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Fs-Sets, Fs-Points, and A Representation Theorem

Vaddiparthi Yogeswara* Biswajit Rath Ch.Ramasanyasi Rao*** K.V. Umakameswari*** and D. Raghu Ram*****

Abstract : In this paper, we establish one of the composition of relations [17] between collection of all subsets of the Fs-points set (FSP(A)) [17] and collection of Fs-subsets of A[17] is identity and other composition contains identity. Already we observed [17] one of the relations is a meet complete homomorphism and the other is a join complete homomorphism [17]. Here we search relations between Fs-complemented sets and complemented constructed crisp sets via these homomorphisms. Also we prove a representation theorem between Fs-subsets of A and crisp subsets of FSP(A) and lastly study some Categorical properties between Categories Fs-set with objects- Fs-sets and morphisms-Fs-functions and set.
Keywords : Fs-set, Fs-subset, Fs-complement, Fs-Function, Fs-point, category of Fs-sets, functor between category of Fs-sets.

1. I. INTRODUCTION

Ever since Zadeh [8] introduced the notion of fuzzy sets in his pioneering work, several mathematicians studied numerous aspects of fuzzy sets.

Murthy[19] introduced *f*-sets in order to prove Axiom of choice for fuzzy sets. The following example shows why the introduction of *f*-set theory is necessitated. Let A be non-empty and consider a diamond lattice $L = \{0, \alpha \parallel \beta, 1\}$. Define two fuzzy sets f and g from A into L such that $f(x) = \alpha$ and $g(x) = \beta$. Here both f and g are nonempty fuzzy sets. The Cartesian product of f and g from A into L is given by $(f \times g)(x) = f(x) \wedge g(x) = \alpha \wedge \beta = 0$. That is, $f \times g$ is a empty set. Even though both f and g are non-empty fuzzy sets, their fuzzy Cartesian product is empty showing that the failure of Axiom of choice in L-fuzzy set theory [1]. The collection of all f-subsets of a given *f*-set with Murthy's definition [19] f-complement [22] could not form a compete Boolean algebra. Vaddiparthi Yogeswara , G.Srinivas and Biswajit Rath introduced the concept of Fs-set and developed the theory of Fs-sets in order to prove collection of all Fs-subsets of given Fs-set is a complete Boolean algebra under Fs-unions, Fs-intersections and Fs-complements. The Fs-sets they introduced contain Boolean valued membership functions .They are successful in their efforts in proving that result with some conditions. In papers [12] and [13] Vaddiparthi Yogeswara, Biswajit Rath and S.V.G.Reddy introduced the concept of Fs-Function between two Fs-subsets of given Fs-set and defined an image of an Fs-subset under a given Fs-function. Also they studied the properties of images under various kinds of Fs-functions.

In the paper [17], we constructed a crisp Fs-points set FSP(A) for given Fs-set A and established a pair of relations between collection of all Fs-subsets of a given Fs-set A and collection of all crisp subsets of Fs-points set FSP(A) of the same Fs-set A and proved one of the relations is a meet complete

* Associate Professor Dept. Mathematics, GIT, GITAM University, Visakhapatnam-530045, A.P State, India. *vaddiparthyy@yahoo.com*
** Asst. Professor Dept. of Applied Mathematics, GIS, GITAM University, Visakhapatnam-530045, A.P, India. *urwithbr@gmail.com*
*** Research Scholar :Dept. of Mathematics, GIS, GITAM University, Visakhapatnam 530045, A.P, India. *ams.mathematics@gmail.com, uma.mathematics@gmail.com, draghuram84@gmail.com*

homomorphism and the other is a join complete homomorphism and searched some properties between Fs-complemented sets and complemented constructed crisp sets via these homomorphisms[17].

In this paper we establish a representation theorem between Fs-subsets of A and crisp subsets of FSP(A) and study some more properties between these -homomorphism and lastly study some Categorical properties between Categories FSSET with objects- Fs-sets and morphisms-Fs-functions and SET. The detailed definitions of Fs-point and FSP (A) for given Fs-set A are discussed before defining those relations mentioned above. For smooth reading of paper, the theory of Fs-sets and Fs-functions in brief is dealt with in first two sections. We denote the largest element of a complete Boolean algebra L_A [1.1] by M_A or 1. We denote Fs-union and crisp set union by same symbol $\cup$ and similary Fs-intersection and crisp set intersection by the same symbol $\cap$. For all lattice theoretic properties and Boolean algebraic properties one can refer Szasz [3], Garret Birkhoff[4],Steven Givant • Paul Halmos[2] and Thomas Jech[5]

2. II. FS-SETS

1. Definition : Let U be a universal set, $A_1 \subseteq U$ and let $A \subseteq U$ be non-empty. A four tuple

$$A = (A_1, A, \overline{A} (\mu_{1A_1}, \mu_{2A}), L_A)$$

is said be an Fs-set if, and only if

(*a*) $A \subseteq A_1$

(*b*) L_A is a complete Boolean Algebra

(*c*) $\mu_{1A_1} : A_1 \to L_A$, $\mu_{2A} : A \to L_A$, are functions such that $\mu_{1A_1} \mid A \geq \mu_{2A}$

2. Definition : Fs-subset

Let $A = (A_1, A, \overline{A} (\mu_{1A_1}, \mu_{2A}), L_A)$ and $B = (B_1, B, \overline{B} (\mu_{1B_1}, \mu_{2B}), L_B)$ be a pair of Fs-sets. B is said to be an Fs-subset of A, denoted by $B \subseteq A$, if, and only if

(*a*) $B_1 \subseteq A_1$, $A \subseteq B$

(*b*) L_B is a complete subalgebra of L_A or $L_B \leq L_A$

(*c*) $\mu_{1B_1} \leq \mu_{1A_1} \mid B_1$, and $\mu_{2B} \mid A \geq \mu_{2A}$

3 . Proposition: Let B and A be a pair of Fs-sets such that $B \subseteq A$. Then $\overline{B}x \leq \overline{A}x$ is true for each $x \in A$

3.1. Remark : For some L_X, such that $L_X \leq L_A$ a four tuple $X = (X_1, X, \overline{X}(\mu_{1X_1}, \mu_{2X}), L_X)$ is not an Fs-set if, and only if

(*a*) $X \not\subset X_1$ or

(b) $\mu_{1X_1} x \not\geq \mu_{2X} x$, for some $x \in X \cap X_1$

Here onwards, any object of this type is called an Fs-empty set of first kind and we accept that it is an Fs-subset of B for any $B \subseteq A$.

