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AXISYMMETRIC DISTRIBUTION OF THICK CIRCULAR PLATE IN VISCOTHERMOELASTIC WITH MASS DIFFUSION AND PHASE LAGS

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Abstract: The present problem is concerned with an axisymmetric problem of thick circular plate in a viscothermoelastic diffusive body within the context of dual-phase-lag diffusion (DPLD) and dual-phase-lag heat transfer (DPLT) models. The lower and upper surfaces of the thick plate are traction free and are subjected to particular types of thermal and chemical potential sources to depict the utility of the solution obtained. The solution has been found by using Laplace and Hankel transform technique and without using the potential functions, a direct approach has been used. The analytical expressions of components of stresses, displacement and chemical potential have been computed in the transformed domain. The resulting quantities in the physical domain have been obtained by using numerical inversion technique. Numerically simulated results have been depicted graphically. The effect of viscosity is shown on the various physical quantities. From the present investigation, some particular cases of results are also deduced.

Keywords: Dual phase lag, isotropic viscothermoelastic, Laplace Transform, Hankel Transform, plane axisymmetric, diffusion.

1. INTRODUCTION

Classical Fourier heat conduction law has very short timescales and small dimensions, due to this it implies an infinitely fast propagation of a thermal signal. This signal is violated in ultra-fast heat conduction system. Catteno [1] and Vernotte [2] studied a thermal wave with a single phase lag. In this, the temperature gradient was given by $q+\tau_q\frac{\partial q}{\partial t}=-k\nabla T$, after a certain elapsed time. Here τ_q is the relaxation time required for thermal physics to take account of hyperbolic effect within the medium. When $\tau_q > 0$, the thermal wave with a finite speed of $\sqrt{\alpha/\tau_q}$, propagates through the medium, where α is known as thermal diffusivity. The thermal wave has an infinite speed when τ_q approaches zero. This leads to reduction of the single phase lag model to traditional Fourier model. Tzou [3] proposed the dual phase lag model of heat conduction $q+\tau_q\frac{\partial q}{\partial t}=-k\left(\tau_t\frac{\partial}{\partial t}\nabla T+\nabla T\right)$, here

∇T is known as temperature gradient of the material at a point P. This corresponds to the heat flux vector q at the $t + \tau_q$ and k is known as thermal conductivity of the material. Due to microstructural interactions, the delay time τ_t is caused. This is referred as the phase lag of temperature gradient. Whereas due to the fast transient effects of thermal inertia, there is another delay time τ_q which is referred as the phase lag of heat flux. When $\tau_t = 0$, the model proposed by Tzou [3] is referred as single phase model. Various attempts have been done for the evolution of an explicit mathematical solution, which has been applied to equation of heat conduction under DPL model. The stability of two different mathematical hyperbolic models proposed by Tzou have been compared by Quintanilla [4]. El-Karamany and Ezzat [5] studied dual-phase-lag thermoelasticity theory and further proved the uniqueness and reciprocal theorems and also established variational principle.

Different authors discussed different types of problems in viscoelasticity. Freudenthal [6] pointed out that many solids exhibit viscous effects when subjected to dynamic loading. To describe the viscoelastic behaviour of a material, Kelvin-Voigt model is used. This is the macroscopic mechanical model. When the deformation is time dependent, Kelvin-Voigt model shows delayed elastic response due to stress. Iesan and Scalia [7] studied some theorems in the theory of thermoviscoelasticity. Sharma, Sharma and Bhargava [8] analysed effect of viscosity on wave propagation in anisotropic thermoelastic with Green-Naghdi theory Type-II and Type-III. AI-Basyouni, Mahmoud and Alzahrani [9] discussed effect of rotation, magnetic field and a periodic loading on radial vibrations thermo-viscoelastic non-homogeneous media. Arefi and Zenkour [10] discussed nonlocal electro-thermo-mechanical analysis.

