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Estimation of Fractal Dimension: A Survey with Numerical Experiments and Software Description

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Abstract: A unifying discussion of basic concepts for the numerical estimation of fractal dimensions and multifractal spectra is presented. For fractal dimensions focus is on box-counting methods, multifractal spectra are introduced by generalized dimensions. Due to limited resolution of empirical data, a discrepancy between analytical and numerical estimation of fractal dimensions can be observed, the corresponding problem of validation is discussed. Numerical estimation is supported by implemented software which can be downloaded. Numerical experiments are presented and simplify the application of the software, which contains numerous examples.

Mathematical Subject Classification: 28A80, 65Z05.

Keywords: Fractal dimension, Box-counting method, Extended counting method, Generalized dimensions, Multifractal spectrum, Numerical methods and software.

1. Introduction

There are many research directions were the theory of fractals occupies a significant part: plasma physics, turbulence, porous media, biological systems, image compression, data traffic, financial markets, neuroscience, etc. Theoretical concepts such as deterministic and random fractals, multifractal measures, fractal dimensions, singularity spectra and fractional analysis play an important role in the present-day development of mathematics and natural science (see, for example, [2, 5, 7, 8, 18, 19, 20, 31]).

Smooth lines, surfaces and polyhedrons are usually not regarded as fractals. Specific for fractals is some irregularity which cannot be described properly by the concepts of traditional geometry. Fractals reveal a fine structure with details on several smaller resolution-distances or scales. On these scales sometimes a self similarity is present, this may exist only approximately or in a random sense. Real examples of fractals are fronds of the fern, cloud boundaries, coastlines, Brownian motion of particles in a liquid, the surface of the human brain cortex, or fMRI time series measuring brain activity. There is no strict mathematical definition separating a fractal from a non-fractal [7], accordingly the quantification of the degree of fractality, the fractal dimension, can be applied also to smooth geometric objects. But for non-fractals usually traditional geometry with minor numerical expenditure is equally informative.

The main property of a fractal is its fractal dimension, which is typically non-integer in contrast to the traditional topological dimension. For instance, the fractal dimension of the Cantor set on the real line is

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0.63, while its topological dimension is 0. The topological dimension of the Sierpinski carpet on the plane and the Menger sponge in space is equal to 1, but their fractal dimensions are equal to 1.58 and 2.73, respectively. The concept of dimension, and of a fractal dimension in particular, is not trivial. Like in case of the fractal, the fractal dimension is not defined uniquely, there exist many non equivalent definitions [3]. However a list of characteristic properties of a fractal dimension can be given [7, 26]. A fractal dimension $FD(F) \in \mathbb{R}$ of a set F has at least the following properties:

1. motion invariance: $FD(f(F)) = FD(F)$, if f is a translation or a rotation
2. scale invariance: $FD(f(F)) = FD(F)$, if f is a similarity or an affine transformation
3. monotony: if $E \subseteq G$, then $FD(E) \leq FD(G)$
4. maximum property: $FD(E \cup G) = \max\{FD(E), FD(G)\}$
5. extension of topological dimension: If M is a smooth m-dimensional manifold (curve, surface, etc.), then $FD(M) = m$.

Fractal dimension characterizes the irregularity or complexity of the whole set under investigation and is governed by the most complex area of the set because of property 4. A refinement of the concept of a fractal is the multifractal. There the degree of irregularity in the object under consideration changes locally. Real examples are the energy dissipation in turbulent flows or the mass distribution of gold on earth. To quantify multifractality of e.g. a time series, a local measure of the singularity behavior of the series (the Holder exponent α) is introduced. The value of the singularity spectrum $f(\alpha)$ quanties the *FD* of the time points which correspond to α [7].

The analytical definitions of FDs and the spectrum use a limit for "scale $\rightarrow$ 0" assuming that the set is known for arbitrary fine detail. Real data with "scale > 0" need often sophisticated numerical methods to approximate *FD* or the spectrum which may pose restrictions to their applicability. For instance, the numerical calculation of the frequently used box counting dimension is based on a linear regression of a log-log plot. For self-similar irregular data this plot is approximately linear, however there exist fractals without this property [26]. Numerical analysis of fractal and multifractal structures, and, the estimation of their fractal dimensions or spectra is a significant research problem. In spite of the variety of developed numerical methods (see, for example, [7, 29]) the problem of their validity remains topical.

In the following we present a unifying discussion of basic concepts and introduce software with examples. The software is available on the internet, see [35], and shall encourage the researchers in different applied disciplines to make use of methods quantifying complexity. Our algorithms were developed through several years of research on fractal dimensions, see, for example, [10, 11, 16, 22, 23, 24, 26, 27, 28].

2. Description of Numerical Methods

In 1919 Felix Hausdorff presented his concept of dimension [13]. The Hausdorff dimension of a bounded set $A \subset R^n$ is defined as a (unique) real number

$$\dim_H A = \sup\{d : S_d(A) = \infty\} = \inf\{d : S_d(A) = 0\},$$

where

$$S_d(A) \lim_{\varepsilon \to 0} \inf \sum_i [\operatorname{diam}(B_i)]^d,$$

and infimum in the last equality is taken over all collections of open balls $\{B_i\}$ in R^n with diameters not greater than ε whose union covers A. For numerical analysis it is hard to use this definition directly. That is why many other concepts and specific computational methods where proposed to study fractal properties

of sets. One of the approaches that enable us to measure fractal dimension is connected with the concept of the box(-counting) dimension and the box-counting method.