4. Definition : An Fs-subset $Y = (Y_1, Y, \overline{Y}(\mu_{1Y_1}, \mu_{2Y}), L_Y)$ of A, is said to be an Fs-empty set of second kind if, and only if

(*a*) $Y_1 = Y$

(*b*) $L_Y \leq L_A$

(*c*) $\overline{Y} = 0$

4.1. Remark : We denote Fs-empty set of first kind or Fs-empty set of second kind by Φ_A.

5. Definition : Let

$$B_1 = (B_{11}, B_1, \overline{B}_1 (\mu_{1B_{11}}, \mu_{2B_1}), L_{B_1}) \text{ and}$$
$$B_2 = (B_{12}, B_2, \overline{B}_2 (\mu_{1B_{12}}), \mu_{2B_2}), L_{B_2} \text{ be a pair of Fs-subsets.}$$

We say that B_1 and B_2 are equal, denoted by $B_1 = B_2$ if, only if

(*a*) $B_{11} = B_{12}$, $B_1 = B_2$

(*b*) $L_{B_1} = L_{B_2}$

(*c*) (*a*) ($\mu_{1B_{11}} = \mu_{1B_{12}}$ and $\mu_{2B_1} = \mu_{2B_2}$) or (*b*) $\overline{B}_1 = \overline{B}_2$

5.1. Remark : We can easily observed that 3(*a*) and 3(*b*) not equivalent statements.

6. Proposition : $B_1 = (B_{11}, B_1, \overline{B}_1 (\mu_{1B_{11}}, \mu_{B_1}), L_{B_1})$

and $B_2 = (B_{12}, B_2, \overline{B}_2 (\mu_{1B_{12}}, \mu_{B_2}), L_{B_2})$

are equal if, only if $B_1 \subseteq B_2$ and $B_2 \subseteq B_1$

7. Definition of Fs-union for a given pair of Fs-subsets of A:

Let $B = (B_1, B, \overline{B}(\mu_{1B_1}, \mu_{2B}), L_B)$ and

$C = (C_1, C, \overline{C}(\mu_{1C_1}, \mu_{2C}), L_C)$,

be a pair of Fs-subsets of A. Then, the Fs-union of B and C, denoted by $B \cup C$ is defined as

$$B \cup C = D = (D_1, D, \overline{D}(\mu_{1D_1}, \mu_{2D}), L_D), \text{ where}$$

(*a*) $D_1 = B_1 \cup C_1$, $D = B \cap C$

(b) $L_D = L_B \vee L_C$ = complete subalgebra generated by $L_B \cup L_C$

(*c*) $\mu_{1D_1} : D_1 \to L_D$ is defined by

$\mu_{1D_1} x = (\mu_{1B_1} \vee \mu_{1C_1}) x$

$\mu_{2D} : D \to L_D$ is defined by

$\mu_{2D} x = \mu_{2B} x \wedge \mu_{2C} x$

$\overline{D} : D \to L_D$ is defined by

$\overline{D}x = \mu_{1D_1} x \wedge (\mu_{2D} x)^c$

8. Proposition : $B \cup C$ is an Fs-subset of A.

9. Definition of Fs-intersection for a given pair of Fs-subsets of A:

Let $B = (B_1, B, \overline{B} (\mu_{1B_1}, \mu_{2B}), L_B)$

and $C = (C_1, C, \overline{C}(\mu_{1C_1}), \mu_{2C}), L_C)$

be a pair of Fs-subsets of A satisfying the following conditions:

(*a*) $B_1 \cap C_1 \supseteq B \cup C$

(*b*) $\mu_{1B1} \; x \wedge \mu_{1C_1} x \geq (\mu_{2B} \vee \mu_{2C})x$, for each $x \in A$

Then, the Fs-intersection of B and *t*, denoted by $B \cap C$ is defined as

$$B \cap C = \varepsilon = (E_1, E, \overline{E} (\mu_{1E} , \mu_{2E}), L_E), \text{ where}$$

(*a*) $E_1 = B_1 \cap C_1$, $E = B \cup C$

(*b*) $L_E = L_B \wedge L_C = L_B \cap L_C$

(*c*) $\mu_{1E_1} : E_1 \to L_E$ is defined by $\mu_{1E_1} x = \mu_{1B_1} x \wedge \mu_{1C_1} x$

$\mu_{2E} : E \to L_E$ is defined by

$\mu_{2E} x = (\mu_{2B} \vee \mu_{2C})x$

$\overline{E} : E \to L_E$ is defined by

$\overline{E}x = \mu_{1E_1} x \wedge (\mu_{2E} x)^c$.

9.1. Remark : If (*i*) or (*ii*) fails we define $B \cap C$ as $B \cap C = \Phi_A$, which is the Fs-empty set of first kind.

2.10. Proposition : For any Fs-subsets B, C and D of $A = (A_1, A, \overline{A} \; (\mu_{1A_1}, \mu_{2A}), L_A)$, the following associative laws are true:

(*a*) $B \cup (C \cup D) = (B \cup C) \cup D$

(*b*) $B \cap (C \cap D) = (B \cap C) \cap D$, whenever Fs-intersections exist.

11. Arbitrary Fs-unions and arbitrary Fs-intersections:

Given a family $(B_i)_{i \in I}$ of Fs-subsets of

$A = (A_1, A, \overline{A}(\mu_{1A_1}, \mu_{2A}), L_A)$, where

$B_i = (B_{1i}, B_i, \overline{B}_i (\mu_{1B_{1i}}, \mu_{2B_i}), L_{B_i})$, for any $i \in I$

12. Definition of Fs-union is as follows

Case (1) : For $I = \Phi$, define Fs-union of $(B_i)_{i \in I}$, denoted by $\cup_{i \in I} B_i$ as $\cup_{i \in I} B_i = \Phi_A$, which is the Fs-empty set

Case (2) : Define for $I \neq \Phi$, Fs-union of $(B_i)_{i \in I}$ denoted by $\cup_{i \in I} B_i$ as follow

$$\bigcup_{i \in I} B_i = B = (B_1, B, \bar{B}(\mu_{1B_1}, \mu_{2B}), L_B)$$

where

(*a*) $B_1 = \cup_{i \in I} B_{1i}$, $B = \cap_{i \in I} B_i$

(*b*) $L_B = \vee_{i \in I} L_{B_i}$ = complete subalgebra generated by $\cup L_i$ $(L_i = L_{B_i})$

$\mu_{1B_1} : B_1 \to L_B$ is defined by

$\mu_{1B_1} x = (\vee_{i \in I} \mu_{1B_{1i}})\, x = \vee_{i \in I} x\, \mu_{1B_{1i}} x$, where

$I_x = \{i \in I | x \in B_i\}$

$\mu_{2B} : B \to L_B$ is defined by $\mu_{2B}\, x = (\wedge_{i \in I} \mu_{2B_i})\, x = \wedge_{i \in I} \mu_{2B_i}\, x$

$\bar{B} : B \to L_B$ is defined by $\bar{B}x = \mu_{1B_1}\, x \wedge (\mu_{2B}\, x)^c$

12.1. Remark : We can easily show that (*d*) $B_1 \supseteq B$ and $\mu_{1B_1} | B \geq \mu_{2B}$.

13. Definition of Fs-intersection:

Case (1) : For $I = \Phi$, we define Fs-intersection of $(B_i)_{i \in I}$, denoted by $\cap_{i \in I} B_i$ as $\cap_{i \in I} B_i = A$

Case (2) : Suppose $\cap_{i \in I} B_{1i} \supseteq \cup_{i \in I} B_i$ and $\wedge_{i \in I} \mu_{1B1i} \,|\, (\cup_{i \in I} B_i) \geq \vee_{i \in I} \mu_{2B_i}$

Then, we define Fs-intersection of $(B_i)_{i \in I}$, denoted by $\cap_{i \in I} B_i$ as follows

$$\bigcap_{i \in I} B_i = C = (C_1, C, \bar{C}\, (\mu_{1C_1}), \mu_{2C}), L_C)$$

(*a*) $C_1 = \cap_{i \in I} B_{1i}$, $C = \cup_{i \in I} B_i$

(*b*) $L_C = \wedge_{i \in I} L_{B_i}$

(*c*) $\mu_{1C_1} : C_1 \to L_C$ is defined by $\mu_{1C_1}\, x = (\wedge_{i \in I} \mu_{1B1i})\, x = \wedge_{i \in I}\, \mu_{1B1i}\, x$