The process in which the particles spontaneously move from higher concentration region to the lower concentration region, is known as diffusion. This happens in response to a concentration gradient, which is expressed as the change in concentration due to change in position. To accomplish isotope separation, the transfer of heat across a thin gas or liquid is utilized by thermal diffusion.

Podstrigach [11] analyzed the differential equations of the problem of thermodiffussion in isotropic deformable solids. Podstrigach and Pavlina [12, 13] gave the general relationships based upon the thermodynamics of solid solutions and discussed the fundamental equations of plane thermodiffusion problem. Nowacki [14, 15, 16, 17, 18] did the further work in a series of papers by giving various theorems. Sharma [19] discussed reflection of plane waves in thermodiffusive elastic half? space with voids. Tripathi *et al.* [20] studied a problem of generalized thermoelastic diffusion. In which a thick circular plate was analysed with axisymmetric heat supply. Kumar, Devi and Sharma [21] discussed plane waves and fundamental solution in a modified couple stress generalised thermoelastic with mass diffusion. Kumar, Sharma and Lata [22] discussed effects of the diffusion and thermal phase-lags with an axisymmetric heat supply in a plate. Zenkour [23] analysed effects of phase-lags and variable thermal conductivity in a thermoviscoelastic solid with a cylindrical cavity. Zenkour *et al.* [24] studied two-temperature dual-phase-lags theory in a thermoelastic solid half-space due to an inclined load. Abbas and Marin [25] discussed a two-dimensional generalized thermoelastic diffusions problem due to laser pulse.

Here in this problem, by using two diffusion phase-lags, a generalized form of mass diffusion equation has been introduced. The delayed time needed for the diffusion of the mass flux is represented by one-phase-lag of diffusing mass flux vector. Whereas the delayed time needed for the establishment of the potential gradient is represented by another phase-lag of chemical potential. The basic equations for the isotropic viscothermoelastic diffusion medium in the case of dual-phase-lag diffusion (DPLD) and dual-phase-lag heat transfer (DPLT) models in axisymmetric form are presented. The components of stresses, temperature, displacements, mass concentration and chemical potential are obtained by using Laplace and Hankel transform technique. A direct approach without the use of potential functions is applied here. Numerical computation is performed by using a numerical inversion technique and the resulting quantities subjected to chemical potential and thermal sources are shown graphically. The various effects of thermal phase-lags diffusion and viscosity have been shown on the various physical quantities.

2. BASIC EQUATIONS

The equations of motion, heat conduction and mass diffusion in a homogeneous isotropic thermoelastic solid with DPLD and DPLT models in the absence of body forces, heat sources and mass diffusion sources are

$$(\lambda+\mu)\nabla(\nabla.u)+\mu\nabla^2 u-\beta_1\nabla T-\beta_2\nabla C=\rho\ddot{u} \tag{1}$$

$$\left(1+\tau_t\frac{\partial}{\partial t}\right)K\,T_{,ii}=\left(1+\tau_q\frac{\partial}{\partial t}+{\tau_q}^2\frac{\partial^2}{\partial t^2}\right)\left[C_E\rho\dot{T}+T_0\beta_1\dot{e}_{kk}+aT_0\dot{C}\right] \tag{2}$$

$$\left(1+\tau_p\frac{\partial}{\partial t}\right)(D\beta_2\nabla^2(\nabla.u)+Da\nabla^2 T-Db\nabla^2 C)+\frac{\partial}{\partial t}\left(1+\tau_\eta\frac{\partial}{\partial t}+{\tau_\eta}^2\frac{\partial^2}{\partial t^2}\right)C=0 \tag{3}$$

and constitutive relations are

$$\sigma_{ij}=2e_{ij}\mu+\delta_{ij}(e_{kk}\lambda-T\beta_1-\beta_2 C) \tag{4}$$

$$\rho\,T_0\,S=\left(1+\tau_q\frac{\partial}{\partial t}+{\tau_q}^2\frac{\partial^2}{\partial t^2}\right)\,(C_E\rho T+T_0\beta_1 e_{kk}+aT_0 C) \tag{5}$$