Below we describe a few estimators of the generalized dimensions (Rényi dimensions), including box-counting, information and correlation dimensions.

2.1 Box-Counting Method and its Modifications

According to [7] p. 41, the definition of box dimension "goes back at least to the 1930s and it has been variously termed Kolmogorov entropy, entropy dimension, capacity dimension (a term best avoided in view of potential theoretic associations), metric dimension, logarithmic density and information dimension".

Definition of box dimension: If a bounded set A is a subset of R^n, then the box dimension is defined as a limit (assuming its existence)

$$\dim_B A = \lim_{\varepsilon \to 0} \frac{\ln N_*(\varepsilon)}{-\ln \varepsilon}, \tag{1}$$

where $N_*(\varepsilon)$ is the smallest number of sets of diameter ε which can cover A. It can be shown that

$$0 \le \dim_H A \le \dim_B A \le n,$$

and

$$\dim_B A = n - \lim_{\delta \to 0} \frac{\ln \mathrm{vol}(A_\delta)}{\ln \delta}, \tag{2}$$

where $\mathrm{vol}(A_\delta)$ is the n-dimensional volume of the δ-neighborhood

$$A_\delta = \{x \in R^n : |x - y| \le \delta \ \text{ for some } \ y \in A\}.$$

The quantity (2) also is often called Minkowski or Minkowski-Bouligand dimension (see details in [7], p. 46).

Obviously, direct usage of (1) for numerical estimation of box dimension is meaningless. Consider n-dimensional cube A with volume $V = X^n$ and assume that X/ε is integer. Then $N_*(\varepsilon) = (X/\varepsilon)^n$ and (1) gives the following "approximation" r of $\dim_B A = n$:

$$r = n\left(1 - \frac{\ln X}{\ln \varepsilon}\right) = n - \frac{\ln V}{\ln \varepsilon}.$$

So, the value r is strongly dependent on the size of the set and it can not be used as a reliable approximation of dimension n. This simple consideration brings to conclusion, that to estimate dimension we ought to compute $N_*(\varepsilon)$ for several values of ε.

Two-scale box counting dimensions: In what follows, we assume that the inclusive n-dimensional space is represented as a union of disjoint n-dimensional cubes (boxes) of size ε, and we can count number $N(\varepsilon)$ of all cubes, which contain points of the set A. Informally, the statement "the box-counting dimension of the fractal A is equal to d" means that the 'd-dimensional volume' V of the bounded set A is finite, non-zero and can be approximated by

$$V \approx \varepsilon^d N(\varepsilon), \tag{3}$$

at least for small ε. If we know $N(\varepsilon_1)$ and $N(\varepsilon_2)$ for two different 'scales' ε_1, ε_2, then (3) gives us the following estimator

$$d \approx \dim_B[\varepsilon_1, \varepsilon_2](A) = \frac{\ln[N(\varepsilon_1)/N(\varepsilon_2)]}{\ln[\varepsilon_2/\varepsilon_1]}, \tag{4}$$

while

$$V \approx \varepsilon_1^{\dim_B[\varepsilon_1,\varepsilon_2](A)} N(\varepsilon_1) = \varepsilon_2^{\dim_B[\varepsilon_1,\varepsilon_2](A)} N(\varepsilon_2). \tag{5}$$

A numerical method based on (4) we shall call the *two-scale box-counting method*. In contrast to the value $\ln N(\varepsilon)/|\ln \varepsilon|$, the two-scale box counting method is stable with respect to the size of the set.

Extended counting method: A modification of the two-scale box counting method was proposed in the papers [26, 27, 28]. This modification, called *extended counting method*, can be described in the following way. Assume that ε is the size of n-dimensional cubes of a given fine box grid. Let us fix a natural number M (which is a parameter of the extended counting method) and consider cubes of size $M\varepsilon$ containing M^n boxes of the fine grid. Such larger cubes will be called *exploratory cubes*. For an exploratory cube E by N_E we denote the number of boxes of the fine grid belonging to E which contain at least one point of set A. The value

$$d_E = \frac{\log N_E}{\log M}$$

can be interpreted as the two-scale box-counting dimension of $A \cap E$ with $\varepsilon_1 = \varepsilon$ and $\varepsilon_2 = M\varepsilon$. According to the extended counting method dimension of A is estimated by

$$x\dim[M](A) = \max d_E = \max \frac{\log N_E}{\log M}, \tag{6}$$

where maximum is taken over all exploratory cubes E. One of the basic ideas underlying the extended counting method is to reproduce the *maximum property* of fractal dimension:

$$\dim(A_1 \cup A_2) = \max\{\dim(A_1), \dim(A_2)\}.$$

The detailed study and applications of the extended counting method are presented in [27]. Note, that estimator (6), called below x-dimension, depends on two parameters: size of atomic cubes ε and exploratory factor M. A fast implementation of the method is given in [26].

Conventional box-counting method and its "adjusted" modification. Similar to the two-scale method, the conventional (multi-scale) box counting is based on relation (3):

$$d \ln \varepsilon - \ln V \approx -\ln N(\varepsilon). \tag{7}$$

The values $N(\varepsilon_i)$ are computed for several ε_i and then d (together with $\ln V$) is estimated by the method of least squares. Typically $\varepsilon_i = \varepsilon_{i-1}/2$ and several tricks are known to minimize errors and computational costs (see, for example, [14, 17, 32, 33]).