$\mu_{2C} : C \to L_C$ is defined by $\mu_{2C}\, x = (\vee_{i \in I} \mu_{2B_i})\, x = \vee_{i \in I}\, x\, \mu_{2B_i}\, x$, where, $I_x = \{i \in I \,|\, x \in B_i\}$

$\bar{C} : C \to L_C$ is defined by $\bar{C}x = \mu_{1C_1} x \wedge \mu_{2C}\, x)^c$

Case (3): $\cap_{i \in I} B_{1i} \not\supseteq \cup_{i \in I} B_i$ or $\wedge_{i \in I} \mu_{1B_{1i}} \,|\, (\cup_{i \in I} B_i) \not\geq \vee_{i \in I} \mu_{2B_i}$

We define $$\bigcap_{i \in I} B_i = \Phi_A$$

13. 1. Lemma : For any Fs-subset $B = (B_1, B, \bar{B}(\mu_{1B_1}, \mu_{2B}), L_B)$

and $B \subseteq B_i = (B_{1i}, B_i, \bar{B}_i\, (\mu_{1B_{1i}}, \mu_{2B_i}), L_{B_i})$

for each $i \in I$. $\cap_{i \in I} B_i$ exists and $B \subseteq \cap_{i \in I} B_i$

14. Proposition : $(L(A), \cap)$ is $\wedge$-complete lattics.

14.1. Corollary : For any Fs-subset B of A, the following results are true

(*a*) $\Phi_A \cup B = B$

(*b*) $\Phi_A \cap B = \Phi_A$.

15. Proposition : $(L(A), \cup)$ is $\vee$-complete lattics.

15.1. Corollary : $(L(A), \cup, \cap)$ is a complete lattice with $\vee$ and $\wedge$

16. Proposition : Let $B = (B_1, B, \bar{B}(\mu_{1B_1}, \mu_{2B}), L_B)$,

$C = (C_1, C, \bar{C}(\mu_{1C_1}, \mu_{2C}), L_C)$

and $D = (D_1, D, \bar{D}(\mu_{1D_1}, \mu_{2D}), L_D)$.

Then $B \cup (C \cap D)$

$= (B \cup C) \cap (B \cup D)$ provided $C \cap D$ exists.

17. Proposition: Let $B = (B_1, B, \overline{B}(\mu_{1B_1}, \mu_{2B}), L_B)$,

$C = (C_1, C, \overline{C}(\mu_{1C_1}, \mu_{2C}), L_C)$

and $D = (D_1, D, \overline{D}(\mu_{1D_1}), \mu_{2D}), L_D)$.

Then $B \cap (C \cup D) = (B \cap C) \cup (B \cap D)$

provided in R.H.S $(B \cap C)$ and $(B \cap D)$ exists.

18. Definition of Fs-complement of an Fs-subset :

Consider a particular Fs-set $A = (A_1, A, \overline{A}(\mu_{1A_1}, \mu_{2A}), L_A)$, $A \neq \Phi$, where

(*a*) $A \subseteq A_1$

(*b*) $L_A = [0, M_A]$, $M_A = \vee \overline{A}A = \vee_{a \in A} \overline{A}a$

(*c*) $\mu_{1A_1} = M_A$, $\mu_{2A} = 0$,

$\overline{A}x = \mu_{1A_1} x \wedge (\mu_{2A}\, x)^c = M_A$, for each $x \in A$

Given $B = (B_1, B, \overline{B}(\mu_{1B_1}, \mu_{2B}, L_B)$. We define Fs-complement of B, denoted by B^{C_A} for B = A and $L_B = L_A$ as follows:

$B^{C_A} = D = (D_1, D, \overline{D}(\mu_{1D_1}, \mu_{2D}, L_D)$, where

(*a*) $D_1 = C_A B_1 = B_1^c \cup A$, $D = B = A$

(*b*) $L_D = L_A$

(*c*) $\mu_{1D_1} : D_1 \rightarrow L_A$, is defined by $\mu_{1D_1} x = M_A$

$\mu_{2D} : A \rightarrow L_A$, is defined by $\mu_{2D}\, x = \overline{B}x = \mu_{1B_1} x \wedge (\mu_{2B} x)^c$

$\overline{D} : A \rightarrow L_A$, is defined by $\overline{D}x = \mu_{1D_1} x \wedge (\mu_{2D}\, x)^c = M_A \wedge (\overline{B}x)^c = (\overline{B}x)^c$.

19. Proposition: $A^{C_A} = \Phi_A$

20. Definition: Define $(\Phi_A)^{C_A} = A$

21. Proposition : For $B = (B_1, B, \overline{B}(\mu_{1B_1}, \mu_{2B}), L_B)$,

$C = (C_1, C, \overline{C}(\mu_{1C_1}, \mu_{2C}), L_C)$,

which are non Fs-empty sets and $B = C = A$, $L_B = L_C = L_A$

(*a*) $B \cap B^{C_A} = \Phi_A$

(*b*) $B \cup B^{C_A} = A$

(*c*) $(B^{C_A})^{C_A} = B$

(*d*) $B \subseteq C$ if and only if $C^{C_A} \subseteq B^{C_A}$

22. Proposition : Fs-De-Morgan's laws for a given pair of Fs-subsets:

For any pair of Fs-sets $B = (B_1, B, \overline{B}(\mu_{1B_1}, \mu_{2B}), L_B)$

and $C = (C_1, C, \overline{C}(\mu_{1C_1}, \mu_{2C}), L_C)$,

with $B = C = A$

and $L_B = L_C = L_A$, we will have

(*a*) $(B \cup C)^{C_A} = B^{C_A} \cap C^{C_A}$ if $(\overline{B}x)^c \wedge (\overline{C}x)^c \leq [(\mu_{1B_1} x)^c \vee \mu_{2C}\, x] \wedge [(\mu_{1C1}\, x)^c \vee \mu_{2B}\, x]$, for each $x \in A$

(*b*) $(B \cap C)^{C_A} = B^C \cup C^{C_A}$, whenever $B \cap C$ exists.

23. Fs-De Morgan laws for any given arbitrary family of Fs-sets:

Proposition : Given a family of Fs-subsets $(B^i)_{i \in I}$ of

$$A = (A_1, A, \overline{A}(\mu_{1A_1}, \mu_{2A}), L_A), \text{ where}$$

$$L_A = [0, M_A].\ \mu_{1A_1}$$

$$= M_A,\ \mu_{2A}$$

$$= 0,\ \overline{A}x$$

$$= M_A$$

(*a*) $(\cup_{i \in I} B_i)^{C_A} = \cap_{i \in I} B_i^{C_A}$, for $I \neq \Phi$, where $B_i = (B_{1i}, B_i, \overline{B}_i\,(\mu_{1B_{1i}}, \mu_{2B_i}), L_{B_i}$ and
(1) $B_i = A$, $L_{B_i} = L_A$ provided $\wedge_{i \in I} \overline{B}_i\, x)^c \leq \wedge_{i,j \in I}[(\mu_{1B_{1i}}\, x)^c \vee \mu_{2B_j} x]$

(*b*) $(\cap_{i \in I} B_i)^{C_A} = \cup_{i \in I} B_i^{C_A}$, whenever $\cap_{i \in I} B_i$ exist

3. FS-FUNCTIONS

1. Definition : A Triplet (f_1, f, Φ) is said to be is an Fs-Function between two given Fs-subsets

$$B = (B_1, B, \overline{B}(\mu_{1B_1}, \mu_{2B}), L_B)$$

and

$$C = (C_1, C, \overline{C}(\mu_{1C_1}, \mu_{2C}), L_C)$$

of A, denoted by (f_1, f, Φ):

$$B = (B_1, B, \overline{B}(\mu_{1B_1}, \mu_{2B}), L_B)$$

$$C = (C_1, C, \overline{C}\,(\mu_{1C_1}, \mu_{2C}), L_C)$$

if, and only if (using the diagrams).

