THE p-QUASIHYPONORMALITY OF THE GENERALIZED ALUTHGE TRANSFORMATION

S. SATHIYABAMA

ABSTRACT: Let $T = U|T|^p$ be the polar decomposition of p-Quasihyponormal operator for $0 < p < 1$. Then the operator $|\tilde{T}_{s,t}| = |T|^s U |T|^t$, $0 < s < 1, 0 < t < 1$ is $(\frac{1}{2(s+t)})$ -Quasihyponormal and the operator $|\tilde{T}_{r,t}| = |T|^t U |T|^{r-t}$, $0 < r$, $t < 1$, $r \geq t$ is q-Quasihyponormal where $q = \max \{p + r + t, 2\}$.

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1. INTRODUCTION

Let H denotes a separable complex infinite dimensional Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. An operator T on a Hilbert space H is said to be p-hyponormal if $(T^* T)^p \geq (TT^*)^p$ for a positive number p and if $p = \frac{1}{2}$, then T is semi-hyponormal. An operator T on a Hilbert space H is said to be p-quasihyponormal if $T^*((T^*T)^p - (TT^*)^p)T \geq 0$ for $p > 0$. If $p = 1$, then T is quasihyponormal and if $p = \frac{1}{2}$, then T is semi-quasihyponormal. Every p-quasihyponormal operator is a q-quasihyponormal operator for $q \leq p$. S.C. Arora and Pramod Arora [2] introduced p-quasihyponormal operator and they studied some properties of p-Quasihyponormal using the operator $|\tilde{T}| = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$, $T = U|T|$ is the polar decomposition of T. MI Young Lee and Sang Hun Lee [3] studied various properties of p-Quasihyponormal for the operator $|\tilde{T}_\varepsilon| = |T|^\varepsilon U |T|^{1-\varepsilon}$, $0 < \varepsilon \leq \frac{1}{2}$. Takashi Yoshino [6] defined more generally for any s and t such as $s \geq 0$ and $t \geq 0$ $\tilde{T}_{s,t} = |T|^s U |T|^t$ the p-hyponormality of the Aluthge transform of T. Throughout this paper we denote $R(T)$, the range space of T and we consider $0 < p < 1$.

Furuta's Inequality: [5]

Let A and B be bounded self-adjoint operators such that $A \geq B \geq 0$. Then for each $r \geq 0$,

$$(B^r A^u B^r)^{\frac{1}{q}} \geq B^{(\frac{u+2r}{q})}$$

$$A^{(\frac{u+2r}{q})} \geq (A^r B^u A^r)^{\frac{1}{q}}$$

for each u and q such that $u \geq 0$, $q \geq 1$ and $(1 + 2r)\, q \geq s + 2r$.

Lemma 1: For $U|T|^p$, $R(\tilde{T}_{s,t}) \subset R(|T|^p), 0 < s, t < 1$.

Proof: By [2, Theorem 2.1] $R(\tilde{T}_{s,t}) \subset R(|T|^s)$.

Since $|T|^p$ is positive for all 0 < r < 1, therefore we have $R(\tilde{T}_{s,t}) \subset R(|T|^p)$.

Theorem 1: Let T = $U|T|^p$ be the polar decomposition of a *p*-Quasihyponormal operator and *U* is unitary. Then $\tilde{T}_{s,t} = |T|^s U |T|^t$ is $(\frac{1}{2(s+t)})$-Quasihyponormal for any $0 < s < 1$, $0 < t < 1$.

