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An observer based dynamic output feedback controller for discrete system with input saturation and time varying delay

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Abstract: This paper is concerned with the stability analysis of discrete system subjected to input saturation and time varying state delay. An observer based dynamic output feedback controller is used to determine observer and controller gain for stabilizing the above system. A delay dependent Lyapunov function, based on reciprocal convex approach, is used to derive the stability criterion. An optimization algorithm is also proposed to maximize the domain of attraction for discrete systems with input saturation and interval like time varying delay using linear matrix inequality (LMI). A numerical example is also provided to show the effectiveness of the proposed theory.

Keywords: Actuator Saturation; Delay-dependent Stability; Linear Matrix Inequality (LMI); Discrete Systems.

1. INTRODUCTION

Time delay is inevitable in practical, industrial and engineering systems which deteriorates the performance and causes instability. Delay often occurs due to limited speed of information processing and data transmitting among different parts of physical system [1-4]. The stability analysis of time delay discrete systems has been extensively investigated in literature [5-10]. Delay dependent stability conditions based on LMI technique have been derived in [5-9] while a new less conservative reciprocal convex approach has been discussed in [34-36].

In industry and real time applications, controller design is very challenging issue for the systems subjected to actuator saturation. Saturation nonlinearity leads to degradation of system stability and responsible for poor performance of closed loop system. An anti-windup technique which is used to tackle saturation nonlinearity has received the great momentum in literature [11-19]. In [18] convex hull approach has been used to tackle the saturation nonlinearity which is less conservative as compared to existing techniques. The issue of stability analysis and control synthesis problems incorporated with actuator saturation and time delay has appeared in [20-26] using LMI technique. Some real time application oriented results for jet engine compressor; flight control system, automatic train etc. are reported for facilitating the exposition [27-30].

However, in real time systems, it is not necessary that all state variables are available while aforementioned works presume that all states of systems are available. In contrast, much less attention has been paid on observer based technique to deal with time delay and saturation nonlinearity [31-32]. In [31] an observer has been appended with anti-windup compensator to diminish the effect of saturation nonlinearity in continuous system while [32] has designed a robust observer based output feedback controller for continuous time delayed system with saturation.

This paper presents an observer based output feedback controller for discrete systems with time delay and saturation nonlinearity. A delay dependent Lyapunov function is used for stability analysis using reciprocal convex approach. The stability conditions are derived in terms of LMI. The saturation behavior is described by convex hull so this approach is less conservative. The domain of attraction is also estimated.

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The rest of the paper is organized as follows. In Sect. 2 problem is formulated and some useful Lemmas are given. In Sect. 3 an observer based dynamic output feedback controller is designed for asymptotic stability of underlying system and new condition for stability is derived for the system subjected to interval type time varying delay and actuator saturation using LMI technique. The domain of attraction is also estimated. The numerical example is given in Sect. 4 to show the validity of proposed approach.

2. PROBLEM FORMULATION AND PRELIMINARIES

The notations used in this paper are:

Notations

$\Re^{m\times n}$	set of $m \times n$ real matrices
$\Re^{m}$	set of $m \times 1$ real matrices
0	null matrix or null vector
I	identity matrix of appropriate dimension
$\lambda_{max}(\Omega)$	maximum eigenvalue of any given matrix Ω
$diag\ \{a_1, a_2, ..., a_n\}$	diagonal matrix with diagonal elements $a_1, a_2, ..., a_n$
*	symmetric terms in symmetric matrix
$\|\cdot\|$	norm of matrix or vector

Consider a discrete time system in presence of actuator saturation and delay

$$\boldsymbol{x}(k+1) = \boldsymbol{A}_p\boldsymbol{x}(k) + \boldsymbol{A}_{dp}\boldsymbol{x}(k-d(k)) + \boldsymbol{B}_p\ sat(\boldsymbol{u}(k)) \tag{1a}$$

$$\boldsymbol{y}(k) = \boldsymbol{C}_p\boldsymbol{x}(k) + \boldsymbol{D}_p sat(\boldsymbol{u}(k)) \tag{1b}$$

where

$k \in z_+$ and z_+ denotes the set of nonnegative integers. The $x(k) \in \Re^n$ is the state vector of the given system. The $u(k) \in \Re^p$ is input vector while $y(k) \in \Re^q$ is output vector.

Matrices $A_p \in \Re^{n\times n}$, $A_{dp} \in \Re^{n\times n}$, $B_p \in \Re^{n\times p}$, $C_p \in \Re^{q\times n}$ and $D_p \in \Re^{q\times p}$ are known constant matrices.

The time varying delay satisfies

$$d_1 \le d(k) \le d_2 \tag{2}$$

where d_1 and d_2 are constant nonnegative integers representing the lower and upper delay bounds respectively.

Note- It is assumed that $(A_p + A_{dp}, B_p)$ must be stablizable and (A_p, C_p) must be detectable.

A linear observer based dynamic output feedback controller for system (1) is proposed as

$$\hat{\boldsymbol{x}}_c(k+1) = \boldsymbol{A}_p\hat{\boldsymbol{x}}_c(k) + \boldsymbol{B}_p sat(\boldsymbol{u}(k)) - \boldsymbol{L}(\ \boldsymbol{y}(k) - \hat{\boldsymbol{y}}(k)) \tag{3a}$$

$$\hat{\boldsymbol{y}}(k) = \boldsymbol{C}_p\hat{\boldsymbol{x}}_c(k) + \boldsymbol{D}_p sat(\boldsymbol{u}(k)) \tag{3b}$$

To stabilize the system (1) an observer based dynamic output feedback controller is proposed in the following form

$$\boldsymbol{u}(k) = \bar{\boldsymbol{K}}\ \hat{\boldsymbol{x}}_c(k) \tag{4}$$

where $\hat{x}_c(k) \in \Re^n$ is estimate of $x(k)$ state vector. The observer output, controller gain and observer gain are $\hat{y}(k) \in \Re^q$, $\bar{\boldsymbol{K}} \in \Re^{p\times n}$ and $\boldsymbol{L} \in \Re^{n\times q}$ respectively.