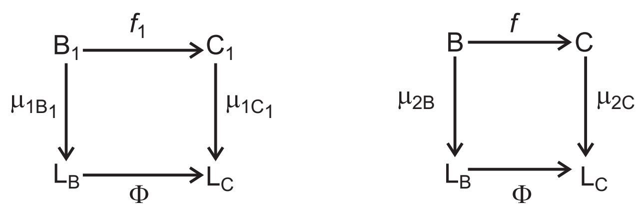

Figure 1: Fs-function $\overline{f}$ $B \to C$

(*a*) $f_1|_B = f$ is onto

(*b*) $\Phi : L_B \to L_c$ is complete homomorphism

(f_1, f, Φ) is denoted by $\overline{f}$

2. Proposition : (*i*) $\mu_{1C_1}|_C \circ f_1|_B \geq \mu_{2C} \circ f$

(*ii*) $\Phi \circ \mu_{1B_1}|_B \geq \Phi \circ \mu_{2B}$

3. Def : Increasing Fs-function

$\overline{f}$ is said to be an increasing Fs- function, and denoted by $\overline{f}_i$ if, and only if(using fig-1)

(*a*) $\mu_{1C_1}|_C \circ f_1|_B \geq \Phi \circ \mu_{1B_1}$

(*b*) $\mu_{2C} \circ f \leq \Phi \circ \mu_{2B}$

4. Proposition : $\Phi \circ (\mu_{2B}\, x)^c = [(\Phi \circ \mu_{2B})x]^c$

5. Proposition: $\Phi \circ \overline{B} \leq \overline{C} \circ f$, provided $\overline{f}$ is an increasing Fs-function

6. Def : Decreasing Fs-function

$\overline{f}$ is said to be decreasing Fs-function denoted as $\overline{f}_d$ and if and only if

(*a*) $\mu_{1C_1}|_C \circ f_1|_B \leq \Phi \circ \mu_{1B_1}$

(*b*) $\mu_{2C} \circ f \geq \Phi \circ \mu_{2B}$

7. Proposition : $\Phi \circ \overline{B} \geq \overline{C} \circ f$, provided $\overline{f}$ is a decreasing Fs-function

8. Def : Preserving Fs- function

$\overline{f}$ is said to be preserving Fs-function and denoted as $\overline{f}_p$ if, and only if

(*a*) $\mu_{1C_1}|_C \circ f_1|_B = \Phi \circ \mu_{1B1}$

(*b*) $\mu_{2C} \circ f = \Phi \circ \mu_{2B}$

9. Proposition : $\Phi \circ \overline{B} = \overline{C} \circ f$, provided $\overline{f}$ is Fs- preserving function

10. Def : Composition of two Fs-function

Given two Fs-functions $\overline{f} : B \to C$ and $\overline{g} : C \to D$. We denote composition of $\overline{g}$ and $\overline{f}$ as $\overline{g} \circ \overline{f}$ and define as $(\overline{g} \circ \overline{f}) = (g_1, g, \Psi) \circ (f_1, f, \Phi) = [g_1 \circ f_1, g \circ f, \Psi \circ \Phi]$

4. FS-POINT

1. Definition We define an object, for $b \in A$, $\beta \in L_A$ such that $\beta \leq \bar{A}b$ – denoted by (b, β) as follows

$$(b, \beta) = (B_1, B, \bar{B}(\mu_{1B1}, \mu_{2B}), L_B),$$

where $A \subseteq B \subseteq B_1 \subseteq A_1, L_B \leq L_A$,

such that $\mu_{1B_1} x, \mu_{2B} x \in L_B$,

$$\alpha \leq \mu_{1A_1} x, \forall x \in A_1, \beta \in L_A$$

$$\mu_{1B_1} x = \begin{cases} \mu_{2A} x, & x\, b, x \in A \\ b \vee \mu_{2A} b, & x = b \\ \alpha, & x \notin A, x \in A_1 \end{cases}$$

and
$$\mu_{2B} x = \begin{cases} \mu_{2A} x, & x \in A \\ \alpha, & x \notin A, x \in B \end{cases}$$

2. Lemma:

(*a*) $\beta \leq \mu_{1A_1} b$ and $\beta \leq (\mu_{2A} b)^c$

(*b*) $\mu_{1B_1} b \geq \mu_{2B} b$

(*c*) $\mu_{1B_1} b \leq \mu_{1A_1} b$

(*d*) $\mu_{2B} b \geq \mu_{2A} b$

(*e*) $\bar{B}b = \beta$

(*f*) (b, β) is Fs-subset of A

Here onward (b, β)-which is an Fs-subset of A, we call a (b,β) objects of A.

3. Definition of a relation between objects:

For any (b, β)objects $B_1 = (B_{11}, B_1, \bar{B}_1 (\mu_{1B_{11}}, \mu_{2B_1}), L_{B_1})$

and $B_2 = (B_{12}, B_2, \bar{B}_2 (\mu_{1B_{12}}, \mu_{2B_2}), L_{B_2})$ of,

we say that $B_1\, R(b, \beta)\, B_2$ if, and only if

$$\mu_{1B_{11}} x = \mu_{2B_1} x, x \neq b$$

and $\forall x \in B_1$ and $\mu_{1B_{12}} x = \mu_{2B_2} x, x \neq b$

and $\forall x \in B_2$ and $\mu_{1B_{11}} b = \mu_{1B_{12}}$

$$b = \beta \vee \mu_{2A} b \text{ and } \mu_{2B_1}$$

$$b = \mu_{2B_2}$$

$$b = \mu_{2A} b.$$

4. Theorem : $R(b, \beta)$ is an equivalence relation.

5. Definition of Fs-point : The equivalence class corresponding to $R(b, \beta)$ is denoted by χ_b^β or (b, β).We define this χ_b^β is an Fs point of A.

Set of all Fs-point of A is denoted by FSP(A).

6. Definition : Let $G \subseteq FSP(A)$.

(*a*) G is said to be closed under stalks if, and only if $\chi_b^\beta \in G, \alpha \leq \beta \Rightarrow \chi_b^\alpha \in G$

(*b*) G is said to be closed under supremums if and only if $M \subseteq L_A, \chi_b^\beta \in G, \forall \beta \in M \Rightarrow \chi_b^{\vee M} \in G$, $\vee M = \vee_{\beta \in M} \beta$

(*c*) G is said to be S-closed if, and only if G is closed under both stalks and supremums.