Relation (3) can be rewritten using a function $c(\varepsilon)$

$$V = c(\varepsilon)\varepsilon^d N(\varepsilon), \tag{8}$$

with $c(\varepsilon) \to 1$ for $\varepsilon \to 0$. It seems reasonable to take $c(\varepsilon) = \exp(\varepsilon)$ with $\alpha < 0$, as we assume that $c(\varepsilon) < 1$ because the 'd-dimensional volume' of the set should be less than the 'd-dimensional volume' of the covering boxes. Then (7) transforms to

$$d \ln \varepsilon - \ln V + \alpha\varepsilon = -\ln N(\varepsilon). \tag{9}$$

If $N(\varepsilon_i)$ are known for three or more different values of ε, then all three parameters d, $\ln V$ and α can be found by least squares (with constraints $d > 0$, $V > 0$, $\alpha < 0$). Such a method to find box dimension d will be called "adjusted".

2.2 Generalized Dimensions, the Renyi Entropy, and the Multifractal Spectrum

For a set $A \subset R^n$ with a probability measure P concentrated on A the generalized dimensions $\dim_q(A, P)$ are defined [29] by formula

$$\dim_q(A, P) = \lim_{\varepsilon \to 0} \frac{S_q(\varepsilon)}{|\ln \varepsilon|}, \qquad q \in (-\infty, +\infty), \tag{10}$$

if the limit exists. Here

$$S_q(\varepsilon) = \ln \left(\left[\sum_i p_i^q(\varepsilon) \right]^{\frac{1}{1-q}} \right), \qquad q \neq 1, \tag{11}$$

is the Renyi entropy of a discrete distribution with probabilities $p_i(\varepsilon)$ and

$$S_1(\varepsilon) = -\sum_i p_i(\varepsilon) \ln p_i(\varepsilon), \tag{12}$$

is the well-known entropy that can be obtained from (11) as $q \to 1$. The sums in (11) and (12) are taken over a lattice of disjoint n-dimensional cubes U_i^{ε} of size ε, and $p_i(\varepsilon)$ is a simplied notation for $P(U_i^{\varepsilon})$, $\sum_i p_i(\varepsilon) = 1$.

It can be shown using the Jensen's inequality that the Renyi entropy S_q and the generalized dimension $D_q = \dim_q(A, P)$ are nonincreasing functions with respect to q,

$$S_\infty \le S_q \le S_{q'} \le S_{-\infty}, \quad D_\infty \le D_q \le D_{q'} \le D_{-\infty}, \quad \text{for} \quad q > q', \tag{13}$$

and $S_q = S_{q'}$ for $q \neq q'$ if and only if the discrete distribution p is uniform. The value S_∞ in (13) is called min-entropy (or the Chebyshev entropy), $S_\infty = \min_i [-\ln p_i(\varepsilon)] = -\ln \max_i p_i(\varepsilon)$, S_0 is called Hartley's entropy, S_1 is called the Kolmogorov entropy, and $S_{-\infty} = \max_i [-\ln p_i(\varepsilon)] = -\ln \min_i p_i(\varepsilon)$.

Box dimension corresponds to $q = 0$, and for $q = 1$, 2 the generalized dimensions are called information and correlation dimensions, respectively:

$$\dim_1(A, P) = \lim_{\varepsilon \to 0} \frac{-\max \sum_i p_i(\varepsilon) \ln p_i(\varepsilon)}{|\ln \varepsilon|}, \tag{14}$$

$$\dim_2(A, P) = \lim_{\varepsilon \to 0} \frac{-\max \sum_i p_i^2(\varepsilon)}{|\ln \varepsilon|}. \tag{15}$$

Correlation dimension D_2 is associated with the probability $P_2(\varepsilon)$ that the distance between two points sampled from A according to P is smaller than ε:

$$P_2(\varepsilon) \approx \lim_{N \to +\infty} \frac{1}{N^2} f(\varepsilon, N) \approx \sum_i p_i^2(\varepsilon) \sim \varepsilon^{D_2}.$$

The second term in this relation, where $f(\varepsilon, N)$ is the number of pairs (from a set of N points) with distance smaller than ε, is called *correlation integral* [9].

Numerical estimation of generalized dimensions. As we discussed before, the formal definitions are not used directly for numerical estimation. Let i be a number of a cube of ε-lattice, and assume now that we know which 'part' $p_i(\varepsilon) \in [0, 1]$ of the set A is contained in this cube, while $\sum p_i(\varepsilon) = 1$ for the sum over all disjoint cubes. Instead of d-dimensional volume (3) now we consider

$$V_q \approx \varepsilon^d \left[\sum_i p_i^q(\varepsilon) \right]^{\frac{1}{1-q}}, \qquad q \neq 1, \tag{16}$$

$$V_1 \approx \varepsilon^d \exp\left(- \sum_i p_i(\varepsilon) \ln p_i(\varepsilon) \right). \tag{17}$$

(Note, that for an "ordinary" set $A \in R^n$ with volume V and uniform probability distribution P on A we have $V_q = V$ for all q.) Then the analog of (4) gives the following two-scale estimators for the generalized dimensions $\dim_q(A, P)$.

$$d = \frac{1}{q-1} \frac{\ln \left[\sum_i p_i^q(\varepsilon_1) \Big/ \sum_i p_i^q(\varepsilon_2) \right]}{\ln [\varepsilon_1 / \varepsilon_2]}, \quad q \neq 1, \tag{18}$$

$$d = \frac{\sum_{i=1} p_i(\varepsilon_1) \ln p_i(\varepsilon_1) - \sum_{i=1} p_i(\varepsilon_2) \ln p_i(\varepsilon_2)}{\ln [\varepsilon_1 / \varepsilon_2]}, \quad (q = 1). \tag{19}$$

Obviously, the generalized dimensions can be found by least squares from (7) and (9), where $N(\varepsilon)$ should be substituted by $\exp(S_q(\varepsilon))$.

