

## LINEAR STOCHASTIC DIFFERENTIAL EQUATIONS WITH ANTICIPATING INITIAL CONDITIONS

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ABSTRACT. In this paper we use the new stochastic integral introduced by Ayed and Kuo [1] and the results obtained by Kuo, Sae-Tang and Szozda [10] to find a solution to a drift-free linear stochastic differential equation with anticipating initial condition. Our solution is based on well-known results from classical Itô theory and anticipative Itô formula results from [10]. We also show that the solution obtained by our method is consistent with the solution obtained by the methods of Malliavin calculus, see e.g., [3].

### 1. Introduction

The aim of the present paper is to establish a solution to linear stochastic differential equation with an anticipating initial condition of a certain form, namely

$$\begin{cases} dX_t = \alpha_t X_t dB_t + \beta_t X_t dt & t \in [a, b] \\ X_a = p(B_b - B_a). \end{cases} \quad (1.1)$$

In the case with  $X_a = x \in \mathbb{R}$ , it is a well-known fact that the unique solution is given by

$$X_t = x \exp \left\{ \int_a^t \alpha_s dB_s + \int_a^t \left( \beta_s - \frac{1}{2} \alpha_s^2 \right) ds \right\}. \quad (1.2)$$

For details, see for example, [6, Section 11.1]. The significance of our result lays in the fact that the solution  $X_t$  of Equation (1.1) is an anticipating stochastic process and it cannot be obtained by the classical tools from the Itô theory of stochastic integration. Instead, we use the integral of adapted and instantly independent processes introduced by Ayed and Kuo [1, 2] and further developed by Kuo, Sae-Tang and Szozda [8, 9, 10]. In contrast to results obtained by Esunge [4] and Buckdahn and Nualart [3], our results do not rely on white noise analysis or Malliavin calculus and are anchored in basic probability theory.

The reminder of this paper is organized as follows. In Section 2 we recall all the necessary definitions and previous results used in the rest of the paper. Section 3 contains a simple example that illustrates our methods and Section 4 presents our main result, Theorem 4.1. We conclude with some examples in Section 5.

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## 2. Preliminary Definitions

In this section we set the notation and recall several definitions used in the remainder of this work.

We denote by  $C^k(\mathbb{R})$  the space of all functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  that are  $k$  times continuously differentiable, and by  $C^\infty(\mathbb{R})$  the space of functions whose derivatives of all orders exist and are continuous. The space of all smooth functions whose Maclaurin series converges for all  $x \in \mathbb{R}$  is denoted by  $\mathcal{M}^\infty$ , that is

$$\mathcal{M}^\infty = \left\{ f \in C^\infty(\mathbb{R}) \left| f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x)}{k!} x^k \text{ for all } x \in \mathbb{R} \right. \right\},$$

where  $f^{(k)}(x)$  stands for the  $k$ -th derivative of  $f(x)$ .

We denote by  $\mathcal{S}(\mathbb{R})$  the Schwartz class of rapidly decreasing functions, that is

$$\mathcal{S}(\mathbb{R}) = \left\{ f \in C^\infty(\mathbb{R}) \left| \sup_{x \in \mathbb{R}} |x^n f^{(m)}(x)| < \infty, \text{ for all } m, n \in \mathbb{N} \right. \right\}. \quad (2.1)$$

It is a well known fact that  $\mathcal{S}(\mathbb{R})$  is closed under the Fourier transform, which we define as  $\hat{f}(\zeta) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \zeta} dx$ , with the inverse Fourier transform given by  $f(x) = \int_{\mathbb{R}} \hat{f}(\zeta) e^{2\pi i x \zeta} d\zeta$ . In this setting, we have the following property of the Fourier transform

$$\left( \widehat{\frac{d}{dx} f(x)} \right)(\zeta) = 2\pi i \zeta \hat{f}(x). \quad (2.2)$$

Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space,  $B_t$  be a standard Brownian motion on  $(\Omega, \mathcal{F}, P)$  and  $(\mathcal{F}_t)_{t \in [0, \infty)}$  be a right-continuous, complete filtration such that:

- (1) for each  $t \in [0, \infty)$ , the random variable  $B_t$  is  $\mathcal{F}_t$ -measurable;
- (2) for all  $s$  and  $t$  such that  $0 \leq s < t$ , the random variable  $B_t - B_s$  is independent of  $\mathcal{F}_s$ .