7. Theorem : Arbitrary intersection of S-closed subset is S-closed

8. Definition : Let $G \subseteq FSP(A)$.

$$\text{Define } G^{\sim} = \Phi_A \text{ if } G = \Phi.$$

Otherwise $G^{\sim} = \cup_{\chi_b^\beta \in G} \chi_b^\beta$

Define $B = (B_1, B, \overline{B}(\mu_{1B_1}, \mu_{2B}), L_B)$, where

$$B_1 \supseteq B = \{b \mid \chi_b^\beta \in G\},$$

$$LB = \vee_{\chi_b^\beta \in G} L_\beta, \mu_{1B_1} b$$

$$= \vee_{\chi_b^\beta \in G} (\beta \vee \mu_{2A} b), \mu_{2B} b = \mu_{2A} b$$

$$\overline{B}b = \mu_{1B_1} b \wedge (\mu_{2B}\, b)^c$$

$$= \vee_{\chi_b^\beta \in G} (\beta \vee \mu_{2A} b) \wedge (\mu_{2A} b)^c$$

$$= \left[\left(\vee_{\chi_b^\beta \in G} \beta\right) \vee \mu_{2A} b\right] \wedge (\mu_{2A} b)^c$$

$$= \left(\left(\vee_{\chi_b^\beta \in G} \beta\right) \wedge (\mu_{2A} b)^c\right) \vee (\mu_{2A} b \wedge (\mu_{2A} b)^c)$$

$$= \vee_{\chi_b^\beta \in G} (\beta \wedge (\mu_{2A} b)^c) \vee 0$$

$$= \vee_{\chi_b^\beta \in G} (\beta \wedge (\mu_{2A} b)^c)$$

$$= \vee_{\chi_b^\beta \in G} \beta$$

9. Theorem : $G^{\sim} = B$

10. Definition : For any $B \subseteq A$

Define $B^{\sim} = \Phi$

if $B = \Phi_A$

Let $B = (B_1, B, \overline{B}(\mu_{1B_1}, \mu_{2B}), L_B)$ and $B \neq \Phi_A$

Define $B^{\sim} = \{\chi_b^\beta \mid b \in B, \beta \in L_B, \beta \leq \overline{B}b\}$

11. Theorem : $A = \cup_{\chi_b^\beta \in FSP(A)} \chi_b^\beta$

12. Lemma : $A^{\sim} = FSP(A)$

13. Theorem: $B^{\sim}$ is S-closed.

14. Theorem: For any $G \subseteq FSP(A)$, $G \subseteq G^{\sim\sim}$

15. Theorem : Let A be an Fs-set. Then the following are equivalent for any $G \subseteq FSP(A)$

16. Theorem : For any B_1 and B_2 such that $B_1 \subseteq B_2 \subseteq A$, $B_1^{\sim} \subseteq B_2^{\sim}$ provided $B_1 = B_2$

where $B_1 = (B_{11}, B_1, \overline{B}_1 (\mu_{1B_{11}}, \mu_{2B_1}), L_{B_1})$

and $B_2 = (B_{12}, B_2, \overline{B}_2 (\mu_{1B_{12}}, \mu_{2B_2}), L_{B_2})$

16.1. Corollary : $B \subseteq A \Rightarrow FSP(B) \subseteq FSP(A)$

17. Result : $B_1 \subseteq B_2$ implies $B_1 \subseteq B_2 \cup B_3$ for any Fs-subset B_3

18. Result : $\chi_b^\beta \subseteq G^{\sim}$ for any $\chi_b^\beta \in G$ such that $G \subseteq FSP(A)$.

19. Recall : 1.16 for any Family $(G_i)_{i \in I}$ of Fs-subsets of A such that $G_i \subseteq G$, $\cup_{i \in I} G_i \subseteq G$.

20. Proposition : $G_1^{\sim} \subseteq G_2^{\sim}$ for any two subsets G_1 and G_2 of FSP(A), such that $G_1 \subseteq G_2$.

21. Theorem: For any Fs-subset B of an Fs-set A, $B^{\sim\sim} = B$.

22. Theorem : $(B \cap C)^{\sim} = B^{\sim} \cap C^{\sim}$

for any Fs-subsets $B = (B_1, B, \overline{B}(\mu_{1B_1}, \mu_{2B}), L_B)$

and $C = (C_1, C, \overline{C}(\mu_{1C_1}, \mu_{2C}), L_C)$

of A such that $B = C$.

23. Proposition: For any family of Fs-subset $(B_i)_{i \in I}$ of A, $(\cap_{i \in I} B_i)^{\sim} = \cap_{i \in I} B_i^{\sim}$ provided all B_i's are equal for each $i \in I$

24. Theorem : $(G_1 \cup G_2)^{\sim} = G_1^{\sim} \cup G_2^{\sim}$ for any subsets G_1 and G_2 of FSP(A),

25. Theorem : $(\cup_{i \in I} G_i)^{\sim} = \cup_{i \in I} G_i^{\sim}$ for any family $(G_i)_{i \in I}$ of subsets of FSP(A).

25.1. Remark : Observe that χ_c^0 is always an Fs-subset of B *i.e.* $\chi_c^0 \in B^{\sim}$ *i.e.* $\chi_c^0 \notin (B^{\sim})^c$

26. Theorem : For $B = (B_1, B, \bar{B}(\mu_{1B_1}, \mu_{2B}), L_B) \subseteq A$, B = A and $L_A = L_B$, $(B^{C_A})^{\sim} \subseteq (B^{\sim})^c$

27. Theorem : $(G^{\sim})^{C_A} \subseteq (G^c)^{\sim}$ for any $G \subseteq$ FSP(A), where $A = (A_1, A, \bar{A}(\mu_{1A_1}, \mu_{2A}), L_A)$, $\mu_{1A_1} = M_A$, $\mu_{2A} = 0$ and $L_A = [0, M_A]$.

5. A REPRESENTATION THEOREM FOR FS-SETS

1. Let

$$A = (A_1, A, \bar{A}(\mu_{1A_1}, \mu_{2A}), L_A)$$

be an Fs-set and L(A) be set of all Fs-subsets

$$B_i = (B_{1i}, B_i, \bar{B}_i (\mu_{1B_{1i}}, \mu_{2B_i}), L_{B_i})$$

with

$$B_i = A \text{ of } A.$$

Let PFSP(A) be the set of all subsets of FSP (A).

$$\Phi : L(A) \rightarrow PFSP(A).$$

Define

$$B \rightarrow B^{\sim}$$

$$\Psi : PFSP(A) \rightarrow L(A).$$

Define

$$G \rightarrow G^{\sim}$$

Then the following are true

(*a*) $\Psi\Phi B = B$ or $\Psi\Phi = 1$

(*b*) $G \subseteq \Phi\Psi G$ or $\Phi\Psi \supseteq 1$

(*c*) Image of $\Phi = \{G \subseteq FSP(A) | G \text{ is S-closed}\}$

(*d*) $\Phi(B^{C_A}) \subseteq (\Phi B)^c$, where $A = (A_1, A, \bar{A}\ (\mu_{1A_1}, \mu_{2A}), L_A)$, $\mu_{1A_1} = M_A$, $\mu_{2A} = 0$ and $L_A = [0, M_A]$

(*e*) $\Psi G)^{C_A} \subseteq \Psi(G^c)$, where $A = (A_1, A, \bar{A}\ (\mu_{1A_1}, \mu_{2A}), L_A)$, $\mu_{1A_1} = M_A$, $\mu_{2A} = 0$ and $L_A = [0, M_A]$