The estimation error is given as

$$\boldsymbol{e}(k) = \boldsymbol{x}(k) - \hat{\boldsymbol{x}}_c(k) \tag{5}$$

Now, defining the extended state vector

$$\xi(k) = \begin{bmatrix} \boldsymbol{x}(k) \\ \boldsymbol{e}(k) \end{bmatrix} \in \Re^{2n} \tag{6}$$

Using (1)-(5), the closed loop system is written as

$$\xi(k+1) = \boldsymbol{A}\xi(k) + \boldsymbol{A}_d\xi(k - d(\mathrm{k})) + \boldsymbol{B}\, sat(\boldsymbol{K}\xi(k)) \tag{7}$$

where

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_p & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}_p + \boldsymbol{L}\boldsymbol{C}_p \end{bmatrix},\ \boldsymbol{B} = \begin{bmatrix} \boldsymbol{B}_p \\ \boldsymbol{0} \end{bmatrix},\ \boldsymbol{K} = \begin{bmatrix} \bar{\boldsymbol{K}} & -\bar{\boldsymbol{K}} \end{bmatrix},\ \boldsymbol{A}_d = \begin{bmatrix} \boldsymbol{A}_{dp} & \boldsymbol{0} \\ \boldsymbol{A}_{dp} & \boldsymbol{0} \end{bmatrix} \tag{8}$$

To establish the proposed criteria in this work following preliminaries are required-

Lemma 1 [33]. For any constant matrix $\boldsymbol{W} \in \Re^{m\times m}$ with $\boldsymbol{W} = \boldsymbol{W}^T > \boldsymbol{0}$, integers $l_1 < l_2$, vector function $\omega : \{l_1, l_1+1, ..., l_2\} \to \Re^m$ such that the sums concerned are well defined, then

$$(l_2 - l_1 + 1)\sum_{i=l_1}^{l_2} \omega^T(i) W \omega(i) \geq \left(\sum_{i=l_1}^{l_2} \omega(i)\right)^T W \left(\sum_{i=l_1}^{l_2} \omega(i)\right) \tag{9}$$

Lemma 2 [34-36]. For any vectors σ_1, σ_2 matrices T, S and real numbers $\alpha_1 \geq 0$, $\alpha_2 \geq 0$, satisfying

$$\begin{bmatrix} \boldsymbol{T} & \boldsymbol{S} \\ * & \boldsymbol{T} \end{bmatrix} \geq \boldsymbol{0},\ \alpha_1 + \alpha_2 = 1, \tag{10a}$$

$$\sigma_k = 0 \ if\ \sigma_k = 0\ (k = 1, 2) \tag{10b}$$

then

$$-\frac{1}{\alpha_1}\sigma_1^T \boldsymbol{T}\sigma_1 - \frac{1}{\alpha_2}\sigma_2^T \boldsymbol{T}\sigma_2 \leq -\begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix}^T \begin{bmatrix} \boldsymbol{T} & \boldsymbol{S} \\ * & \boldsymbol{T} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix} \tag{10c}$$

Lemma 3: [18]. To embed the saturation nonlinearity within convex hull the set is defined as

$$sat(\boldsymbol{K}\xi(k)) \in co\left\{\left(\boldsymbol{D}_x\boldsymbol{K} + \boldsymbol{D}_x^-\boldsymbol{H}\right)\xi(k), \quad x = 1, 2, ..., 2^p\right\} \tag{11}$$

for two given gain matrices $\boldsymbol{K}$ and $\boldsymbol{H} = [\boldsymbol{H}_1 \quad \boldsymbol{H}_2] \in \Re^{p\times 2n}$ and all $\xi(k) \in \Re^{2n}$ satisfying $|\boldsymbol{H}_s\xi(k)| \leq 1,\ \ s = 1, 2, ..., p$

where

$\boldsymbol{H}_s$ – s^{th} row of the matrix $\boldsymbol{H}$

$co\{.\}$- convex hull.

Taking a set $\Psi \in \Re^{p\times p}$ comprises of diagonal matrices with diagonal elements are 1 or 0. For example if $p = 2$, then $\Psi = \{\boldsymbol{D}_1, \boldsymbol{D}_2, \boldsymbol{D}_3, \boldsymbol{D}_4\}$

$$= \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \tag{12a}$$

The set Ψ contains 2^p elements $\boldsymbol{D}_x$ for every $x = 1, 2, \ldots, 2^p$ and $\boldsymbol{D}_x^- = \boldsymbol{I}_p - \boldsymbol{D}_x$ is also an element in Ψ.