Following Ayed and Kuo [1], we say that a stochastic process  $X_t$  is *instantly independent* with respect to  $(\mathcal{F}_t)_{t \in [0, \infty)}$  if for each  $t \in [0, \infty)$ , the random variable  $X_t$  is independent of  $\mathcal{F}_t$ . Recall that if  $f_t$  is adapted and  $\varphi_t$  is instantly independent with respect to  $(\mathcal{F}_t)$ , the Itô integral of the product of  $f$  and  $\varphi$  is defined as the limit

$$\int_a^b f_t \varphi_t dB_t = \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=0}^n f_{t_{i-1}} \varphi_{t_i} (B_{t_i} - B_{t_{i-1}}), \quad (2.3)$$

whenever the limit exists in probability. Note that if  $\varphi \equiv 1$ , then the integral defined in Equation (2.3) reduces to the ordinary Itô integral for adapted processes. This kind of integral was first introduced by Ayed and Kuo [1, 2] and studied further by Kuo, Sae-Tang and Szozda [8, 9, 10].

Following the notation of [6], we denote by  $L_{\text{ad}}^2(\Omega \times [a, b])$  the space of all adapted stochastic processes  $X_t$  such that  $\mathbb{E} \left[ \int_a^b X_t^2 dB_t \right] < \infty$ . It is a well-known fact that the Itô integral is well-defined for processes from  $L_{\text{ad}}^2(\Omega \times [a, b])$ .

As in the Itô theory of stochastic integration, the key tool used in this work will be the Itô formula. We state below one of the results of Kuo, Sae-Tang

and Szozda [10] where the authors provide several formulas of this type. Multi-dimensional version and further generalizations of Itô formulas together with an anticipation version of the Girsanov theorem can be found in [7].

**Theorem 2.1** ([10, Corollary 6.2]). *Suppose that  $\theta(t, x, y) = \tau(t)f(x)\varphi(y)$ , where  $\tau \in C^1(\mathbb{R})$ ,  $f \in C^2(\mathbb{R})$ , and  $\varphi \in \mathcal{M}^\infty$ . Let*

$$X_t = \int_a^t \alpha_s dB_s + \int_a^t \beta_s ds,$$

where  $\alpha, \beta \in L_{ad}^2(\Omega \times [a, b])$ . Then

$$\begin{aligned} \theta(t, X_t, B_b - B_a) &= \theta(a, X_a, B_b - B_a) + \int_a^t \frac{\partial \theta}{\partial x}(s, X_s, B_b - B_a) dX_s \\ &+ \frac{1}{2} \int_a^t \frac{\partial^2 \theta}{\partial x^2}(s, X_s, B_b - B_a) (dX_s)^2 \\ &+ \int_a^t \frac{\partial^2 \theta}{\partial x \partial y}(s, X_s, B_b - B_a) (dX_s)(dB_s) \\ &+ \int_a^t \frac{\partial \theta}{\partial t}(s, X_s, B_b - B_a) ds. \end{aligned} \quad (2.4)$$

Equivalently, we can write the Equation (2.4) in a differential form as

$$\begin{aligned} d\theta(t, X_t, B_b - B_a) &= \frac{\partial \theta}{\partial x}(t, X_t, B_b - B_a) dX_t + \frac{1}{2} \frac{\partial^2 \theta}{\partial x^2}(s, X_t, B_b - B_a) (dX_t)^2 \\ &+ \frac{\partial^2 \theta}{\partial x \partial y}(t, X_t, B_b - B_a) (dX_t)(dB_t) \\ &+ \frac{\partial \theta}{\partial t}(t, X_t, B_b - B_a) ds. \end{aligned} \quad (2.5)$$

### 3. A Motivational Example

In this section, we present an example that illustrates the method for obtaining a solution of Equation (1.1). We begin with the simplest possible case of Equation (1.1), that is we set  $\alpha \equiv 1, \beta \equiv 0$  and  $p(x) = x$ , and restrict our considerations to the interval  $[0, 1]$ . Thus we wish to find a solution to

$$\begin{cases} dX_t = X_t dB_t, & t \in [0, 1] \\ X_0 = B_1. \end{cases} \quad (3.1)$$