Proof: If *T* is a *p*-Quasihyponormal operator, then we have

$$T^*((T^*T)^p - (TT^*)^p)T \geq 0.$$

This implies that

$$|T|^p U^* \left(|T|^{2p^2} - \left(U|T|^{2p} U^*\right)^p\right) U|T|^p \geq 0.$$

$$|T|^p U^* \left(|T|^{2p^2} - \left(U|T|^{2p^2} U^*\right)\right) U|T|^p \geq 0$$

This is equivalent to

$$|T|^p \left(U^* |T|^{2p^2} U - |T|^{2p^2}\right) |T|^p \geq 0.$$

Thus on $R(|T|^p)$,

$$U^* |T|^{2p^2} U \geq |T|^{2p^2}$$
$$U^* |T|^{2p^2} U \geq |T|^{2p^2} \geq U|T|^{2p^2} U^*$$

Let $A = U^* |T|^{2p^2} U, B = |T|^{2p^2}, C = U|T|^{2p^2} U^*$

$$r = \frac{t}{2p^2}, \quad u = \frac{2s}{2p^2}, \quad q = \frac{1}{2(s+t)},$$

Since in Furuta's inequality

$$\left(1 + \frac{t}{p^2}\right) 2(s+t) \geq \frac{s}{p^2} + \frac{t}{p^2} \qquad \text{and} \qquad 2(s+t) \geq 1,$$

We have on $R(\tilde{T}_{s,t})$,

$$\begin{aligned}
\left(\tilde{T}_{s,t}^{*}\tilde{T}_{s,t}\right)^{\frac{1}{2(s+t)}} &= \left(|T|^{t}U^{*}|T|^{2s}U|T|^{t}\right)^{\frac{1}{2(s+t)}} \\
&= \left(B^{\frac{t}{2p^2}}A^{\frac{2s}{2p^2}}B^{\frac{t}{2p^2}}\right)^{\frac{1}{2(s+t)}} \\
&= B^{\left(\frac{2t}{2p^2}+\frac{2s}{2p^2}\right)\left(\frac{1}{2(s+t)}\right)} \\
&= B^{\left(\frac{s+t}{p^2}\right)\left(\frac{1}{2(s+t)}\right)} \\
&= B^{\frac{1}{2p^2}} \\
&= |T| \qquad (1)
\end{aligned}$$

Similarly,

Since $(1+\frac{s}{p^2})2(s+t) \geq \frac{t}{p^2}+\frac{s}{p^2}$ and $2(s+t) \geq 1$, we have

$$\begin{aligned}
\left(\tilde{T}_{s,t}\tilde{T}_{s,t}^{*}\right)^{\left(\frac{1}{2(s+t)}\right)} &= \left(|T|^{s}U|T|^{2t}U^{*}|T|^{s}\right)^{\left(\frac{1}{2(s+t)}\right)} \\
&= \left(B^{\frac{s}{2p^2}}C^{\frac{2t}{2p^2}}B^{\frac{s}{2p^2}}\right)^{\left(\frac{1}{2(s+t)}\right)} \\
&\leq B^{\left(\frac{s+t}{p^2}\right)\left(\frac{1}{2(s+t)}\right)} \\
&\leq |T| \qquad (2)
\end{aligned}$$

By (1) and (2) we have

$$\left(\tilde{T}_{s,t}^{*}\tilde{T}_{s,t}\right)^{\left(\frac{1}{2(s+t)}\right)} \geq |T| \geq \left(\tilde{T}_{s,t}\tilde{T}_{s,t}^{*}\right)^{\left(\frac{1}{2(s+t)}\right)}.$$

On $R(\tilde{T}_{s,t})$

$$\tilde{T}_{s,t}^{*}\left(\left(\tilde{T}_{s,t}^{*}\tilde{T}_{s,t}\right)^{\left(\frac{1}{2(s+t)}\right)} - \left(\tilde{T}_{s,t}\tilde{T}_{s,t}^{*}\right)^{\left(\frac{1}{2(s+t)}\right)}\right)\tilde{T}_{s,t} \geq 0.$$

Hence $\tilde{T}_{s,t}$ is $(\frac{1}{2(s+t)})$-Quasihyponormal.

Theorem 2: Let $T = U|T|^{p}$ be the polar decomposition of a p-Quasihyponormal operator and U is unitary. For $0 < r$, $t < 1$, $r \geq t$ and Let $q = \max\{p + r + t, 2\}$ and $\tilde{T}_{r,t} = |T|^{r}U|T|^{r-t}$. Then $\tilde{T}_{r,t}$ is q-Quasihyponormal.