An ellipsoid $\varepsilon(\boldsymbol{P}, 1)$ is characterized as following for $\boldsymbol{0} < \boldsymbol{P} \in \Re^{2n\times 2n}$

$$\varepsilon(\boldsymbol{P},1) = \left\{ \xi(k) \in \Re^{2n} : \xi^T(k)\boldsymbol{P}\xi(k) \le 1 \right\} \tag{12b}$$

A polyhedral set $\ell(H)$ is construed as:

$$\ell(H) = \left\{ \xi(k) \in \Re^{2n} : \left| \boldsymbol{H}_s \xi(k) \right| \le \rho, \;\; s = 1, 2, \ldots, p \right\} \tag{12c}$$

$\boldsymbol{H}_s$ is the s^{th} row of the matrix $\boldsymbol{H}$ and ρ represents the level of saturation.

When $\xi(k) \in \ell(H)$, it follows from Lemma 3 that

$$sat(\boldsymbol{K}\xi(k)) \in co\left\{ \left(\boldsymbol{D}_x \boldsymbol{K} + \boldsymbol{D}_x^- \boldsymbol{H} \right) \xi(k), \quad x = 1, 2, \ldots, 2^p \right\} \tag{13}$$

Next, using (11-13), the closed loop system given by (7) can be represented as

$$\xi(k+1) = \tilde{\boldsymbol{A}}\xi(k) + \boldsymbol{A}_d \xi(k - d(k)) \tag{14}$$

where

$$\tilde{\boldsymbol{A}} = \boldsymbol{A} + \boldsymbol{B}\boldsymbol{D}_x \boldsymbol{K} + \boldsymbol{B}\boldsymbol{D}_x^- \boldsymbol{H} \text{ for } x = 1, 2, \ldots, 2^p.$$

An estimate of domain attraction is represented by $E_\delta \subset \Gamma$ where

$$E_\delta \triangleq \left\{ \phi_\xi(k), \; -d_2 \le k \le 0 : \max \left\| \phi_\xi(k) \right\| \le \delta \right\} \tag{15a}$$

with initial condition $\xi_0 = \phi_\xi(k) \;\; -d_2 \le k \le 0$ be $\phi(k, \xi_0)$ and domain of attraction of the origin is

$$\Gamma \triangleq \left\{ \phi_\xi(k), \; -d_2 \le k \le 0 : \lim_{k\to\infty} \phi_\xi(k, \xi_0) = 0 \right\} \tag{15b}$$

3. MAIN RESULTS

The main results of the paper are stated as follows.

3.1. Robust Stabilization without External Disturbances

Theorem 1: *For given scalars* d_1, d_2 *satisfying* $0 < d_1 < d_2$, *if there exist symmetric matrices* $0 < P \in \Re^{2n\times 2n}$, $0 < Q_k$ $(k = 1, 2, 3)$ $\in \Re^{2n\times 2n}$, $0 < Z_k$ $(k = 1, 2)$ $\in \Re^{2n\times 2n}$, *controller gain matrix* $\bar{\boldsymbol{K}} \in \Re^{p\times n}$, *observer gain matrix* $\boldsymbol{L} \in \Re^{n\times q}$, *matrix* $\boldsymbol{H} \in \Re^{p\times 2n}$ *and matrix* $\mathbf{S}$ *with appropriate dimension satisfying the following set of LMIs* (*16-18*)

$$\boldsymbol{\varphi} = \begin{bmatrix} \boldsymbol{Z}_2 & \boldsymbol{S} \\ * & \boldsymbol{Z}_2 \end{bmatrix} \ge \boldsymbol{0} \tag{16}$$

$$\begin{bmatrix} \boldsymbol{P} & \boldsymbol{H}_s^T \\ * & \rho^2 \end{bmatrix} > \boldsymbol{0}, \quad s = 1, 2, ..., p \tag{17}$$

$$\begin{bmatrix} \boldsymbol{\mu}_{11} & \boldsymbol{0} & \boldsymbol{Z}_1 & \boldsymbol{0} & \tilde{\boldsymbol{A}}^T & \boldsymbol{D}_1(\tilde{\boldsymbol{A}}-\boldsymbol{I})^T & \boldsymbol{D}_{12}(\tilde{\boldsymbol{A}}-\boldsymbol{I})^T \\ * & \boldsymbol{\mu}_{22} & \boldsymbol{Z}_2-\boldsymbol{S}^T & \boldsymbol{Z}_2-\boldsymbol{S} & \boldsymbol{A}_d^T & \boldsymbol{D}_1\boldsymbol{A}_d^T & \boldsymbol{D}_{12}\boldsymbol{A}_d^T \\ * & * & -\boldsymbol{Q}_1-\boldsymbol{Z}_1-\boldsymbol{Z}_2 & \boldsymbol{S} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ * & * & * & -\boldsymbol{Q}_2-\boldsymbol{Z}_2 & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ * & * & * & * & -2\boldsymbol{X}_1+\boldsymbol{X}_1\boldsymbol{P}\boldsymbol{X}_1 & \boldsymbol{0} & \boldsymbol{0} \\ * & * & * & * & * & -2\boldsymbol{X}_2+\boldsymbol{X}_2\boldsymbol{Z}_1\boldsymbol{X}_2 & \boldsymbol{0} \\ * & * & * & * & * & * & -2\boldsymbol{X}_3+\boldsymbol{X}_3\boldsymbol{Z}_2\boldsymbol{X}_3 \end{bmatrix} < \boldsymbol{0} \tag{18}$$

with

$$\boldsymbol{\mu}_{11} = -\boldsymbol{P} + \sum_{k=1}^{3}\boldsymbol{Q}_k + \boldsymbol{D}_{12}\boldsymbol{Q}_3 - \boldsymbol{Z}_1, \; \boldsymbol{\mu}_{22} = -\boldsymbol{Q}_3 + \boldsymbol{S} + \boldsymbol{S}^T - 2\boldsymbol{Z}_2, \; \tilde{\boldsymbol{A}} = \boldsymbol{A} + \boldsymbol{B}\boldsymbol{D}_x\boldsymbol{K} + \boldsymbol{B}\boldsymbol{D}_x^-\boldsymbol{H} \text{ for } x = 1, 2, ..., 2^p, \tag{19}$$

