, x_2, \ldots, x_n\}$. Let α, α_1, α_2, β be the thresholds assume one of the values from the domain D where D is constructed using the fuzzy sets A and B.

3. ROUGH SET APPROACH ON A FUZZY SET WITH THRESHOLD

Let Z be any partition of U, say $\{X_1, X_2, \ldots, X_t\}$. For the given fuzzy set A, the lower and upper approximations with respect to α can be defined as $A_\alpha = \underline{A[\alpha]}$ and $A^\alpha = \overline{A[\alpha]}$ respectively.

Here, by using the properties of rough sets, the following propositions can be obtained [6].

1. $(A \cup B)^\alpha = A^\alpha \cup B^\alpha$ 2. $(A \cap B)_\alpha = A_\alpha \cap B_\alpha$ 3. $(A \cup B)_\alpha \supseteq A_\alpha \cup B_\alpha$
4. $(A \cap B)^\alpha \subseteq A^\alpha \cap B^\alpha$ 5. $(A^c)^\alpha = (A_{1-\alpha})^c$ 6. $(A^c)_\alpha = (A^{1-\alpha})^c$

These properties can be illustrated by the following example.

Example 3.1: Consider the universe of discourse U={a,b,c,d,e,f} with the partition Z = {{a,b}, {c}, {d,f}, {e}} given by the equivalence relation R. Let A=(0.2,0.4,0.3,0.5,0.7,0) and B=(0.6,0.8,1,0.4,0.6,0.6). Then D=(0,1)-{0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8}. Let $\alpha \in D$, say, $\alpha = 0.45$

Then $A[\alpha]$={d,e} and B[a]={a,b,c,e,f}. Hence A_α={e}; A^α ={d,e,f}; B_α = {a,b,c,e} and B^α = U.

$A \cup B$=(0.6, 0.8, 1, 0.5, 0.7, 0.6); $(A \cup B)[\alpha]$=U. Hence $(A \cup B)_\alpha = (A \cup B)^\alpha$ = U. But, $A_\alpha \cup B_\alpha$ = {a, b, c, e} and $A^\alpha \cup B^\alpha$ = U. Hence, $(A \cup B)^\alpha = A^\alpha \cup B^\alpha$ and $(A \cup B)_\alpha \supseteq A_\alpha \cup B_\alpha$ $A \cap B$ = (0.2, 0.4, 0.3, 0.4, 0.6, 0); $(A \cap B)[\alpha]$={e}. Hence $(A \cap B)_\alpha = (A \cap B)^\alpha$ = {e}. But, $A_\alpha \cap B_\alpha$ = {e} and $A^\alpha \cap B^\alpha$={d,e,f}. Hence, $(A \cap B)_\alpha = A_\alpha \cap B_\alpha$ and $(A \cap B)^\alpha \subseteq A^\alpha \cap B^\alpha$ A^c = (0.8,0.6,0.7,0.5,0.3,1). $A^c[\alpha] = A^c[0.45]$ = {a,b,c,d,f}. Hence $(A^c)_\alpha = (A^c)^\alpha$

= {a,b,c,d,f}. Now, A[1–α]={e}. Hence, $A_{1-\alpha} = A^{1-\alpha} = \{e\} \Rightarrow (A_{1-\alpha})^c = (A^{1-\alpha})^c$ ={a,b,c,d,f}. Therefore, $(A^c)^\alpha = (A_{1-\alpha})^c$ and $(A^c)_\alpha = (A^{1-\alpha})^c$

4. PROPER ROUGH APPROXIMATION OF A FUZZY SET

In general, it is observed that $A_\alpha \subseteq A[\alpha] \subseteq A^\alpha$. The boundary region is ambiguous. In this section, a tool is derived to reduce the ambiguity.

Theorem 4.1: For any two fuzzy sets A and B with the given threshold α, whenever $A \subset B$, any one the following results follows:

(a) A^α and B_α are not comparable

(b) $A^\alpha = B^\alpha$ and $A_\alpha = B_\alpha$

(c) $A^\alpha \subseteq B_\alpha$

Proof: If A^α and B_α are not comparable or $A^\alpha \subseteq B_\alpha$ then there is nothing to prove. Suppose that $A^\alpha \supset B_\alpha$. Then the proof can be classified into four cases.

Case 1: If $A[\alpha] \subseteq B_\alpha \subset A^\alpha \subseteq B[\alpha]$, then B_α becomes the upper approximation of A, which leads the contradiction.

Case 2: If $A[\alpha] \subseteq B_\alpha \subseteq B[\alpha] \subseteq A^\alpha$ and $B_\alpha \subset A^\alpha$, then B_α becomes the upper approximation of A, which leads the contradiction.

Case 3: If $B_\alpha \subseteq A[\alpha] \subseteq A^\alpha \subseteq B[\alpha]$ and $B_\alpha \subset A^\alpha$, then A^α becomes the lower approximation of B, which leads the contradiction.