Proof: If T is a p-quasihyponormal operator, then we have

$$U^* \, |T|^{2p^2} \, U \geq |T|^{2p^2} \geq U\,|T|^{2p^2} \, U^*.$$

Let $A = U^* \,|\,T\,|^{2p^2} \, U, B = |\,T\,|^{2p^2}, C = U\,|\,T\,|^{2p^2} \, U^*, q = \max\{p+r+t, 2\}$

$$\begin{aligned}
\left(\tilde{T}^*_{r,t}\tilde{T}_{r,t}\right)^q &= \left(|T|^{r-t}\, U^* \,|T|^{2t}\, U\,|T|^{r-t}\right)^{\left(\frac{1}{q}\right)} \\
&= \left(B^{\frac{r-t}{2p^2}} A^{\frac{2t}{2p^2}} B^{\frac{r-t}{2p^2}}\right)^{\left(\frac{1}{q}\right)} \\
&= \left(B^{\frac{r-t}{2p^2}} \; B^{\frac{2t}{2p^2}} \; B^{\frac{r-t}{2p^2}}\right)^{\left(\frac{1}{q}\right)} \\
&\geq B^{\left(\frac{r-t}{p^2}+\frac{t}{p^2}\right)\left(\frac{1}{q}\right)} \\
&= B^{\frac{r}{p^2 q}} \\
&\geq |T|^{\frac{2r}{q}} \qquad (3)
\end{aligned}$$

Since $(1+2\,\frac{r-t}{2p^2})q \geq \frac{2t}{2p^2} + 2\,\frac{r-t}{2p^2}$ and $q \geq 1$

Similarly

$$\begin{aligned}
\left(\tilde{T}_{r,t}\tilde{T}^*_{r,t}\right)^q &= \left(|T|^{t}\, U\,|T|^{2(r-t)}\, U^* \,|T|^{t}\right)^{\left(\frac{1}{q}\right)} \\
&= \left(B^{\frac{t}{2p^2}} C^{\frac{2(r-t)}{2p^2}} B^{\frac{t}{2p^2}}\right)^{\left(\frac{1}{q}\right)} \\
&\leq \left(B^{\frac{r-t}{2p}} B^{\frac{2t}{2p}} B^{\frac{r-t}{2p}}\right)^{\left(\frac{1}{q^1}\right)} \\
&= B^{\left(2\,\frac{r-t}{2p^2}+\frac{2t}{2p^2}\right)\left(\frac{1}{q}\right)} \\
&\leq B^{\left(\frac{r-t}{p^2}+\frac{t}{p^2}\right)\left(\frac{1}{q}\right)} \\
&= B^{\frac{r}{p^2 q}} \\
&\leq |T|^{\frac{2r}{q}} \qquad (4)
\end{aligned}$$

Since $(1+2\frac{t}{2p^2})q \geq \frac{2(r-t)}{2p^2} + 2\frac{t}{2p^2}$ and $q \geq 1$

From (3) and (4) we have

$$\left(\tilde{T}_{r,t}^{*}\tilde{T}_{r,t}\right)^{q} \geq \left|T\right|^{\frac{2r}{q}} \geq \left(\tilde{T}_{r,t}\tilde{T}_{r,t}^{*}\right)^{q}.$$

Hence on $R(\tilde{T}_{r,t})$

$$\left(\tilde{T}_{r,t}^{*}\tilde{T}_{r,t}\right)^{q} \geq \left(\tilde{T}_{r,t}\tilde{T}_{r,t}^{*}\right)^{q}.$$

This implies that

$$\tilde{T}_{r,t}^{*}\left(\left(\tilde{T}_{r,t}^{*}\tilde{T}_{r,t}\right)^{q} - \left(\tilde{T}_{r,t}\tilde{T}_{r,t}^{*}\right)^{q}\right)\tilde{T}_{r,t} \geq 0.$$

Hence $\tilde{T}_{r,t}$ is q-quasihyponormal.

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S. Sathiyabama
Department of Mathematics,
Kongu Engineering College,
Perundurai, Erode-638052,
Tamilnadu, India.
E-mail: sathiyabam@gmail.com