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_p & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}_p + \boldsymbol{L}\boldsymbol{C}_p \end{bmatrix}, \; \boldsymbol{K} = \begin{bmatrix} \bar{\boldsymbol{K}} & -\bar{\boldsymbol{K}} \end{bmatrix}, \; d_{12} = d_2 - d_1, \; \boldsymbol{D}_1 = d_1\boldsymbol{I}, \; \boldsymbol{D}_{12} = d_{12}\boldsymbol{I} \tag{20}$$

then the system given by (*14*) *is stabilized by the observer based output feedback controller and observer given by* $\bar{\boldsymbol{K}}$ *and* $\boldsymbol{L}$ *respectively.*

The estimated domain of attraction for (*14*) *is represented by* $\Gamma_\delta \le 1$

where

$$\Gamma_\delta = \delta^2[(\lambda_{\max}(\boldsymbol{P}) + d_1\lambda_{\max}(\boldsymbol{Q}_1) + d_2\lambda_{\max}(\boldsymbol{Q}_2) + 0.5(d_2 - d_1 + 1)(d_2 + d_1)\lambda_{\max}(\boldsymbol{Q}_3) + 0.5(d_1)^2$$
$$(1 + d_1)\lambda_{\max}(\boldsymbol{Z}_1) + 0.5(d_2 - d_1)^2(d_2 + 1 + d_1)\lambda_{\max}(\boldsymbol{Z}_2))] \tag{21}$$

Proof:

Define

$$\boldsymbol{\eta}(k) = \boldsymbol{\xi}(k+1) - \boldsymbol{\xi}(k) \tag{22}$$

Consider a quadratic Lyapunov functional

$$v(\boldsymbol{\xi}(k)) = \sum_{k=1}^{5} v(\boldsymbol{\xi}(k)) \tag{23}$$

$$v_1(\boldsymbol{\xi}(k)) = \boldsymbol{\xi}^T(k)\boldsymbol{P}\boldsymbol{\xi}(k) \tag{24}$$

$$v_2(\boldsymbol{\xi}(k)) = \sum_{r=k-d_1}^{k-1} \boldsymbol{\xi}^T(r)\boldsymbol{Q}_1\boldsymbol{\xi}(r) \tag{25}$$

$$v_3(\boldsymbol{\xi}(k)) = \sum_{r=k-d_2}^{k-1} \boldsymbol{\xi}^T(r)\boldsymbol{Q}_2\boldsymbol{\xi}(r) \tag{26}$$

$$v_4(\xi(k)) = \sum_{s=-d_2}^{-d_1} \sum_{r=k+s}^{k-1} \xi^T(r)\boldsymbol{Q}_3\xi(r) \tag{27}$$

$$v_5(\xi(k)) = d_1 \sum_{\theta=-d_1+1}^{0} \sum_{r=k-1+\theta}^{k-1} \boldsymbol{\eta}^T(r)\boldsymbol{Z}_1\boldsymbol{\eta}(r) + d_{12} \sum_{\theta=-d_2+1}^{-d_1} \sum_{r=k-1+\theta}^{k-1} \boldsymbol{\eta}^T(r)\boldsymbol{Z}_2\boldsymbol{\eta}(r) \tag{28}$$

The forward difference of Lyapunov functional along trajectories of system (14) is given as

$$\Delta v\left(\xi(k)\right) = v(\xi(k+1)) - v(\xi(k)) \tag{29}$$

where

$$\begin{aligned}\Delta v(\xi(k)) &= \boldsymbol{\xi}^T(k+1)\boldsymbol{P}\xi(k+1) - \boldsymbol{\xi}^T(k)\boldsymbol{P}\xi(k) + \xi^T(k)\boldsymbol{Q}_1\xi(k) - \xi^T(k-d_1)\,\boldsymbol{Q}_1\xi(k-d_1)\\
&+\xi^T(k)\boldsymbol{Q}_2\xi(k) - \xi^T(k-d_2)\boldsymbol{Q}_2\xi(k-d_2) + (d_{12}+1)\xi^T(k)\boldsymbol{Q}_3\xi(k)\\
&-\sum_{s=-d_2}^{-d_1} \xi^T(k+s)\boldsymbol{Q}_3\xi(k+s) + d_1^2\boldsymbol{\eta}^T(k)\boldsymbol{Z}_1\boldsymbol{\eta}(k) + d_{12}^2\boldsymbol{\eta}^T(k)\boldsymbol{Z}_2\boldsymbol{\eta}(k)\\
&-d_1 \sum_{\theta=-d_1+1}^{0} \boldsymbol{\eta}^T(k+\theta-1)\boldsymbol{Z}_1\boldsymbol{\eta}(k+\theta-1) - d_{12} \sum_{\theta=-d_2+1}^{-d_1} \boldsymbol{\eta}^T(k+\theta-1)\boldsymbol{Z}_2\boldsymbol{\eta}(k+\theta-1)\end{aligned} \tag{30}$$

Note that the term

$$-\sum_{s=-d_2}^{-d_1} \xi^T(k+s)\boldsymbol{Q}_3\xi(k+s) \le -\xi^T(k-d(k))\boldsymbol{Q}_3\xi(k-d(k)) \tag{31}$$