Case 4: If $B_\alpha \subseteq A[\alpha] \subset B[\alpha] \subseteq A^\alpha$ then as $A \subset B$, $A^\alpha \subseteq B^\alpha$ and $A_\alpha \subseteq B_\alpha$. Hence A^α becomes the upper approximation of B, which forces $A^\alpha = B^\alpha$ and $A_\alpha = B_\alpha$.

Consider any algebra of fuzzy subsets Σ which contains all the elements of X. Clearly, $\sigma\left(U/R\right)$ is contained in Σ.

Definition 4.2: A fuzzy set F in Σ is said to be α_x essential with respect to U/R if

(a) $F \subset C$ for any $C \in \Sigma$ $\Rightarrow$ $F^\alpha \subseteq C_\alpha$ or $F^{1-\alpha} \subseteq C_{1-\alpha}$

(b) $C \subset F$ for any $C \in \Sigma$ $\Rightarrow$ $C^\alpha \subseteq F_\alpha$ or $C^{1-\alpha} \subseteq F_{1-\alpha}$

Theorem 4.3: If A is α_x essential with respect to U/R then A^c is also roughly essential with respect to U/R

Proof: Let $A^c \subset F$ where $F \in \Sigma$. As Σ is the algebra, $F^c \subset A$. Since A is α_x essential with respect to U/R, $(F^c)^\alpha \subseteq A_\alpha$. But $(F^c)^\alpha = (F_{1-\alpha})^c$. Therefore, $(F_{1-\alpha})^c \subseteq A_\alpha$. Hence, $(A_\alpha)^c \subseteq F_{1-\alpha.}$ Since, $(A_\alpha)^c = (A^c)^{1-\alpha}$, we have $(A^c)^{1-\alpha} \subseteq F_{1-\alpha}$ ...(4.3.1)

Let $F \subset A^c \Rightarrow A \subset F^c \Rightarrow A^\alpha \subseteq (F^c)_\alpha \Rightarrow A^\alpha \subseteq (F^{1-\alpha})^c$

$\Rightarrow F^{1-\alpha} \subseteq (A^\alpha)^c \Rightarrow F^{1-\alpha} \subseteq (A^c)_{1-\alpha}$ ...(4.3.2)

From (4.3.1) and (4.3.2), A^c is α_x essential with respect to U/R

Theorem 4.4: If A and B are α_x essential with respect to U/R then $A \cup B$ is α_x essential with respect to U/R.

Proof: Let $A \cup B \subset F$. Then $A \subset F$ and $B \subset F$, which implies $A^\alpha \subseteq F_\alpha$; $B^\alpha \subseteq F_\alpha \Rightarrow A^\alpha \cup B^\alpha \subseteq F_\alpha \Rightarrow (A \cup B)^\alpha \subseteq F_\alpha$ …(4.4.1)

Now, suppose that $F \subset A \cup B$. Then F can be written as $F=[F \cap A] \cup [F \cap B]$. Let $F_A = F \cap A$ and $F_B = F \cap B$, which give $F_A \subset A$ and $F_B \subset B$. As A and B are α_x essential, $(F_A)^\alpha \subseteq A_\alpha$ and $(F_B)^\alpha \subseteq B_\alpha$. Hence, $(F_A)^\alpha \cup (F_B)^\alpha \subseteq A_\alpha \cup B_\alpha$. Hence, $(F_A \cup F_B)^\alpha \subseteq (A \cup B)_\alpha$. Therefore, $F^\alpha \subseteq (A \cup B)_\alpha$. ...(4.4.2)

Hence, from 4.4.1 and 4.4.2, it can be seen that the union of two essential sets with respect to U/R is again essential with respect to U/R.

Hence, the collection of all α_x essential sets with respect to U/R form an algebra in Σ which contains all the elements of U/R. Here, as Σ is the algebra of fuzzy sets, the sub algebra of essential with respect to U/R in Σ which can be denoted by $C_{U/R}$. Now, by applying strong α cut, each element of $C_{U/R}$ can be defuzzified. Denote C(U/R) be the collection all such defuzzified elements from $C_{U/R}$. Here, it can be noticed that each element of Z lies in C(U/R). For convenience, we denote the elements of C(U/R) by $\{A_1, A_2, \ldots, A_s\}$ Now the refinement Z' of the partition Z can be obtained by the following algorithm.

4.5. Algorithm

Algorithm Refinement

1. M= C(U/R)-{φ}; Z'={}
2. Compute the cardinality of each element of M
3. If the cardinality of each element is zero, goto 7
4. Locate a set, say A, with least positive cardinality
5. Z'=Z' ∪ {A}
6. M = {B-A/ B∈M}; goto 2
7. list Z'
8. end

Using the above algorithm, the refinement Z' is obtained. As Z' is the refinement of Z, the rough fuzzy approximation of any fuzzy input reduces the ambiguity.

5. CONCLUSION

This paper deals with the theoretical approach of obtaining rough fuzzy approximation, which is more efficient than the usual rough fuzzy approximations.

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