Proof : We Already proved that Φ, Ψ are increasing and Φ is a meet complete homomorphism and Ψ is join complete homomorphism [17]

(*a*) Follows from 4.21

(*b*) Follows from 4.14

(*c*) $G \in$ LHS = image of Φ implies. $\Phi B = B^{\sim} = G$ for some $B \subseteq A$ and $B^{\sim}$ is always S-closed from (4.13) implying $B^{\sim} = G \in$ RHS

$G \in$ RHS implies $G(^{\sim\sim} = G$ or $\Phi\Psi G = G$ from (4.15). That is, $\Phi(\Psi G) = G$ so that $G \in$ LHS

(*d*) Follows from 4.26

(*e*) Follows from 4.27

1.1. Example : Let

$$A = (A_1, A, \bar{A}(\mu_{1A_1}, \mu_{2A}), L_A),$$

where

$$A_1 = \{a, b\},$$

$$A = \{a\},$$

$$\mu_{1A_1} = 1, \mu_{2A} = 0$$

and

$$L_A = \{0, \alpha \parallel \beta, 1\}$$

Suppose

$$B = \chi_a^{\alpha}$$

and

$$C = \chi_a^{\beta}.$$

Then

$$B^{\sim} = \{\chi_a^0, \chi_a^{\alpha}\}$$

and

$$C^{\sim} = \{\chi_a^0, \chi_a^{\beta}\}$$

And

$$B^{\sim} C^{\sim} = \{\chi_a^0,, \chi_a^{\alpha}, \chi_a^{\beta}\}$$

Here

$$B \cup C = \chi_a^{\alpha} \cup \chi_a^{\beta}$$

$= \chi_a^{1}$ implying $(B \cup C)^{\sim}$

$= \{\chi_a^{0}, \chi_a^{\alpha}, \chi_a^{\beta}, \chi_a^{1}\}$

So that, $(B \cup C)^{\sim} \neq B^{\sim} \cup C^{\sim}$

2. Theorem : Φ is a join complete homomorphism, if and only if $L_A = \{0, 1\}$

Proof : If L_A contains more than two elements, then there exist $\beta \in L_A$ such that $\beta \neq 0$, $\beta \neq 1$ and β^c exists such that $\beta^c \neq 0$ and $\beta^c \neq 1$ so that $\beta \parallel \beta^c$.

Hence Φ cannot be a join complete homomorphism by above example, a contradiction.

Hence $L_A = \{0,1\}$

Conversely suppose $L_A = \{0,1\}$

Consider a nonempty family of nonempty Fs-subset $(B_i)_{i \in I}$, we have to show that $(\cup_{i \in I} B_i)^{\sim} = \cup_{i \in I} B_i^{\sim}$

Clearly RHS $\subseteq$ LHS …(1)

Let $\chi_b^{\beta} \in$ LHS.

Then $\chi_b^{\beta} \subseteq_{i \in I} B_i$, here all possible values of β are 0 and 1.

For $\beta = 0$

consider $B_i = (B_{1i}, B_i, \overline{B}_i\,(\mu_{1B1_i}), \mu_{2B_i}), L_{B_i})$ such that $b \in B_i$

Define $\chi_b^{0} = (B_{1i}, B_i, \overline{C}_i\,(\mu_{1C1_i}, \mu_{2C_i}), L_{C_i})$

$= C_i$

where $\mu_{1C1_i} = \mu_{2C_i}$,

$L_{C_i} = \{0, 1\}$

Clearly $\chi_b^{0} \subseteq B_i$ so that $\chi_b^{0} \in B_i^{\sim} \subseteq \cup_{i \in I} B_i^{\sim} =$ RHS

Hence LHS $\subseteq$ RHS …(2)

For $\beta = 1$ consider

$B_i = (B_{1_i}, B_i, \overline{B}_i\,(\mu_{1B1_i}, \mu_{2B_i}), L_{B_i})$ such that $b \in B_i$

Define $\chi_b^{1} = (B_{1_i}, B_i, \overline{C}_i\,(\mu_{1C1_i}, \mu_{2C_i}), L_{Ci})$

$= C_i$ where

$\mu_{1C1_i} x = \mu_{2C_i} x, \forall x \neq b$,

$\mu_{1C1_i} b = 1$,

$\mu_{2C_i} x = 0$,

$L_{Ci} = \{0, 1\}$

Clearly $\chi_b^{1} \subseteq B_i$

$\Rightarrow$ $\chi_b^{0} \in B_i^{\sim} \subseteq \cup_{i \in I} B_i^{\sim}$

$=$ RHS

Hence LHS $\subseteq$ RHS …(3)

From (1), (2) and (3), LHS = RHS

3. Theorem : Ψ is a meet complete homomorphism if and only if L_A is singleton.

Proof : Suppose Ψ is a meet complete homomorphism

Let $\beta \in$ LA such that $\beta \neq 0$

Let $G_1 = \{\chi_b^{1}, \chi_b^{\beta}\}$,

$G_2 = \{\chi_b^{0}, \chi_b^{\beta c}\}$

$G_1 \cap G_2 = \Phi$

$\Rightarrow$ $(G_1 \cap G_2)^{\sim} = \Phi_A$

$G_1^{\sim} = \chi_b^{1} \cup \chi_b^{\beta}$

$$= \chi_b^{\,1}, G_2^{\sim}$$
$$= \chi_b^{\,0} \cup \chi_b^{\,\beta c}$$
$$= \chi_b^{\,\beta c}$$
$$\Rightarrow \quad G_1^{\sim} \cap G_2^{\sim} = \chi_b^{\,1} \cap \chi_b^{\,\beta c}$$
$$= \chi_b^{\,\beta c} \qquad (\because \chi_b^{\,1} \supseteq \chi_b^{\,\beta c})$$
$$\therefore \quad (G_1 \cap G_2)^{\sim} \neq G_1^{\sim} \cap G_2^{\sim}, \text{ which is contradiction.}$$

So Ψ is not a meet complete homomorphism

Conversely, suppose L_A is singleton. To prove Ψ is a meet complete homomorphism

Suppose Ψ is not a meet complete homomorphism. Then there exist a nonempty family $(G_i)_{i \in I}$ such that $(\cap_{i \in I} G_i)^{\sim} \not\subset \cap_{i \in I} G_i^{\sim}$. Then there exist $\chi_b^{\,\beta} \subseteq$ RHS such that $\chi_b^{\,\beta} \not\subset$ LHS and $\beta \neq 0$ and $\beta \in L_A$, contradicting L_A is singleton

Hence Ψ is a meet complete homomorphism.

4. Proposition : Given B, then $B^{\sim} = G$

$$\Rightarrow \quad B = G^{\sim}$$

Proof: $B^{\sim} = G$

i.e $\Phi(B) = G$

$$\Rightarrow \quad \Psi\Phi(B) = \Psi(G)$$
$$\Rightarrow \quad 1(B) = \Psi(G) \text{ from 4.28}(e) \text{ that is,}$$
$$B = G^{\sim}.$$

5. Proposition: Given G is S-closed, then

$$G^{\sim} = B$$
$$\Rightarrow \quad B^{\sim} = G$$

Proof : Given G is S-closed implies $G = G^{\sim\sim}$

i.e. $\Phi\Psi(G) = G$

Let $B = G^{\sim}$

i.e. $B = \Phi(G)$

$$\Rightarrow \quad \Psi(B) = \Psi\Phi(G)$$
$$\Rightarrow \quad \Psi(B) = G$$

6. Lemma : $(G_1 \cap G_2)^{\sim} = G_1^{\sim} \cap G_2^{\sim}$,

for any two S-closed subsets G_1 and G_2 of FSP(A).