From Lemma 1, we have the following relation

$$-d_1 \sum_{\theta=-d_1+1}^{0} \boldsymbol{\eta}^T(k+\theta-1)\boldsymbol{Z}_1\boldsymbol{\eta}(k+\theta-1) \le -\left[\xi^T(k) - \xi^T(k-d_1)\right]\boldsymbol{Z}_1\left[\xi(k) - \xi(k-d_1)\right] \tag{32}$$

Also the term

$$\begin{aligned}-d_{12} \sum_{\theta=-d_2+1}^{-d_1} \boldsymbol{\eta}^T(k+\theta-1)\boldsymbol{Z}_2\boldsymbol{\eta}(k+\theta-1) &= -d_{12} \sum_{\theta=-d(k)+1}^{-d_1} \boldsymbol{\eta}^T(k+\theta-1)\boldsymbol{Z}_2\boldsymbol{\eta}(k+\theta-1)\\
&-d_{12} \sum_{\theta=-d_2+1}^{-d(k)} \boldsymbol{\eta}^T(k+\theta-1)\boldsymbol{Z}_2\boldsymbol{\eta}(k+\theta-1)\end{aligned} \tag{33}$$

Employing Lemma 1 on (33)

$$\begin{aligned}-d_{12} \sum_{\theta=-d_2+1}^{-d_1} \boldsymbol{\eta}^T(k+\theta-1)\boldsymbol{Z}_2\boldsymbol{\eta}(k+\theta-1) &\le \frac{-d_{12}}{(d(k)-d_1)}\left(\sum_{\theta=-d(k)+1}^{-d_1} \boldsymbol{\eta}^T(k+\theta-1)\right)\boldsymbol{Z}_2\left(\sum_{\theta=-d(k)+1}^{-d_1} \boldsymbol{\eta}(k+\theta-1)\right)\\
&-\frac{d_{12}}{(d_2-d(k))}\left(\sum_{\theta=-d_2+1}^{-d(k)} \boldsymbol{\eta}^T(k+\theta-1)\right)\boldsymbol{Z}_2\left(\sum_{\theta=-d_2+1}^{-d(k)} \boldsymbol{\eta}(k+\theta-1)\right)\end{aligned} \tag{34a}$$

$$= -\frac{1}{\dfrac{(d(k)-d_1)}{d_{12}}}\left[\boldsymbol{\xi}^T(k-d_1) - \boldsymbol{\xi}^T(k-d(k))\right]\boldsymbol{Z}_2\left[\boldsymbol{\xi}(k-d_1) - \boldsymbol{\xi}(k-d(k))\right]$$

$$-\frac{1}{\frac{(d_2-d(k))}{d_{12}}}\left[\xi^T(k-d(k))-\xi^T(k-d_2)\right] \boldsymbol{Z}_2\left[\xi(k-d(k))-\xi(k-d_2)\right] \tag{34b}$$

Define,

$$\chi_1(k)=\xi(k-d_1)-\xi(k-d(k)) \tag{35a}$$

$$\chi_2(k)=\xi(k-d(k))-\xi(k-d_2) \tag{35b}$$

From (34) and (35), it is clear that $\chi_1(k)=\mathbf{0}$, if $\frac{(d(k)-d_1)}{d_{12}}=0$ and $\chi_2(k)=\mathbf{0}$, if $\frac{(d_2-d(k))}{d_{12}}=0.$ In view of Lemma 2 and (34), if a matrix $\boldsymbol{S}$ satisfy (16) then

$$-d_{12}\sum_{\theta=-d_2+1}^{-d_1}\boldsymbol{\eta}^T(k+\theta-1)\boldsymbol{Z}_2\boldsymbol{\eta}(k+\theta-1)\le-\begin{bmatrix}\chi_1(k)\\ \chi_2(k)\end{bmatrix}^T\boldsymbol{\varphi}\begin{bmatrix}\chi_1(k)\\ \chi_2(k)\end{bmatrix} \tag{36}$$

Employing (29)-(36), we have the following inequality

$$\Delta v(\xi(k))\le\vartheta^T(k)\Xi\,\vartheta(k)$$

where

$$\Xi=\begin{bmatrix}\boldsymbol{\mu}_{11}+\Xi_{11} & \Xi_{12} & \boldsymbol{Z}_1 & \mathbf{0}\\ * & \boldsymbol{\mu}_{22}+\Xi_{22} & \boldsymbol{Z}_2-\boldsymbol{S}^T & -\boldsymbol{S}+\boldsymbol{Z}_2\\ * & * & -\boldsymbol{Q}_1-\boldsymbol{Z}_1-\boldsymbol{Z}_2 & \boldsymbol{S}\\ * & * & * & -\boldsymbol{Q}_2-\boldsymbol{Z}_2\end{bmatrix} \tag{37}$$

and

$$\vartheta^T(k)=\left[\xi^T(k)\quad \xi^T(k-d(k))\quad \xi^T(k-d_1)\quad \xi^T(k-d_2)\right] \tag{38}$$

$$\Xi_{11}=\tilde{\boldsymbol{A}}^T\boldsymbol{P}\tilde{\boldsymbol{A}}+\boldsymbol{D}_1^2(\tilde{\boldsymbol{A}}-\boldsymbol{I})^T\boldsymbol{Z}_1(\tilde{\boldsymbol{A}}-\boldsymbol{I})+\boldsymbol{D}_{12}^2(\tilde{\boldsymbol{A}}-\boldsymbol{I})^T\boldsymbol{Z}_2(\tilde{\boldsymbol{A}}-\boldsymbol{I}) \tag{39a}$$