$$P=-\beta_2 e_{kk}-aT-bC \tag{6}$$

$$\lambda = \lambda^*(1 + Q_1 \frac{\partial}{\partial t}) \quad (7)$$

$$\mu = \mu^*(1 + Q_2 \frac{\partial}{\partial t}) \quad (8)$$

here λ^*, μ^* and ρ are Lame's constants and density (assumed to be independent of time) respectively. D, P and C are the diffusivity, chemical potential per unit mass and mass concentration respectively. u_i are components of displacement vector **u**. C_E, K are the specific heat at constant strain and the coefficient of thermal conductivity respectively. $T = \vartheta - T_0$ is small temperature increment, where ϑ is the absolute temperature of the medium and T_0 is the reference temperature of the body such that $\left|\frac{T}{T_0}\right| << 1$. a and b are the coefficients describing the measure of thermodiffusion and mass diffusion effect respectively. e_{ij} and σ_{ij} are the components of strain and stress respectively. e_{kk}, S are dilatation and entropy per unit mass, $\beta_1 = (3\lambda + 2\mu)\alpha_t$, $\beta_2 = (3\beta + 2\mu)\alpha_c$, where α_c, α_t are the coefficient of linear diffusion expansion and coefficient of thermal linear expansion. τ_t, τ_η, τ_q, τ_p are phase lag of temperature gradient, phase lag of diffusing mass flux vector, the phase lag of heat flux and phase lag of chemical potential. In all the above equations, a superposed dot means derivative with respect to time and a comma followed by suffix means spatial derivative.

3. FORMULATION AND SOLUTION OF THE PROBLEM

Consider a thick circular plate of thickness 2b which is occupying the space D defined by $0 \le r \le \infty$, $-b \le z \le b$ in viscothermoelastic diffusion with dual phase lag model. . Let the plate be subjected to chemical potential source and an axisymmetric heat supply with stress free boundary which depends on the axial and radial directions of cylindrical co-ordinate system. The chemical potential source of unit magnitude and heat flux are directed along with vanishing components of stress on the lower and upper boundary surfaces along with traction free boundary $z = \pm b$. The viscothermoelastic quantities are required to be determined in a thick circular plate, under the given conditions. T_0 is called initial temperature in thick circular plate. A cylindrical polar co-ordinate system (r, θ, z) has been considered, whose symmetry is about the z -axis. The field component $u_\theta = 0$, as the problem considered is plane axisymmetric. Also, u_r, u_z, C and T are not dependent on θ and we restrict this analysis to two dimensional problem with

$$u = (u_r, 0, u_z) \quad (9)$$

Eqns. (1)-(6), on using (9) take the form

$$\mu\left(\nabla^2-\frac{1}{r^2}\right)u_r+(\lambda+\mu)\frac{\partial e}{\partial r}-\beta_1\frac{\partial T}{\partial r}-\beta_2\frac{\partial C}{\partial r}=\rho\frac{\partial^2 u_r}{\partial t^2} \quad (10)$$

$$\mu\nabla^2 u_z+(\lambda+\mu)\frac{\partial e}{\partial z}-\beta_1\frac{\partial T}{\partial z}-\beta_2\frac{\partial C}{\partial z}=\rho\frac{\partial^2 u_z}{\partial t^2} \quad (11)$$

$$(1+\tau_t\frac{\partial}{\partial t})\,\mathrm{K}\,\nabla^2 T=\left(1+\tau_q\frac{\partial}{\partial t}+\frac{\tau_q{}^2}{2}\frac{\partial^2}{\partial t^2}\right)\left[\rho C_E\dot{T}+\beta_1 T_0\frac{\partial}{\partial t}divu+aT_0\frac{\partial C}{\partial t}\right] \quad (12)$$