Multifractals. If generalized dimensions $\dim_q(A, P)$ are equal for all q, then the set A (with measure P) is called *monofractal*. The concept of *multifractal* (or *singularity*) *spectrum* $f(\alpha)$ is used in so-called multifractal analysis (see, for example, [7, 29]). For self-similar multifractals (see [7], p. 288) a specific value $f(\alpha)$ of the multifractal spectrum at point α for a measure P in R^n, $n \geq 2$, is the Hausdorff dimension of the set of points $x \in R^n$ with *local dimension* (or *Hölder exponent*) equal to α. The Hölder exponent is defined by

$$\lim_{r \downarrow 0} \frac{\log P(B(x, r))}{\log r} = \alpha,$$

where $B(x, r)$ denotes the open ball of radius r centered in x. The generalized dimensions are connected with the multifractal spectrum $f(\alpha)$:

$$f(\alpha(q)) = \tau(q) + q\alpha(q), \tag{20}$$

where function $\alpha(q)$ satisfies

$$\alpha(q) = -\frac{d}{dq} \tau(q), \tag{21}$$

or

$$\frac{d}{d\alpha} [q\alpha - f(\alpha)] = 0,$$

$$\frac{df}{d\alpha} = q,$$

and the values

$$\tau(q) = (1 - q) D_q = f(\alpha(q)) - q\alpha(q),$$

are called *mass exponents*. Equations (20), (21) represent a Legendre transform from the variables q and τ to the variables α and f. Singularity spectrum $f(\alpha)$ is a concave function for a self-similar multifractal with $\max_\alpha f(\alpha) = D_0$; see, for example, [12].

3. A BRIEF DESCRIPTION OF THE SOFTWARE WITH EXAMPLES OF NUMERICAL EXPERIMENTS

In [35] we collected software (MS Windows executables) to estimate fractal dimension for specific classes of time series, surfaces, and general sets in 2D and 3D spaces. Below we present a list of programs that can be downloaded from [35] with a brief description. For multifractals we refer to software from [35, 36, 37].

3.1 Dimension Estimators for Time Series

Program **xDim_ts** is destined to estimate fractal dimension of a time series by the *extended counting method* [26, 27, 28]. Here we realized method X2 with scaling factor 1 and exploratory factor equal to the length of time series divided by 10, see a detailed description of the algorithm in [22]. In program **vDim_ts** the *variance counting method* is realized. This is a simple (two-scale) and fast method based on analysis of variances of the time series increments for the smallest step and the step twice larger. The method is appropriate to estimate Hausdorff dimension for some types of random processes (in particular, Gaussian processes with stationary increments). A study and a few applications of the variance counting method are presented in [23]. About comparison of these two methods realized in programs **xDim_ts** and **vDim_ts** see [22]. The programs are easy to use and they don't need any subsidiary input parameters. Computing time for program **xDim_ts** can be considerably larger than for **vDim_ts**, particularly if the studied sample is large. Additional examples of time series with dimensions 1.1, 1.3, 1.5, 1.7, 1.9 are included in the downloaded zip-file, see Fig. 1. These time series are realizations of the fractional Brownian motion and they can be used to test the programs. A program to simulate realizations of the fractional Brownian motion can be downloaded, as well.

3.2 Estimation of Multifractal Spectra

In a comparative study by Turiel *et al.*, [34] multifractal time series, $s(t)$, $t \in T \subseteq R^+$, are analyzed by four different numerical methods. Benchmark tests favour the gradient modulus wavelet projection method (GMWP). Corresponding software can be downloaded from Turiel's web page [36]. Another source of convenient software can be found on the web page [37]. The GMWP for time series is described in the following: In this method for the derivative $\nabla s(i) = s(i+1) - s(i)$ of signal s, wavelet projections $s \otimes \Psi_r(t) = \int d\tau \left| \nabla s(\tau) \right| \Psi_r\left(\frac{t-\tau}{r}\right) / r$ for different scales r are calculated. The local Holder exponents $\alpha(t)$ can then be estimated by a linear regression of the log-log plot of $s \otimes \Psi_r(t) \sim r^{\alpha(t)}$ for different scales r at every time point t. As analyzing wavelet Ψ the positive Lorentz function $\Psi(t) \sim 1/(1 + [t/r]^2)^\gamma$, for $\gamma = 0.5$ is applied. Positive wavelets allow smaller r, because the minimal resolution of a wavelet projection is restricted by the number of zero-crossings of the wavelet. The singularity spectrum $f(\alpha)$ is estimated finally by the histogram method [7], $f(\alpha) = 1 - \log(\rho(\alpha)/\max_\alpha \rho(\alpha)) / \log(\min r)$, where $\rho(\alpha)$ is the frequency histogram of the set $\{\alpha(t) \mid t \in T\}$. As an example of a GMWP application, in Fig. 2 for two mice the EEG time series are presented: left panel (A) for a normal mouse, right panel (B) for a genetically modified knock out mouse type with epileptic spikes and a higher noise level. The calculated singularity spectra $f(\alpha)$ are presented in panels (C)-(D). For the normal mouse (C) the range of the Holder exponents is roughly [0.7, 0.9], while for the knock out mouse the range is [0.6, 0.95]. The spectrum (indicating for fixed α the fractal dimension of the corresponding set of time points) has for the knock out mouse a distinct shoulder for low α. Low α corresponds to high spikes, the low dimensions of the shoulder to the fact that high spikes are often isolated. Another example for an application of fractal concepts to such data can be found in [24].