$$\Xi_{12}=\tilde{\boldsymbol{A}}^T\boldsymbol{P}\boldsymbol{A}_d+\boldsymbol{D}_1^2(\tilde{\boldsymbol{A}}-\boldsymbol{I})^T\boldsymbol{Z}_1\boldsymbol{A}_d+\boldsymbol{D}_{12}^2(\tilde{\boldsymbol{A}}-\boldsymbol{I})^T\boldsymbol{Z}_2\boldsymbol{A}_d \tag{39b}$$

$$\Xi_{22}=\boldsymbol{A}_d^T\boldsymbol{P}\boldsymbol{A}_d+\boldsymbol{D}_1^2\boldsymbol{A}_d^T\boldsymbol{Z}_1\boldsymbol{A}_d+\boldsymbol{D}_{12}^2\boldsymbol{A}_d^T\boldsymbol{Z}_2\boldsymbol{A}_d \tag{40}$$

Applying Schur's complement on (37), we obtain

$$\Xi_1=\begin{bmatrix}\boldsymbol{\mu}_{11} & \mathbf{0} & \boldsymbol{Z}_1 & \mathbf{0} & \tilde{\boldsymbol{A}}^T & \boldsymbol{D}_1(\tilde{\boldsymbol{A}}-\boldsymbol{I})^T & \boldsymbol{D}_{12}(\tilde{\boldsymbol{A}}-\boldsymbol{I})^T\\ * & \boldsymbol{\mu}_{22} & \boldsymbol{Z}_2-\boldsymbol{S}^T & \boldsymbol{Z}_2-\boldsymbol{S} & \boldsymbol{A}_d^T & \boldsymbol{D}_1\boldsymbol{A}_d^T & \boldsymbol{D}_{12}\boldsymbol{A}_d^T\\ * & * & -\boldsymbol{Q}_1-\boldsymbol{Z}_1-\boldsymbol{Z}_2 & \boldsymbol{S} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ * & * & * & -\boldsymbol{Q}_2-\boldsymbol{Z}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ * & * & * & * & -\boldsymbol{P}^{-1} & \mathbf{0} & \mathbf{0}\\ * & * & * & * & * & -\boldsymbol{Z}_1^{-1} & \mathbf{0}\\ * & * & * & * & * & * & -\boldsymbol{Z}_2^{-1}\end{bmatrix} \tag{41}$$

For any matrices $\mathbf{0}<\boldsymbol{X}_k$ ($k=1,\ 2,\ 3$), [19] we have

$$\begin{cases} -\boldsymbol{P}^{-1} \le -2\boldsymbol{X}_1 + \boldsymbol{X}_1\boldsymbol{P}\boldsymbol{X}_1 \\ -\boldsymbol{Z}_1^{-1} \le -2\boldsymbol{X}_2 + \boldsymbol{X}_2\boldsymbol{Z}_1\boldsymbol{X}_2 \\ -\boldsymbol{Z}_2^{-1} \le -2\boldsymbol{X}_3 + \boldsymbol{X}_3\boldsymbol{Z}_2\boldsymbol{X}_3 \end{cases} \tag{42}$$

In view of (42), it is clear that (41) orients to (18).

It is required to pay attention to the condition of $\Xi_1 < \mathbf{0}$, which implies $\Delta v(\boldsymbol{\xi}(k)) < 0$ for $\boldsymbol{\xi}(k) \neq \mathbf{0}$. So the condition $\Xi_1 < \mathbf{0}$ along with (16)-(17) are sufficient condition for asymptotic stability of the system (14).

The satisfaction of condition stated in (17) signifies that the set $\varepsilon\,(\boldsymbol{P}) = \left\{\xi \in \Re^{2n};\, \boldsymbol{\xi}^T \boldsymbol{P}\, \boldsymbol{\xi} \le 1\right\}$ is included in polyhedral set $\ell(H)$ as defined in (12c). It can be proven that $\varepsilon\,(\boldsymbol{P}) = \left\{\xi \in \Re^{2n};\, \boldsymbol{\xi}^T \boldsymbol{P}\, \boldsymbol{\xi} \le 1\right\}$ is equivalent to (Boyd et al., 1994)

$$\boldsymbol{P} - \boldsymbol{H}_{(s)}{}^T \boldsymbol{H}_{(s)}\rho^{-2} > \mathbf{0},\; s = 1, 2, ..., p \tag{43}$$

Pre and post multiplication of (43) by ξ^T and ξ respectively, it follows that $\xi \in \ell(H)$ for all $\xi \in \varepsilon\,(P)$. The relation (17) is obtained by using Schur's complement on (43).

3.2. Maximization of Domain of Attraction

An optimization procedure to maximize the estimate of domain of attraction is presented in this section.