$$(1+\tau_p\frac{\partial}{\partial t})(D\beta_2\nabla^2 divu+Da\nabla^2 T-Db\nabla^2 C)+\frac{\partial}{\partial t}\left(1+\tau_\eta\frac{\partial}{\partial t}+\frac{\tau_\eta{}^2}{2}\frac{\partial^2}{\partial t^2}\right)C=0 \quad (13)$$

and constitutive relations

$$\sigma_{rr}=2\mu e_{rr}+\lambda e-\beta_1 T-\beta_2 C \quad (14)$$

$$\sigma_{\theta\theta}=2\mu e_{\theta\theta}+\lambda e-\beta_1 T-\beta_2 \quad (15)$$

$$\sigma_{zz}=2\mu e_{zz}+\lambda e-\beta_1 T-\beta_2 C \quad (16)$$

$$\sigma_{rz}=\mu e_{rz},\sigma_{r\theta}=0\ ,\sigma_{z\theta}=0 \quad (17)$$

$$P=-\beta_2 e-aT+bC \quad (18)$$

where $$e=\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{\partial u_z}{\partial z},\quad e_{zz}=\frac{\partial u_z}{\partial z}$$

$$e_{\theta\theta}=\frac{u_r}{r},e_{rr}=\frac{\partial u_r}{\partial r},e_{rz}=\frac{1}{2}\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right) \quad (19)$$

The following dimensionless quantities are introduced to facilitate the solution

$$r'=\frac{w_1^*}{c_1}r,\ (u_r',u_z')=\frac{\omega_1^*}{c_1}(u_r,u_z),\ t'=\omega_1^* t,\quad z'=\frac{\omega_1^*}{c_1}z$$

$$\left(\sigma_{rr}',\ \sigma_{\theta\theta}',\quad \sigma_{zz}',\ \sigma_{rz}'\right)=\frac{1}{T_{0\beta_1}}(\sigma_{rr},\quad \sigma_{\theta\theta},\ \sigma_{zz},\ \sigma_{rz})$$

$$(T',C')=\frac{\beta_1}{\rho c_1^2}(T,C)\ ,\ \left(\tau_q',\tau_t',\tau_p',\tau_\eta'\right)=\omega_1^*(\tau_q,\tau_t,\tau_p,\tau_\eta),\ \omega_1^*=\frac{\rho C_E c_1^2}{K},\ c_1^2=\frac{\lambda+2\mu}{\rho} \quad (20)$$

Using (20) in equations (10)-(13) and thereafter dropping the primes. Then applying the Laplace transform, which is defined by

$$\bar{f}\,(r,\ z,\ s)=\int_0^\infty f(r,\ z,\ t)\ e^{-st}\,dt \quad (21)$$

$$\tilde{f}\,(\xi,\ z,\ s)=\int_0^\infty \hat{f}(r,\ z,\ s)\ \ r\ \ J_n(r\,\xi)\ \ dr \quad (22)$$

on the obtained quantities and solving them, we get

$$\nabla^2\bar{T} + \nabla^2\bar{C} - (\nabla^2 - s^2)\ \bar{e} = 0 \tag{23}$$

$$(\nabla^2 - \frac{\tau_q'}{\tau_t'}ks)\bar{T} - \frac{KaT_0\tau_q's}{\rho C_E\beta_2\tau_t'}\bar{C} - \frac{K{\beta_1}^2T_0}{\rho^2C_Ec_1^2}\frac{\tau_q'}{\tau_t'}s\bar{e}=0 \tag{24}$$

$$Da\nabla^2\frac{\rho c_1^2}{\beta_1}\bar{T} - \left(Db\frac{\rho c_1^2}{\beta_2}\nabla^2 - \frac{s\tau_\eta'kc_1^2}{\tau_p'\beta_2C_E}\right)\bar{C} + D\beta_2\nabla^2\bar{e} = 0 \tag{25}$$

where $\tau_q' = 1 + s\tau_q + \frac{s^2{\tau_q}^2}{2}$, $\tau_\eta' = 1 + s\tau_\eta + \frac{s^2{\tau_\eta}^2}{2}$, $\tau_p' = 1 + s\tau_p$,$\tau_t' = 1 + s\tau_t$