The natural guess for the solution of Equation (3.1) is obtained by putting  $B_1$  for  $x$  in Equation (1.2) to obtain

$$X_t = B_1 \exp \left\{ B_t - \frac{1}{2}t \right\}. \quad (3.2)$$

Using the Itô formula, it is easy to show that the process  $X_t$  is not a solution of Equation (3.1), but it is a solution of

$$dX_t = X_t dB_t + e^{B_t - \frac{1}{2}t} dt, \quad (3.3)$$

which is obviously different from Equation (3.1). The failure of this approach comes the fact that we do not account for the new factor in the equation, namely

$B_1$ . To account for  $B_1$  in Equation (3.1), we can introduce a correction term to  $X_t$  that will counteract the  $dt$  term appearing in Equation (3.3).

Now, we will use the following as an ansatz for the solution of Equation (3.1)

$$X_t = (B_1 - \xi(t)) \exp \left\{ B_t - \frac{1}{2}t \right\}, \quad (3.4)$$

where  $\xi(t)$  is a deterministic function. The reason for this particular choice is simple. We see that the difference between Equations (3.3) and (3.1) is the term  $\exp\{B_t - \frac{1}{2}t\} dt$ , and to counteract this, we need to introduce another  $dt$ -term with the opposite sign. Looking at the Itô formula in Theorem 2.1, we see that we have to introduce a correction factor that depends only on  $t$ .

We use the Itô formula from Theorem 2.1 with  $\theta(t, x, y) = (y - \xi(t))e^{x - \frac{1}{2}t}$ , and

$$\begin{aligned} \theta_t &= -\xi'(t)e^{x - \frac{1}{2}t} - \frac{1}{2}(y - \xi(t))e^{x - \frac{1}{2}t}, \\ \theta_x &= (y - \xi(t))e^{x - \frac{1}{2}t}, \\ \theta_{xx} &= (y - \xi(t))e^{x - \frac{1}{2}t}, \\ \theta_{xy} &= e^{x - \frac{1}{2}t}, \end{aligned}$$

to obtain

$$\begin{aligned} d\theta(t, B_t, B_1) &= (B_1 - \xi(t)) e^{B_t - \frac{1}{2}t} dB_t + \frac{1}{2} (B_1 - \xi(t)) e^{B_t - \frac{1}{2}t} dt \\ &\quad + e^{B_t - \frac{1}{2}t} dt - \left( \xi'(t)e^{B_t - \frac{1}{2}t} + \frac{1}{2}(B_1 - \xi(t))e^{B_t - \frac{1}{2}t} \right) dt \\ &= (B_1 - \xi(t)) e^{B_t - \frac{1}{2}t} dB_t + \left( e^{B_t - \frac{1}{2}t} - \xi'(t)e^{B_t - \frac{1}{2}t} \right) dt. \end{aligned}$$

So for  $X_t = \theta(t, B_t, B_1)$  to be the solution of Equation (3.1), function  $\xi(t)$  has to satisfy the following ordinary differential equation

$$\begin{cases} \xi'(t) = 1, & t \in [0, 1] \\ \xi(0) = 0. \end{cases} \quad (3.5)$$

Thus, with  $\xi(t) = t$ , process  $X_t$  given in Equation (3.4) is a solution to stochastic differential equation (3.1), that is

$$X_t = (B_1 - t) \exp \left\{ B_t - \frac{1}{2}t \right\} \quad (3.6)$$

solves Equation (3.1).

We point out that the solution in Equation (3.6) coincides with the one that can be obtained by methods of Buckdahn and Nualart [3], where in Proposition 3.2 authors state that the unique solution of Equation (3.1) has the form

$$X_t = g(t, x) \Big|_{x=B_1} \exp \left\{ B_t - \frac{1}{2}t \right\},$$

where  $g$  solves partial differential equation

$$\begin{cases} g_t(t, x) = -g_x(t, x), & t \in (0, 1) \\ g(0, x) = x. \end{cases}$$

In our case,  $g(t, x) = x - t$ .