Consider the closed loop system (14) with the initial conditions (15b) then the maximized domain of attraction can be estimated from the following convex optimization problem

Minimize r

where

$$r = w_1 + d_1 w_2 + d_2 w_3 + 0.5(d_2 - d_1 + 1)(d_2 + d_1) w_4 + 0.5 d_1^2 (1 + d_1) w_5 + 0.5(d_2 - d_1)^2 (d_2 + d_1 + 1) w_6 \tag{44}$$

subject to (16)-(18) and

$$w_1\boldsymbol{I} - \boldsymbol{P} \ge 0,\; w_2\boldsymbol{I} - \boldsymbol{Q}_1 \ge 0,\; w_3\boldsymbol{I} - \boldsymbol{Q}_2 \ge 0,\; w_4\boldsymbol{I} - \boldsymbol{Q}_3 \ge 0,\; w_5\boldsymbol{I} - \boldsymbol{Z}_1 \ge 0,\; w_6\boldsymbol{I} - \boldsymbol{Z}_2 \ge 0 \tag{45}$$

has a feasible solution for the weighting parameters $w_k > 0$, $k = 1,\ 2,\ ..., 6$ positive definite symmetric matrices $\boldsymbol{P} \in \Re^{2n\times 2n}$, $\boldsymbol{Q}_k\,(k = 1,2,3) \in \Re^{2n\times 2n}$, $\boldsymbol{Z}_k\,(k = 1,2) \in \Re^{2n\times 2n}$, $\mathbf{0} < \boldsymbol{X}_k$, $(k = 1,\ 2,\ 3)$, controller gain matrix $\bar{\boldsymbol{K}} \in \Re^{p\times n}$, observer gain matrix $\boldsymbol{L} \in \Re^{n\times q}$, matrix $\boldsymbol{H} \in \Re^{p\times 2n}$ and matrix $\boldsymbol{S}$.

From the above optimization technique, the maximized estimate of domain of attraction is given by $\delta_{\max} = \dfrac{1}{\sqrt{\beta_r}}$ with

$$\beta_r = \lambda_{max}(\boldsymbol{P}) + d_1\lambda_{max}(\boldsymbol{Q}_1) + d_2\lambda_{max}(\boldsymbol{Q}_2) + 0.5(d_2 - d_1 + 1)(d_2 + d_1)\lambda_{max}(\boldsymbol{Q}_3) + 0.5 d_1^2(1 + d_1)\lambda_{max}(\boldsymbol{Z}_1) + 0.5(d_2 - d_1)^2(d_2 + d_1 + 1)\lambda_{max}(\boldsymbol{Z}_2) \tag{46}$$

Proof: The satisfaction of relation (45) implies that $w_1\boldsymbol{I} \ge \lambda_{max}(\boldsymbol{P})$, $w_2\boldsymbol{I} \ge \lambda_{max}(\boldsymbol{Q}_1)$, $w_3\boldsymbol{I} \ge \lambda_{max}(\boldsymbol{Q}_2)$, $w_4\boldsymbol{I} \ge \lambda_{max}(\boldsymbol{Q}_3)$, $w_5\boldsymbol{I} \ge \lambda_{max}(\boldsymbol{Z}_1)$ and $w_6\boldsymbol{I} \ge \lambda_{max}(\boldsymbol{Z}_2)$.

Thus, if we minimize (44), estimate of region is being maximized. In other words, the above optimization procedure orients the solutions of (16)-(18) in order to obtain the domain of attraction as large as possible.

4. NUMERICAL EXAMPLE

The effectiveness of the proposed approach is illustrated by the given Example.

Example 1 Consider the discrete time state delayed system is following

$$\boldsymbol{A}_p = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.7 \end{bmatrix}, \boldsymbol{A}_{dp} = \begin{bmatrix} -0.015 & -0.01 \\ 0 & -0.013 \end{bmatrix}, \boldsymbol{B}_p = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}, \boldsymbol{C}_p = \begin{bmatrix} -0.0018 & -0.0018 \end{bmatrix} \tag{47}$$

Using Matlab LMI toolbox [37-38] the LMI conditions (16-18) stated in Theorem 1 and (45) is solved for optimization program to maximize the domain of attraction. The controller gain $\bar{K}$, observer gain $\boldsymbol{L}$ and δ_{max} are obtained as following after solving the above LMIs

Table 1
Computational results

Delay Range $d_1 \le d(k) \le d_2$	δ_{max}	*Controller gain* $\bar{K}$	Observer gain L
$1 \le d(k) \le 21$	0.0012	[0.1277 0.0784]	$\begin{bmatrix} -105.0112 \\ -123.6960 \end{bmatrix}$
$1 \le d(k) \le 15$	0.0016	[0.0795 0.0426]	$\begin{bmatrix} -48.7253 \\ -116.1249 \end{bmatrix}$
$1 \le d(k) \le 10$	0.0029	[0.0690 0.0321]	$\begin{bmatrix} -45.6324 \\ -107.3030 \end{bmatrix}$
$1 \le d(k) \le 5$	0.0081	[0.0661 0.0286]	$\begin{bmatrix} -44.3869 \\ -100.2890 \end{bmatrix}$

Figure 1(a)-1(b) depicts the state trajectories of discrete system given by (1) and their estimated states. The initial conditions are given as $\boldsymbol{x}(k) = \begin{bmatrix} 0.01 \\ -0.06 \end{bmatrix}$ and $\hat{\boldsymbol{x}}_c(k) = \begin{bmatrix} 0.005 \\ 0.013 \end{bmatrix}$.

From Figure 1 it is shown that the actual state of the plant is estimated by observer given by (3) and the control effort is shown in Figure 2 for control bound $-10 \le u \le 10$. It can be seen from the Figure 1 that the error $\boldsymbol{e}(k) = \boldsymbol{x}(k) - \hat{\boldsymbol{x}}_c(k)$ is converges to $\boldsymbol{e}(k) = 0$.