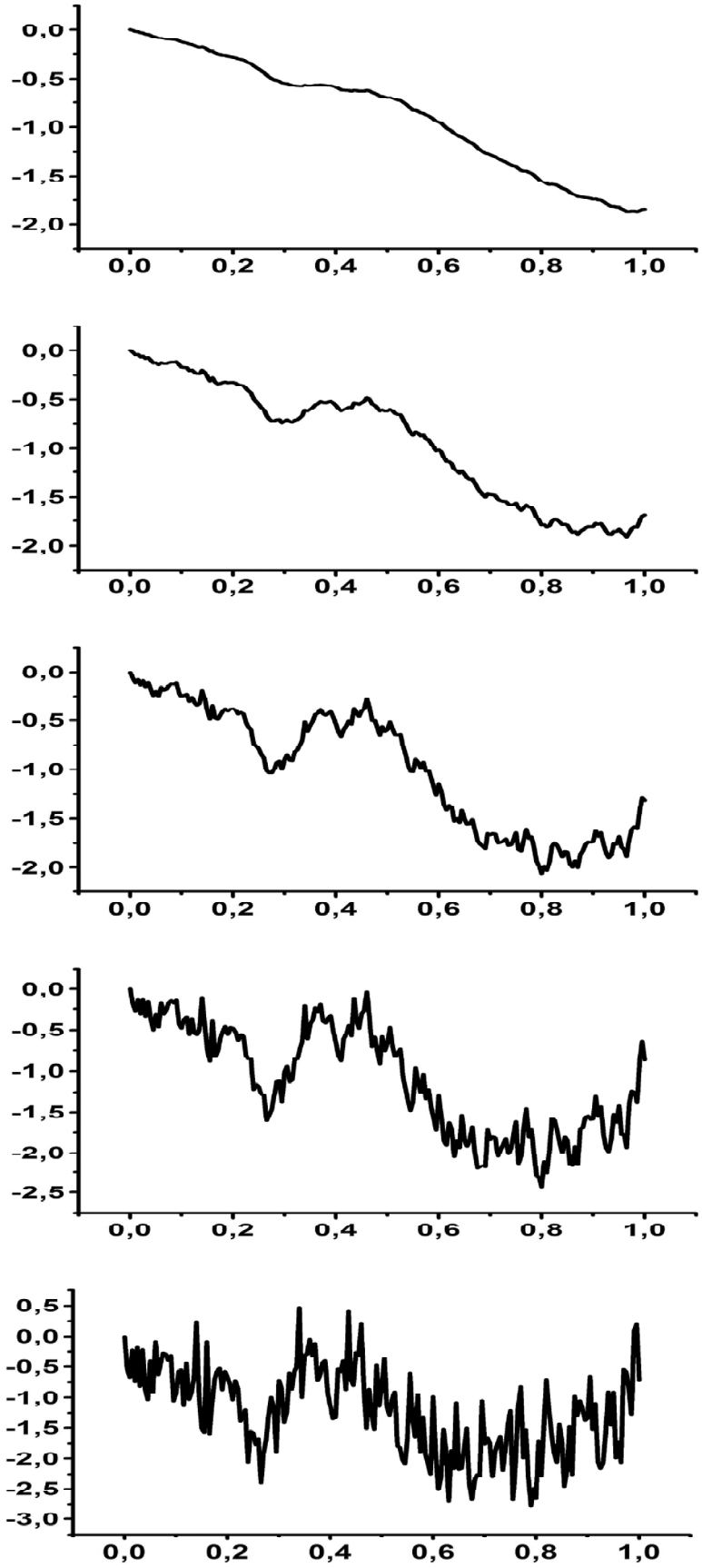

Figure 1: Realizations of the Fractional Brownian Motion with Dimensions 1.1, 1.3 (Two Upper Diagrams), 1.5 (Conventional Brownian Motion is in the Middle), 1.7 and 1.9 (Two Lower Diagrams)