#### 4. General Case

Theorem 4.1 gives the solution to Equation (1.1) for a certain class of coefficients  $\alpha_t$  and  $\beta_t$ , and initial conditions  $p(x)$  with  $x = B_b - B_a$ . The proof of this theorem uses the idea of a correction term introduced in the previous section.

**Theorem 4.1.** *Suppose that  $\alpha \in L^2([a, b])$  and  $\beta \in L^2_{ad}(\Omega \times [a, b])$ . Suppose also that  $p \in \mathcal{M}^\infty \cap \mathcal{S}(\mathbb{R})$ . Then the stochastic differential equation*

$$\begin{cases} dX_t = \alpha_t X_t dB_t + \beta_t X_t dt, & t \in [a, b] \\ X_a = p(B_b - B_a), \end{cases} \quad (4.1)$$

has a unique solution given by

$$X_t = [p(B_b - B_a) - \xi(t, B_b - B_a)] Z_t, \quad (4.2)$$

where

$$\xi(t, y) = \int_a^t \alpha_s p' \left( y - \int_s^t \alpha_u du \right) ds, \quad (4.3)$$

and

$$Z_t = \exp \left\{ \int_a^t \alpha_s dB_s + \int_a^t \left( \beta_s - \frac{1}{2} \alpha_s^2 \right) ds \right\}.$$

*Remark 4.2.* Before we proceed with proof of Theorem 4.1, let us remark that if  $a = 0$ ,  $\alpha_t \equiv \alpha$  and  $\beta_t \equiv \beta$ , that is the coefficients are constant and evolution starts at 0, we can again apply the results of Proposition 3.2 of [3]. In our notation, the above mentioned proposition states that the solution to Equation (4.1) has the form

$$X_t = g(t, B_1) \exp \left\{ \alpha B_t + \left( \beta - \frac{1}{2} \alpha^2 \right) t \right\}, \quad (4.4)$$

where  $g(t, x)$  is the solution of the following partial differential equation

$$\begin{cases} g_t(t, x) = -\alpha g_x(t, x) & t \in (0, b) \\ g(0, x) = p(x). \end{cases} \quad (4.5)$$

Hence in order to show that our solution and the one given by Equation (4.4) coincide, it is enough to show that  $g(t, x) = p(x) - \xi(t, x)$  solves Equation (4.5). Note that in the case of constant coefficients,  $g(t, x) = p(x - \alpha t)$ . Now it is a matter of simple computation to check that  $g$  solves Equation (4.5).

*Proof.* The uniqueness of a solution follows from linearity of Equation (4.1) and standard arguments. To prove the existence of a solution, first observe that  $Z_t$  is a solution of a stochastic differential equation given by

$$\begin{cases} dZ_t = \alpha_t Z_t dB_t + \beta_t Z_t dt, & t \in [a, b] \\ Z_a = 1. \end{cases}$$

Consider

$$\begin{aligned} dX_t &= d \left[ (p(B_b - B_a) - \xi(t, B_b - B_a)) Z_t \right] \\ &= d[p(B_b - B_a) Z_t] - d[\xi(t, B_b - B_a) Z_t], \end{aligned}$$

where

$$\xi(t, y) = \sum_{n=0}^{\infty} \xi_n(t) y^n, \quad \text{for all } t \geq 0, y \in \mathbb{R}. \quad (4.6)$$

Note that since the function  $z\xi_n(t)y^n$  satisfies the assumptions of the Theorem 2.1, we can write

$$\begin{aligned} d(Z_t \xi(t, B_b - B_a)) &= d\left(Z_t \sum_{n=0}^{\infty} \xi_n(t) (B_b - B_a)^n\right) \\ &= \sum_{n=0}^{\infty} d(Z_t \xi_n(t) (B_b - B_a)^n) \\ &= \sum_{n=0}^{\infty} \left[ \xi_n(t) (B_b - B_a)^n dZ_t + Z_t \xi_n'(t) (B_b - B_a)^n dt \right. \\ &\quad \left. + Z_t \xi_n(t) n (B_b - B_a)^{n-1} (dZ_t)(dB_t) \right] \\ &= \xi(t, B_b - B_a) dZ_t + Z_t \frac{\partial \xi}{\partial t}(t, B_b - B_a) dt \\ &\quad + \frac{\partial \xi}{\partial y}(t, B_b - B_a) (dZ_t)(dB_t). \end{aligned} \quad (4.7)$$