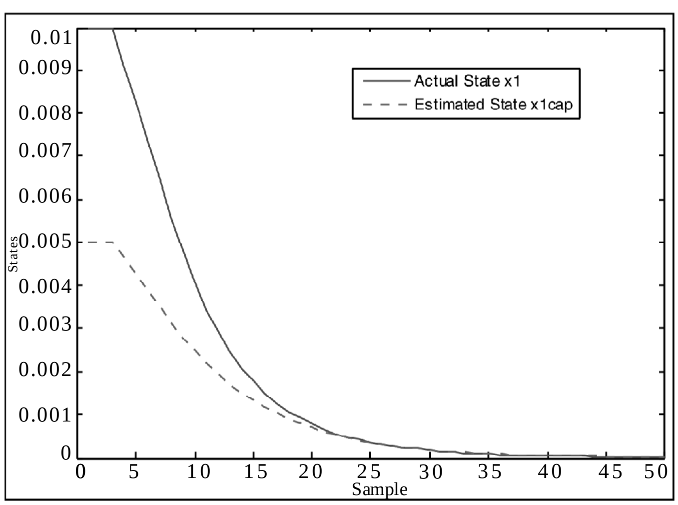

Figure 1: (a) Plant state x1and estimated state by Observer

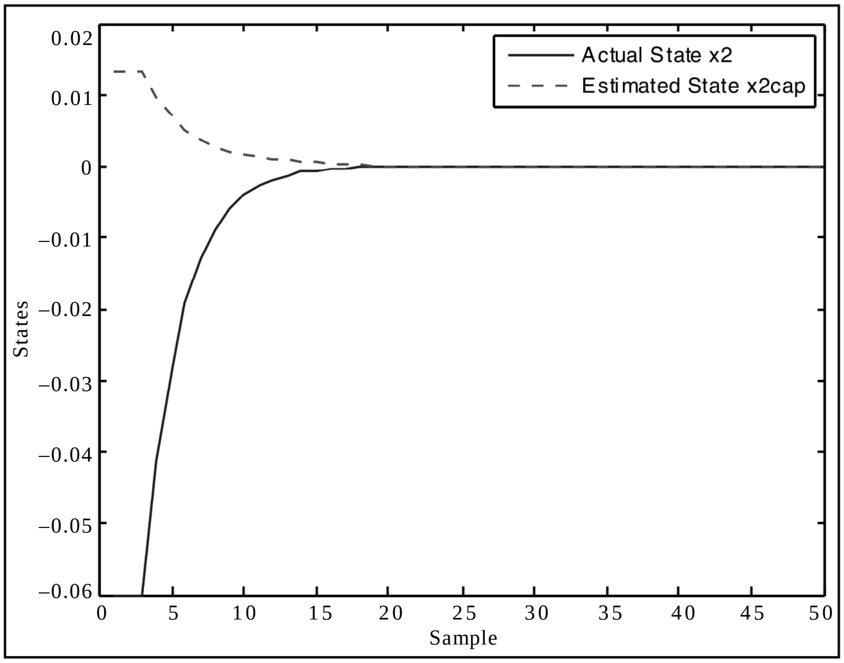

Figure 1 (b) Plant state x2 and estimated state by Observer

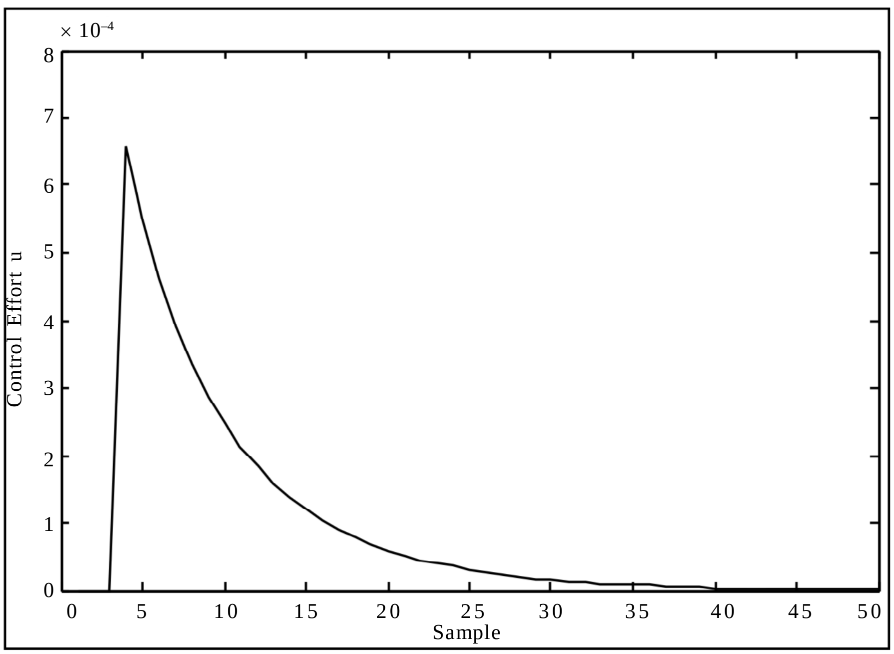

Figure 2: Control Effort

5. CONCLUSION

In this paper, an observer based output feedback approach is used for discrete systems subjected to input saturation and time varying delay. The stability conditions are obtained by using LMI technique with delay dependent Lyapunov functional based on reciprocal convex approach. The gain of observer and controller are obtained from delay dependent LMI conditions to stabilize the system. A less conservative method, based on convex hull approach, is used for representing the input saturation nonlinearity. There is considerable increment in delay range as compared to previous existing results [19]. An algorithm is given to maximize the domain of attraction. To validate the proposed technique a numerical example is also provided.

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