Proof : G_1 is S-closed implies $G_1 = B_1^{\sim}$,

where $B_1 = A$

and
$$B_1 = (B_{11}, B_1, \overline{B}_1\,(\mu_{1B_{11}}, \mu_{2B_1}), L_{B_1})$$
$$= \cup_{\chi_b^{\beta} \in G_1} \chi_b^{\beta}$$
$$B_{11} \supseteq B_1 = A$$
$$= \{b \mid \chi_b^{\,\beta} \in G_1\}, L_{B_1}$$
$$= L_A, \mu_{1B_{11}}\, b$$
$$= \vee_{\chi_b^{\beta} \in G_1} (\beta \vee \mu_{2A} b)$$
$$\mu_{2B_1} b = \mu_{2A}\, b$$

Similarly G_2 is S-closed implies $G_2 = B_2^{\sim}$,

where $B_2 = A$

and $B_2 = (B_{12}, B_2, \overline{B}_2\,(\mu_{1B_{12}}, \mu_{2B_2}), LB_2)$

$= \bigcup_{\chi_b^\beta \in G_2} \chi_b^\beta$

$B_{12} \supseteq B_2 = A$

$= \{b \mid \chi_b^\beta \in G_2\}, L_{B_2}$

$= L_A, \mu_{1B_{12}}\, b$

$= \bigvee_{\chi_b^\beta \in G_2} (\beta \vee \mu_{2A}\, b), 2_{B2}\, b$

$= \mu_{2A}\, b$

Need to show that $\mu_{1B_{11}} \wedge \mu_{1B_{12}} \geq \mu_{2B_1}\, \mu_{2B_2}$

But we have, $\mu_{1B_{11}}\, b \geq \mu_{2A} b$

$= \mu_{2B_1}\, b$

and $\mu_{1B_{12}}\, b \geq \mu_{2A}\, b$

$= \mu_{2B_2}\, b$

$\therefore$ $\mu_{1B_{11}}\, b \wedge \mu_{1B_{12}}\, b \geq \mu_{2A}\, b$

$= \mu_{2B_1}\, b \vee \mu_{2B_2}\, b$

Hence $B_1 \cap B_2$ non-empty.

Now, $G_1 \cap G_2 = B_1^{\sim} \cap B_2^{\sim}$

$= (B_1 \cap B_2)^{\sim}$

from 4.25 so that $(B_1 \cap B_2)^{\sim} = (G_1^{\sim} \cap G_2^{\sim})^{\sim}$

We have for any Fs-subset $H, H^{\sim} = G$

$\Rightarrow$ $H = G^{\sim}$.

Take $H = G_1^{\sim} \cap G_2^{\sim}$

and $G = G_1 \cap G_2$

Hence $(G_1 \cap G_2)^{\sim} = G_1^{\sim} \cap G_2^{\sim}$

7. Theorem : $(\cap_{i \in I} G_i)^{\sim} = \cap_{i \in I} G_i^{\sim}$

for any family $(G_i)_{i \in I}$ of S-closed subsets of FSP(A)

8. Define E : FS-SET$_*$ $\rightarrow$ SET, where $* = i$ or p

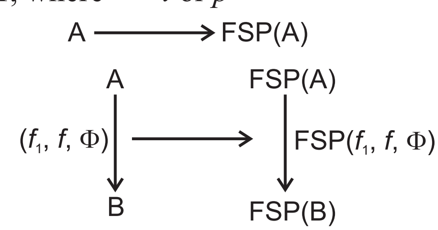

Figure 2

Such that FSP $(f_1, f, \Phi)(\chi_b^\beta) = \chi_{fb}^{\Phi\beta}$. Then, E dense functor.

Proof : For C$\in$ FSSET$_*$ $1_c = (1_{C_1}, 1_C, 1_{LC}) : C \rightarrow C$

$E(1_C)(\chi_b^\beta) = E(1_{C_1}, 1_C, 1_{L_C})\,(\chi_b^\beta)$

$= \text{FSP}\,(1_{C1}, 1_C, 1_{L_C})\,(\chi_b^\beta)$

$= \chi_{1_C b}^{1_{LC}\beta}$

$= \chi_b^\beta$

$= 1_{FSP(C)}\,(\chi_b^\beta)$

$= 1_{E(C)}\,(\chi_b^\beta)$

So that, $E(1_C) = 1_{E(C)}$

For $(f_1, f, \Phi) \in \mathrm{Hom}_i\,(B,C)$ and $(g_1, g, \Psi) \in \mathrm{Hom}_i\,(C, D)$ as in 2.3

$$\mu_{1C_1}|_C \circ f_1|_B \geq \Phi \circ \mu_{1B_1}) \text{ and } \mu_{2C} \circ f \leq \Phi \circ \mu_{2B} f$$

$$\mu_{1D_1}|_D \circ g_1|_B \geq \Psi \circ \mu_{1C_1}) \text{ and } \mu_{2D} \circ g \leq \Psi \circ \mu_{2C}$$

From 2.11, Composition of two increasing Fs-function is increasing, we can have

$$\mu_{1D_1}|_D \circ g_1 f_1|_B \geq \Psi\Phi \circ \mu_{1B_1} \text{ and } \mu_{2D} \circ gf \leq \Psi\Phi \circ \mu_{2B}$$

So that,
$$\bar{D}gfb = \mu_{1D_1} gfb \wedge (\mu_{2D}\, gfb)^c$$
$$\geq (\Psi\Phi \circ \mu_{1B_1} b \wedge [(\Psi\Phi \circ \mu_{2B})b]^c$$
$$E[(g_1, g, \Psi) \circ (f_1, f, \Phi)](\chi_b^{\beta}) = E[g_1 \circ f_1, g \circ f, \Psi \circ \Phi]$$
$$= \chi_{gfb}^{\Psi\Phi\beta}$$
$$= E(g_1, g, \Phi)\left(\chi_{fb}^{\Phi\beta}\right)$$
$$= E(g_1, g, \Psi) \circ E(f_1, f, \Phi)(\chi_b^{\beta})$$

So that
$$E[(g_1, g, \Psi) \circ (f_1, f, \Phi)] = E(g_1, g, \Psi) \circ E(f_1, f, \Phi).$$

Hence E is functor.

Let $B \in (\mathrm{SET})_o$. We have to find $B \in (\mathrm{FSSET})_o$ such that $E(B) = FSP(B)$ is isomorphic with B.

Consider $B = \cup_{b \in B}\, \chi_b^0$

We have $FSP(B) = \{\chi_b^0 \mid b \in B\}$

Define *f* : $FSP(B) \rightarrow B$ by $f(\chi_b^0) = b$

Clearly f is a bijection.