Using Theorem 2.1 and Equation (4.7) we obtain

$$\begin{aligned} dX_t &= p(B_b - B_a) dZ_t + p'(B_b - B_a) (dZ_t)(dB_t) \\ &\quad - \left[ \frac{\partial \xi}{\partial t}(t, B_b - B_a) Z_t dt + \xi(t, B_b - B_a) dZ_t + \frac{\partial \xi}{\partial y}(t, B_b - B_a) (dZ_t)(dB_t) \right] \\ &= [p(B_b - B_a) - \xi(t, B_b - B_a)] dZ_t \\ &\quad + \left[ p'(B_b - B_a) (dZ_t)(dB_t) - \frac{\partial \xi}{\partial t}(t, B_b - B_a) Z_t dt \right. \\ &\quad \left. - \frac{\partial \xi}{\partial y}(t, B_b - B_a) (dZ_t)(dB_t) \right]. \end{aligned}$$

So for  $X_t$  to be a solution of Equation (4.1), we need

$$p'(B_b - B_a) (dZ_t)(dB_t) - \frac{\partial \xi}{\partial t}(t, B_b - B_a) Z_t dt - \frac{\partial \xi}{\partial y}(t, B_b - B_a) (dZ_t)(dB_t) = 0 \quad (4.8)$$

for all  $t \in [a, b]$ . Note that

$$\begin{aligned} (dZ_t)(dB_t) &= (\alpha_t Z_t dB_t + \beta_t Z_t dt)(dB_t) \\ &= \alpha_t Z_t dt. \end{aligned} \quad (4.9)$$

Putting together Equations (4.8) and (4.9) yields

$$p'(B_b - B_a) \alpha_t Z_t dt - \frac{\partial \xi}{\partial t}(t, B_b - B_a) Z_t dt - \frac{\partial \xi}{\partial y}(t, B_b - B_a) \alpha_t Z_t dt = 0,$$

or equivalently,

$$\left[ p'(B_b - B_a) \alpha_t - \frac{\partial \xi}{\partial t}(t, B_b - B_a) - \frac{\partial \xi}{\partial y}(t, B_b - B_a) \alpha_t \right] X_t dt = 0.$$

Hence it is enough to find  $\xi(t, y)$  such that

$$\begin{cases} p'(y)\alpha_t - \frac{\partial \xi}{\partial t}(t, y) - \frac{\partial \xi}{\partial y}(t, y)\alpha_t = 0, & t \in [a, b] \\ \xi(0, y) = 0. \end{cases} \quad (4.10)$$

Thus the problem of finding a solution to the stochastic differential equation (4.1) has been reduced to that of finding a solution to the deterministic partial differential equation (4.10). In order to solve Equation (4.10), we apply the Fourier transform to both sides of Equation (4.10), to obtain

$$\widehat{p}'(\zeta)\alpha_t - \frac{\partial}{\partial t}\widehat{\xi}(t, \zeta) - 2\pi i\zeta\widehat{\xi}(t, \zeta)\alpha_t = 0. \quad (4.11)$$

Note that Equation (4.11) is an ordinary differential equation in  $t$ , with an integrating factor

$$\exp\left\{2\pi i\zeta \int_a^t \alpha_s ds\right\}.$$

Hence Equation (4.11) is equivalent to

$$\frac{\partial}{\partial t}\left(\widehat{\xi}(t, \zeta) \exp\left\{2\pi i\zeta \int_a^t \alpha_s ds\right\}\right) = \widehat{p}'(\zeta)\alpha_t \exp\left\{2\pi i\zeta \int_a^t \alpha_s ds\right\}. \quad (4.12)$$