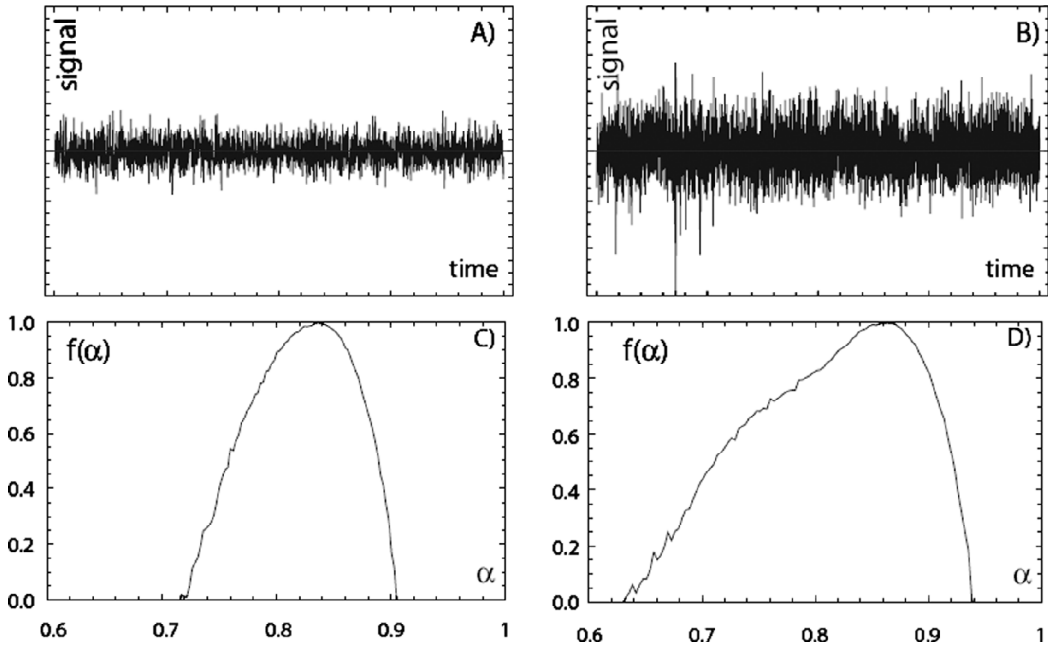

Figure 2: Comparison of EEG Time Series and Their Singularity Spectra for a Normal Mouse (Panels A, C), and for a Knock Out Mouse (Panels B, D).

3.3 Measuring Fractal Dimension of Random Surfaces

Several estimates can be computed by program **get-vDim-2D** to measure fractal dimension of a random field on the plane (i.e. dimension of its graph in 3D space). The estimates are versions of the variance counting method described in [24] and they depend on two input parameters. Two ASCII files are included to test the program. The files contain realizations of random fields with dimensions 2.5 (a specific isotropic Gaussian moving average field) and 2.75 (fractional Brownian motion). In addition, programs to simulate realizations of the isotropic Gaussian moving average field, the fractional Brownian motion and sheet can be downloaded.

Remark 1: To simulate the fractional Brownian motion and sheet we used a conventional algorithm for Gaussian distributions based on the Cholesky decomposition of the correlation matrix. This method is exact, but it is slow and storage-consuming for comparatively large samples. About other simulation algorithms for the fractional Brownian processes and fields see, for example, [21, 25, 30].

Remark 2: Variance counting method from [22, 23, 24] is a simplified (two-scale) variant of a general approach developed for some classes of random processes and fields, see [4, 5, 6, 15] and references therein.

3.4 Estimation of fractal Dimension for Sets on the plane and in space

To estimate fractal dimensions of sets on the plane we propose to use several programs: **012-dim_2D**, **xdim_2D**, **gen-dim-tab_2D**, **gendim2D**. Twoscale methods are realized in programs **012-dim_2D** and **gen-dim-tab_2D** to compute the generalized dimensions, including box, information and correlation dimensions. Program **xdim_2D** can be used to estimate the fractal dimension by the extended counting method, and program **gendim2D** is aimed to compute the generalized dimensions by least squares (including adjusted modifications). Formally, program **gendim2D** covers possibilities of "two-scale" programs **012-dim_2D** and **gen-dim-tab_2D**, but it may turn out to be less convenient in some cases. Several 2D fractal sets (the Sierpinski carpet and its stochastic modifications, the Cantor dust, the fractal percolation model, the Henon attractor) can be simulated by supplementary programs presented in [35].

Example 1: Let us consider a realization of stochastic carpet in Fig. 3. This realization was simulated by six iterations as a 4096 × 4096-matrix with zero-one elements. For each iteration every non-empty square of the previous iteration is divided into 16 smaller squares, then $16 - K$ ($K = 13$) of them are