Eliminating $\bar{T}$, $\bar{e}$ and $\bar{C}$ from equations (23)-(25),we get

$$(\nabla^2 - k_1^2)(\nabla^2 - k_2^2)(\nabla^2 - k_3^2)(\bar{T}\ , \bar{C}\ , \bar{e}) = 0 \tag{26}$$

The solutions of the equation (26) can be written in the form

$$\bar{T} = \sum_{i=1}^3\bar{T}_i,\quad \bar{e} = \sum_{i=1}^3\bar{e}_i\quad ,\bar{C} = \sum_{i=1}^3\bar{C}_i \tag{27}$$

where $\bar{T}_i$, $\bar{e}_i$, and $\bar{C}$ are solutions of the following equation

$$(\nabla^2 - k_i^2)(\bar{T}_i, \bar{e}_i, \bar{C}_i) = 0,\ \text{here } i = 1,2,3 \tag{28}$$

Applying Hankel transform, which is defined by (22) on (28), we get

$$(D^2 - \xi^2 - k_i^2)\left(\bar{T}_i^{\ *}, \bar{e}_i^{\ *}, \bar{C}_i^{\ *}\right) = 0 \tag{29}$$

$$\bar{T}^* = \sum_{i=1}^3 A_i(\xi, s)\ \cosh{(q_iz)} \tag{30}$$

$$\bar{C}^* = \sum_{i=1}^3 d_i\ A_i(\xi, s)\ \cosh{(q_iz)} \tag{31}$$

$$\bar{e}^* = \sum_{i=1}^3 f_i\ A_i(\xi, s)\ \cosh{(q_iz)} \tag{32}$$

$$d_i = \frac{\zeta_{33}q_i^4 - 6\zeta_{21}\zeta_{33}q_i^2 + \zeta_{23}\zeta_{31}}{-q_i^2(\zeta_{22}\zeta_{33} + \zeta_{23}\zeta_{32}) + \zeta_{23}\zeta'_{32}}$$

$$f_i = \frac{-\zeta_{32}q_i^4 + \left(\zeta'_{32} + \zeta_{21}\zeta_{32}\right)q_i^2 - \zeta_{21}\zeta'_{32}}{-q_i^2(\zeta_{22}\zeta_{33} + \zeta_{23}\zeta_{32}) + \zeta_{23}\zeta'_{32}}$$

$q_i = \sqrt{\xi^2 + k_i^2}$, $\zeta_{21} = \frac{\tau_q'}{\tau_t'}ks$, $\zeta_{22} = \frac{KaT_0\tau_q's}{\rho C_E\beta_2\tau_t'}$, $\zeta_{23} = \frac{K{\beta_1}^2T_0}{\rho^2C_Ec_1^2}\frac{\tau_q'}{\tau_t'}s$, $\zeta_{31} = \frac{Da\rho c_1^2}{\beta_1}$, $\zeta_{32} = Db\frac{\rho c_1^2}{\beta_2}$,

$\zeta'_{32} = \frac{s\tau_\eta'kc_1^2}{\tau_p'\beta_2C_E}$, $\zeta_{33} = D\beta_2$

On applying the inversion of Hankel transform on (30), (31) and (32), we obtain

$$\bar{T} = \int_0^\infty \{\sum_{i=1}^3 A_i(\xi, s) \cosh\ (q_i z)\}\ \xi\ J_0(\xi r)\ d\xi \tag{33}$$

$$\bar{C} = \int_0^\infty \{\sum_{i=1}^3 d_i A_i(\xi, s)\ \cosh(q_i z)\}\ \xi\ J_0(\xi r)\ d\xi \tag{34}$$

$$\bar{e} = \int_0^\infty \{\sum_{i=1}^3 f_i\, A_i(\xi,\ s)\ \cosh\ (q_i z)\}\ \xi\ J_0(\xi r)\ d\xi \tag{35}$$