Integration with respect to  $t$  of both sides of Equation (4.12) yields

$$\widehat{\xi}(t, \zeta) \exp\left\{2\pi i\zeta \int_a^t \alpha_s ds\right\} = \widehat{p}'(\zeta) \int_a^t \alpha_s \exp\left\{2\pi i\zeta \int_a^s \alpha_u du\right\} ds + \widehat{C}(\zeta), \quad (4.13)$$

for some function  $\widehat{C}(\zeta) \in \mathcal{S}(\mathbb{R})$ . Thus, the Fourier transform of function  $\xi(t, y)$ , that is a solution of Equation (4.10), is given by

$$\begin{aligned} \widehat{\xi}(t, \zeta) &= \widehat{p}'(\zeta) \int_a^t \alpha_s \exp\left\{-2\pi i\zeta \int_s^t \alpha_u du\right\} ds \\ &\quad + \widehat{C}(\zeta) \exp\left\{-2\pi i\zeta \int_a^t \alpha_s ds\right\}. \end{aligned} \quad (4.14)$$

Now, we apply the inverse Fourier transform to get

$$\begin{aligned} \xi(t, y) &= \int_{\mathbb{R}} \widehat{p}'(\zeta) \int_a^t \alpha_s \exp\left\{-2\pi i\zeta \int_s^t \alpha_u du\right\} ds \exp\{-2\pi iy\zeta\} d\zeta \\ &\quad + \int_{\mathbb{R}} \widehat{C}(\zeta) \exp\left\{-2\pi i\zeta \int_a^t \alpha_s ds\right\} \exp\{2\pi iy\zeta\} d\zeta \\ &= \int_a^t \alpha_s \int_{\mathbb{R}} \widehat{p}'(\zeta) \exp\left\{a\pi i\zeta \left(y - \int_s^t \alpha_u du\right)\right\} d\zeta ds \\ &\quad + \int_{\mathbb{R}} \widehat{C}(\zeta) \exp\left\{a\pi i\zeta \left(y - \int_s^t \alpha_u du\right)\right\} d\zeta \\ &= \int_a^t \alpha_s p' \left(y - \int_s^t \alpha_u du\right) ds + C \left(y - \int_a^t \alpha_s ds\right). \end{aligned}$$

Using the initial condition from Equation (4.10), we see that  $C(y) \equiv 0$ . Hence  $X_t$  as in Equation (4.2) is a solution of Equation (4.1).  $\square$

*Remark 4.3.* Although very tedious, it is straightforward to check that function  $\xi(t, y)$  in Equation (4.3) can be expressed in the form of Equation (4.6).

## 5. Examples

Below we give several examples of stochastic differential equations with either deterministic or anticipating initial conditions. It is interesting to compare the solutions to see how anticipating initial conditions affect the solutions.

**Example 5.1** (Adapted). Equation

$$\begin{cases} dX_t = X_t dB_t + X_t dt \\ X_0 = x \end{cases}$$

has solution given by

$$X_t = x \exp \left\{ B_t + \frac{1}{2}t \right\}.$$

**Example 5.2** (Anticipating, compare with Example 5.1). Equation

$$\begin{cases} dX_t = X_t dB_t + X_t dt \\ X_0 = B_1 \end{cases}$$

has solution given by

$$X_t = (B_1 - t) \exp \left\{ B_t + \frac{1}{2}t \right\}.$$

**Example 5.3** (Anticipating, compare with Example 5.1). Equation

$$\begin{cases} dX_t = X_t dB_t + X_t dt \\ X_0 = e^{B_1} \end{cases}$$

has a solution given by

$$X_t = e^{B_1 - t} \exp \left\{ B_t - \frac{1}{2}t \right\}$$

**Example 5.4** (Adapted). Equation

$$\begin{cases} dX_t = \alpha_t X_t dB_t + \beta_t X_t dt \\ X_0 = x \end{cases}$$

has solution given by

$$X_t = x \exp \left\{ \int_0^t \alpha_s dB_s + \int_0^t \left( \beta_s - \frac{1}{2} \alpha_s^2 \right) ds \right\}.$$

**Example 5.5** (Anticipating, compare with Example 5.4). Equation

$$\begin{cases} dX_t = \alpha_t X_t dB_t + \beta_t X_t dt \\ X_0 = B_1 \end{cases}$$

has solution given by

$$X_t = \left( B_1 - \int_0^t \alpha_s ds \right) \exp \left\{ \int_0^t \alpha_s dB_s + \int_0^t \left( \beta_s - \frac{1}{2} \alpha_s^2 \right) ds \right\}.$$



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