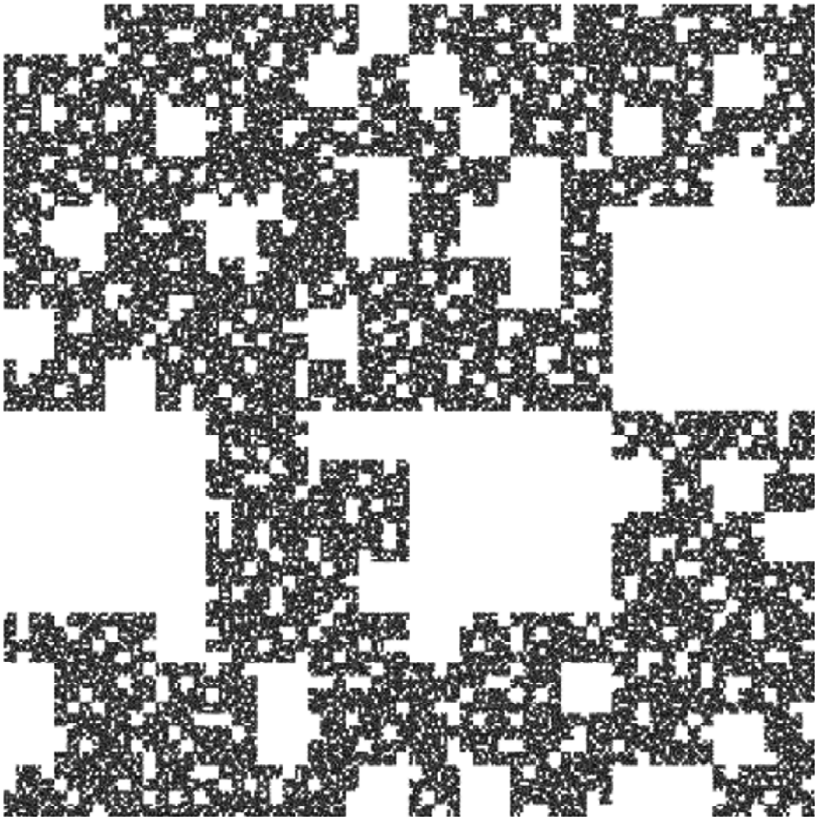

Figure 3: A Realization of a Stochastic Carpet on the Plane with Fractal Dimension 1.85022

randomly chosen and claimed to be empty (i.e., they do not contain points of the set). The estimates of box, information and correlation dimensions calculated by program **gendim2D** are presented in Table 1. The estimates were obtained by least squares using boxes of size 1, 2, 4, and 8. One can see from Table 1 that adjusted estimates based on (9) are more precise. The exact values of the dimensions are equal to $\ln 13/\ln 4 \approx 1.85022$ in this case (for infinite number of iterations). Results for similar numerical experiments with $K = 5$, 8 and respective dimensions $\ln 5/\ln 4 \approx 1.16096$, $\ln 8/\ln 4 = 1.5$ show that adjusted method can considerably improve the estimates of dimensions.

Table 1
Estimates of Box, Information and Correlation Dimensions of Three Realizations of Stochastic Carpets (See Details in the Text)

dimension	*box*	*information*	*correlation*
exact = 1.85022 ($K = 13$)			
least squares	1.82026	1.82729	1.83319
"adjusted" least squares	1.85022	1.85032	1.85038
exact = 1.5 ($K = 8$)			
least squares	1.41147	1.43393	1.4525
"adjusted" least squares	1.49979	1.50006	1.50011
exact = 1.16096 ($K = 5$)			
least squares	1.04981	1.07311	1.09549
"adjusted" least squares	1.16188	1.1628	1.16406

For sets in 3D space we propose two programs: **x-dim3D** to estimate fractal dimension by the extended counting method and **box-dim3D** to estimate the box dimension. A fast box counting algorithm from [14] is realized in program **box-dim3D**, where a cube containing a fractal set is consequently divided in 8, 8^2, ..., 8^{10} boxes. This program generates a table of box dimensions by least squares for different scales and propose an optimal estimate by a heuristic algorithm. In contrast to this realization of box-counting method, program **x-dim3D** requires two input parameters (see description of the method in the previous section). Simulation programs for the Menger sponge (3D analogue of the Sierpinski carpet) and the fractal foam are presented in [35] as well.

Example 2: The Menger sponge (dimension $\log 20 / \log 3 \approx 2.727$), simulated by 3 iterations as an array of size $81 \times 81 \times 81$ with $0 - 1$ elements, was analyzed by programs **x-dim3D** and **box-dim3D**. Program **box-dim3D** gave out value 2.764. The estimates generated by program **x-dim3D** are presented in Fig. 4 for different exploratory factors (the size of atomic cubes is equal to 1).

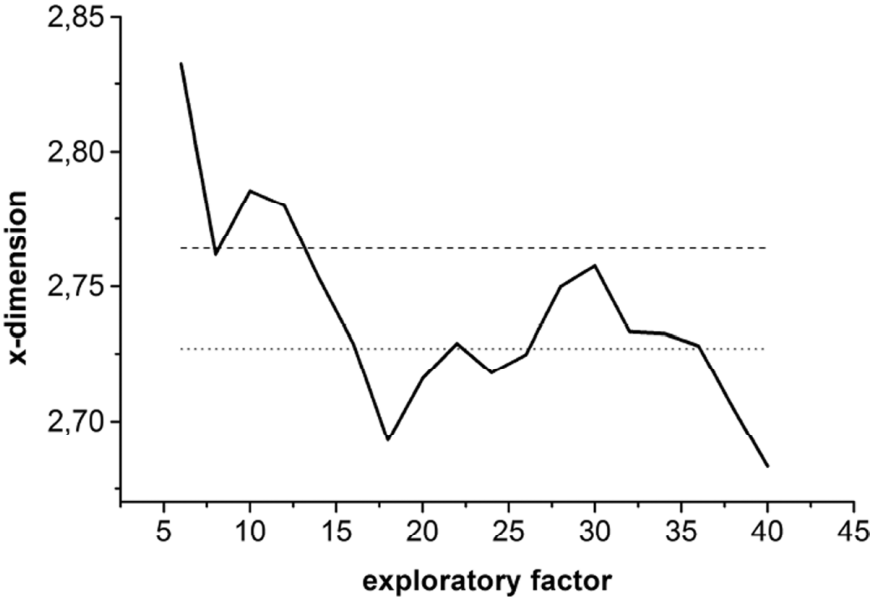

Figure 4: Dependence of *x*-Dimension of the Menger sponge on the Exploratory Factor. Estimate of the Box Dimension is 2.764 (Dash Line). Exact Dimension of the Menger Sponge is Equal to 2.7268 (Dot Line)

Example 3: A cortex surface is presented in [35] as an array of size $175 \times 117 \times 144$ with $0 - 1$ elements (see Fig. 5). Its fractal dimension was estimated by programs **x-dim3D** and **box-dim3D**. Box dimension of the cortex surface is equal to 2.402. The values computed by program **x-dim3D** are shown in Fig. 6 for different exploratory factors and unit atomic cubes.