With use of (10)-(13), (20) and (33)-(35), we get the components of displacement in transformed domain as

$$\bar{u}_r\ (r, z, s) = \int_0^\infty \xi^2 J_1(\xi r)\left[E(\xi, s) \cosh\ (qz) + \sum_{i=1}^3 \frac{\lambda_i}{\left(\frac{\mu q_i^2}{\rho c_1^2} - s^2\right)} \cosh\ (q_i z)\right] d\xi \tag{36}$$

$$\bar{u}_z\ (r, z, s) = \int_0^\infty \xi J_0(\xi r)\left[F\ \ (\xi, s) \sinh\ (qz) - \sum_{i=1}^3 \frac{\lambda_i}{\left(\frac{\mu q_i^2}{\rho c_1^2} - s^2\right)} \sinh\ (q_i z)\right] d\xi \tag{37}$$

where

$$\lambda_i = \left(\frac{\lambda+\mu}{\rho c_1^2} f_i - 1 - d_i\right) A_i\ ,\ \ F(\xi, s) = \frac{\xi^2\ E(\xi,s)}{q}\ ,\ q = \sqrt{\xi^2 + \frac{c_1^2 \rho}{\mu} s^2}$$

Substituting the values of $\bar{u}_r$ and $\bar{u}_z$ in (14)-(17) and using (20) yield stress components and chemical potential

$$\overline{\sigma_{\theta\theta}} = \frac{2\mu}{\beta_1 T_0} \int_0^\infty \xi^2 J_1(\xi r)\left[E(\xi, s)\ cosh(qz) + \sum_{i=1}^3 \frac{\lambda_i}{\left(\frac{\mu q_i^2}{\rho c_1^2} - s^2\right)} cosh(q_i z)\right] d\xi\ + \int_0^\infty \sum_{i=1}^3 \eta_i\ cosh(q_i z)\ \xi J_0(\xi r)\ d\xi \tag{38}$$

$$\overline{\sigma_{rr}} = \frac{2\mu}{\beta_1 T_0} \int_0^\infty \xi^3 \left(\frac{1}{\xi r} J_1(\xi r) - J_0(\xi r)\right)\left[E(\xi, s) \cosh(qz) + \sum_{i=1}^3 \frac{\lambda_i}{\left(\frac{\mu q_i^2}{\rho c_1^2} - s^2\right)} \cosh(q_i z)\right] d\xi + \int_0^\infty \sum_{i=1}^3 \eta_i \cosh(q_i z)\ \xi J_0(\xi r)\ d\xi \tag{39}$$

$$\overline{\sigma_{zz}} = \frac{2\mu}{\beta_1 T_0}\int_0^\infty \xi J_0(\xi r)\left[F(\xi,s)q\cosh(qz) + \sum_{i=1}^{3}\frac{\lambda_i q_i^2}{\left(\frac{\mu q_i^2}{\rho c_1^2}-s^2\right)}\cosh(q_i z)\right]d\xi + \int_0^\infty \sum_{i=1}^{3}\eta_i \cosh(q_i z)\,\xi J_0(\xi r)\,d\xi \tag{40}$$

$$\overline{\sigma_{rz}} = \frac{\mu}{2\beta_1 T_0}\int_0^\infty \xi^2 J_1(\xi r)\left[\left(\frac{q^2-\xi^2}{q}\right)E(\xi,s)\,\mathrm{q}\sinh\,(qz) + 2\sum_{i=1}^{3}\frac{\lambda_i}{\left(\frac{\mu q_i^2}{\rho c_1^2}-s^2\right)}q_i\sinh\,(q_i z)\right]d\xi \tag{41}$$