Hence E is a dense functor

9. Remark : Note that $\chi_b^0 = \Phi_A$ Fs-empty set of second kind if, and only if

$$\chi_b^0 = (D, D, \bar{D}(\mu_{1D_1}, \mu_{2D}), L_D),$$

where $\mu_{1D_1} = \mu_{2D}$

10. Remark : Φ_A-Fs-empty set of second kind can be treated as Fs-point.

11. Definition: Let $F, G{:}A \rightarrow B$ be a pair of functors. A natural transformation η between two functor F and G –denoted by $\eta : F \rightarrow G$ is defined as follows with the help of the following diagram which should be commutative .That is, $Gf \circ \eta_A = \eta_B \circ Ff$

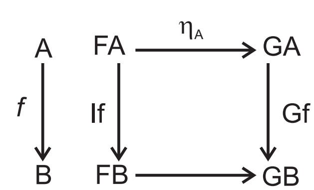

Figure 3

FS-SET = Category with Fs-sets (with complete Boolean algebra valued membership functions) as objects and Fs-functions [3.1] as morphisms between Fs-sets.

FS-SETND = Category with Fs-sets (with non-degenerating complete Boolean algebra valued membership functions) as objects and Fs-functions as morphisms between Fs-sets.

12. Theorem : There is a natural transformation between the functors I and $G \circ E$ where $G \circ E$ composition of functors G-as described below and E in 3.35 and I: $(\mathrm{FS\text{-}SET})_* \rightarrow (\mathrm{FS\text{-}SET})_*$ is the identity functor where $* = i$ or p

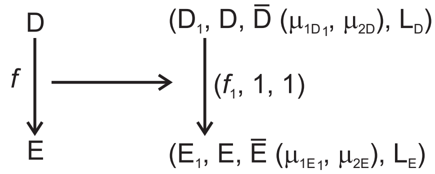

Figure 4

Where $\mu_{1D_1} = \mu_{2D} = \infty$

and $\mu_{1E_1} = \mu_{2E} = \infty$

and $L_D = L_E = 1_\infty$

Proof: E : FF-SSET$_*$ → SET

$$E(f_1, f, \Phi)(\chi_b^{\beta}) = FSP(f_1, f, \Phi)(\chi_b^{\beta}) = \chi_{fb}^{\Phi\beta}$$

$$\text{F-SET}_* \xrightarrow{E} \text{SET} \xrightarrow{G} \text{FS-SET}_*$$

$$\text{F-SET}_* \xrightarrow{G \circ E} \text{FS-SET}), \; I : \text{FS-SET}_* \rightarrow \text{FS-SET}_*$$

$$\eta : I = G \circ E$$

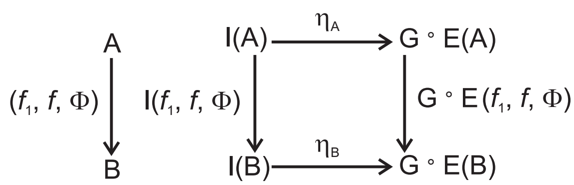

Figure 5

To be proved $G \circ E(f_1, f, \Phi) \circ \eta_A = \eta_B \circ I(f_1, f, \Phi)$, where

$$\eta_A = (C_{A_1}, C_A, C_{L_A}), \text{ where}$$

$$C_{A_1} : A_1 \rightarrow FSP(A)$$

$$C_A : A \rightarrow FSP(A)$$

$$C_{L_A} : L_A \rightarrow \infty$$

$$a_1 \rightarrow \chi_{a_1}^{0}$$

$$a \rightarrow \chi_a^{0}$$

$$\alpha \rightarrow \infty$$

$$\eta_B \rightarrow (C_{B_1}, C_B, C_{LB}), \text{ where}$$

$$C_{B_1} : B_1 \rightarrow FSP(A)$$

$$C_B : B \rightarrow FSP(A)$$

$$C_{L_B} : L_B \rightarrow \infty$$

$$b_1 \rightarrow \chi_{b_1}^{0}$$

$$b \rightarrow \chi_a^{0}$$

$$\beta \rightarrow \infty$$

So that, $\eta_B \circ I(f_1, f, \Phi) : I(A) \rightarrow G \circ E(B)$

$G \circ E(f_1, f, \Phi) \circ \eta_A : I(A) \rightarrow G \circ E(B).$

We have
$$
\begin{aligned}
G \circ E(f_1, f, \Phi) \circ \eta_A &= G \circ E(f_1, f, \Phi) \circ (C_{A_1}, C_A, C_{L_A}) \\
&= G \circ (Ef_1, f, \Phi) \circ (C_{A_1}, C_A, C_{L_A}) \\
&= G(g) \circ (C_{A_1}, C_A, C_{L_A}) \\
&= (g, g, 1_\infty) \circ (C_{A_1}, C_A, C_{L_A}) \\
&= (g \circ C_{A_1}), \circ g \circ C_A, 1_\infty \circ C_{L_A}) \\
\eta_B \circ I(f_1, f, \Phi) &= \eta_B \circ (f_1, f, \Phi) \\
&= (C_{B_1}, C_B, C_{LB}) \circ (f_1, f, \Phi) \\
&= (C_{B_1} \circ f_1, C_B \circ f, C_{LB} \circ \Phi)
\end{aligned}
$$

To be proved

1. $g \circ C_{A_1} = C_{B_1} \circ f_1$
2. $g \circ C_A = C_B \circ f$
3. $1_\infty \circ C_{L_A} = C_{L_B} \circ \Phi$

1. $A_1 \xrightarrow{C_A} FSP(A) \xrightarrow{g} FSP(B)$

$$
\begin{aligned}
(g \circ C_{A_1})\, a_1 &= g(C_{A_1}\, a_1) \\
&= g\left(\chi^0_{a_1}\right) = \chi^0_{fa_1} = \chi^0_{f_1 a 1}
\end{aligned}
$$

1. $L_A \rightarrow \xrightarrow{f_1} B_1 \xrightarrow{CB_1} FSP(B)$

$$
\begin{aligned}
(C_{B_1} \circ f_1)\, a_1 &= C_{B_1}(f_1\, a_1) \\
&= \chi^0_{f_1 a 1}
\end{aligned}
$$

Hence $$g \circ C_{A_1} = C_{B_1} \circ f_1$$

2. $A \xrightarrow{C_A} FSP(A) \xrightarrow{g} FSP(B)$

$$
\begin{aligned}
(g \circ C_A)\, a_1 &= g(C_A a) \\
&= g(\chi_a{}^0) \\
&= \chi_{fa}{}^0
\end{aligned}
$$

$A \xrightarrow{f_1} B \xrightarrow{CB} FSP(B)$

$$
\begin{aligned}
(C_B \circ f)a &= C_B\,(fa) \\
&= \chi^0_{f_1 a 1}
\end{aligned}
$$

Hence $$g \circ C_A = C_B \circ f$$

3. $L_A \xrightarrow{CL_A} \infty \xrightarrow{\infty} \infty$

$$
\begin{aligned}
(1_\infty \circ C_{L_A})\alpha &= 1_\infty\,(C_{LA}\,\alpha) \\
&= 1_\infty\,(\infty) \\
&= \infty
\end{aligned}
$$

$L_A \xrightarrow{\Phi} L_B \xrightarrow{CL_B} \infty$

$$
\begin{aligned}
(C_{LB} \circ \Phi)\alpha &= C_{L_B}\,(\Phi(\alpha)) \\
&= \infty \\
&= (1_\infty \circ C_{L_A})\alpha
\end{aligned}
$$

Hence $$1_\infty \circ C_{L_A} = C_{L_B} \circ \Phi$$

From (1), (2) and (3) clearly $G \circ E(f_1, f, \Phi) \circ \eta_A = \eta_B \circ I(f_1, f, \Phi)$

That is, η from I into $G \circ E$ is a natural transformation.

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