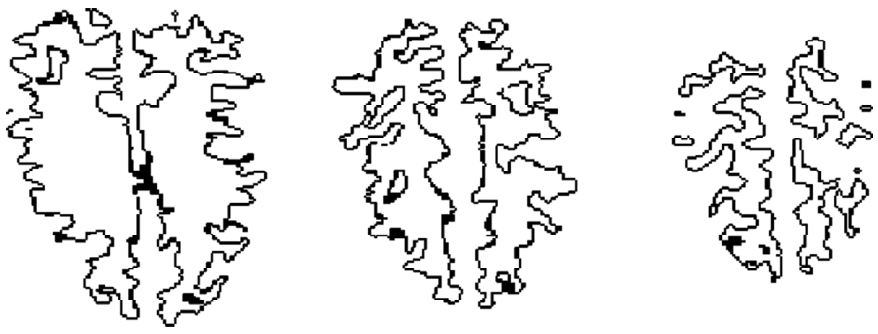

Figure 5: Slices of a Cortex Surface

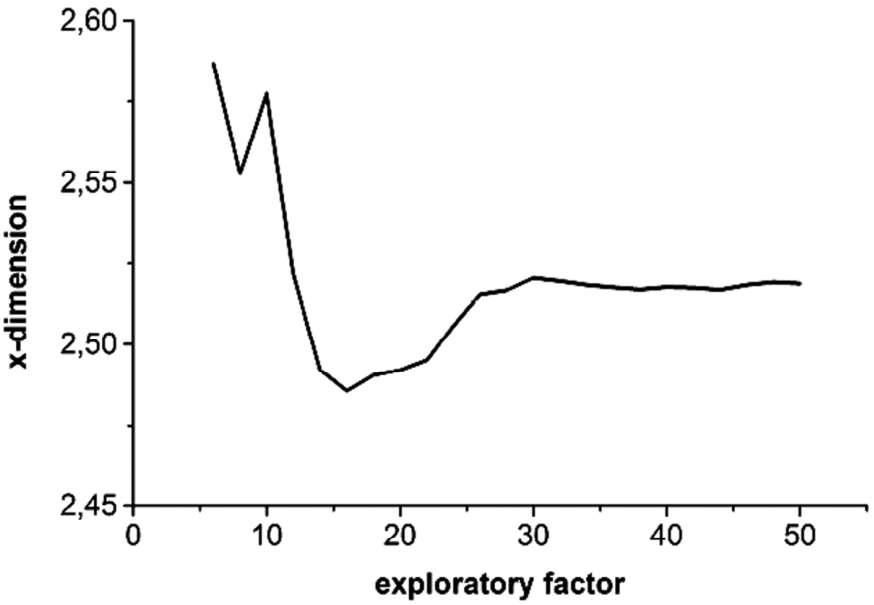

Figure 6: Dependence of *x*-Dimension of the Cortex Surface on the Exploratory Factor

4. Numerical Methods, Parameters and Ambiguities

In the software [35] several numerical methods are implemented to estimate fractal dimensions of time series, sets in the plane, surfaces and sets in 3D space. The computed results are strongly affected by the choice of the numerical method and by their input parameters, examples in section 3 show this clearly. The numerical estimation of fractal dimensions is an ambiguous problem, as there is often no "best" solution.

Let us be more specific and recall basic input parameters and methods presented in our paper and in [36]. For the conventional box-counting method and its adjusted modification it is necessary to set parameters $J, \varepsilon_1, ..., \varepsilon_J$ to solve (7) and (9) by least squares. Formally, J must be equal or larger than 2 for the box-counting ($J = 2$ in case of two-scale methods) and $J \geq 3$ for the adjusted modification. Two parameters must be fixed for the extended counting method: the size of atomic cubes ε_1 and the exploratory factor M which defines the size $\varepsilon_2 = M\varepsilon_1$ of exploratory cubes, see (6). The variance counting method realized in [35] for time series and random surfaces is a two-scale method and it is defined by two parameters, as well. Behavior of the variance counting method depending on the parameters was studied particularly in [22, 23, 24]. In some programs presented in [35] the basic input parameters are hidden, but in many cases they must be fixed manually. The question, what numerical method is more convenient is a point of discussion [26]. Analytically, fractal dimension is a characterization of a set revealing only in the limit $\epsilon \to 0$. No discrete approximation can achieve this limit. Therefore it is hard to validate the quality of numerical algorithms calculating fractality. A close agreement between numerical and analytical dimension may be test-set dependent. Even simple transformations (translation, rotation) of the test-set can change numerical results [7, 11, 26]. A trivial recommendation to remove some ambiguities: If we want to compare fractal dimensions of different sets numerically, the same method with the same input parameters should be used for every set.

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