$$\bar{P}(r,z,s) = \int_0^\infty \sum_{i=1}^{3}\zeta_i \cosh(q_i z)\,\xi J_0(\xi r)\,d\xi \tag{42}$$

where$\eta_i = \left(\frac{f_i-\rho c_1^2-\rho c_1^2 d_i}{\beta_1 T_0}\right)A_i$, $F(\xi,s) = \frac{\xi^2(E(\xi,s)}{q}$, $\zeta_i = \left(-\beta_2 f_i - \frac{\rho c_1^2}{\beta_1} + \frac{b\rho c_1^2}{\beta_2}d_i\right)A_i(\xi,s)$

4. BOUNDARY CONDITIONS

We consider a chemical potential and thermal source along with vanishing of components of stress at the surface $z = \pm\, b$. These can be expressed mathematically as

$$\frac{\partial T}{\partial z} = \pm\, g_0\, F(r,z), \tag{43}$$

$$\sigma_{zz} = 0, \tag{44}$$

$$\sigma_{rz} = \sigma_{rz} = 0, \tag{45}$$

$$P = \delta(t)H(a-r) \tag{46}$$

where $F(r, z) = z^2\, e^{-\omega r}$,

g_0 is known as constant temperature on the boundary, δ ( ) is called Dirac delta function.

$H(\ )$ is the Heaviside unit step function.

On both sides of the boundary conditions (43)-(46), applying Laplace and Hankel transform, we obtain

$$\frac{dT}{dz} = g_0\bar{F}(\xi\ ,z) \tag{47}$$

$$\overline{\sigma_{zz}}\ =0 \tag{48}$$

$$\overline{\sigma_{rz}} = 0 \tag{49}$$

$$\bar{P} = \frac{aJ_1(\xi a)}{\xi} \tag{50}$$

where $$\bar{F}(\xi, z) = \frac{z^2\omega}{(\xi^2+\omega^2)^{3/2}}$$

Substitute the values of $\bar{T}, \overline{\sigma_{zz}}$, $\overline{\sigma_{rz}}$, $\bar{P}$ for (47) - (50), the values of unknown parameters have been obtained as

$$A_1 = \frac{\Delta_1}{\Delta}, \quad A_2 = \frac{\Delta_2}{\Delta}, \quad A_3 = \frac{\Delta_3}{\Delta}, E(\xi, s) = \frac{\Delta_4}{\Delta}$$

where

$$\Delta = \begin{vmatrix} \Delta_{11} & \Delta_{12} & \Delta_{13} & 0 \\ \Delta_{21} & \Delta_{22} & \Delta_{23} & \dfrac{-2\mu}{\beta_1 T_0}\xi^2 q\cos(qb) \\ \Delta_{31} & \Delta_{32} & \Delta_{33} & \dfrac{q^2-\xi^2}{q}\sinh(qb) \\ \Delta_{41} & \Delta_{42} & \Delta_{43} & 0 \end{vmatrix}$$

$\Delta_{1i} = q_i \sinh(q_i b)$, $\Delta_{2i} = (\mu_i q_i^2 + \eta_i)$ cosh?$(q_i b)$, $\Delta_{3i} = \mu_i q_i \sinh(q_i b)$, $\Delta_{4i} = \zeta_i \cosh(q_i b)$, $i = 1, 2, 3$ and Δ_i is obtained from Δ, by interchanging i^{th} column with $\left[g_0 F(\xi, b)\ 0 \quad 0 \quad \dfrac{aJ_1(\xi a)}{\xi} \right]^t$, where t denotes transpose.

5. INVERSION OF DOUBLE TRANSFORM

Inverse of Laplace transform has been obtained by applying the Gaver - Stehfast algorithm, in order to avoid the complexity of the solution in the Laplace transform domain. Gaver [26] and Stehfast [27, 28] derived the formula, which is given below. The inverse $f(t)$ of Laplace transform $\bar{f}(s)$ is approximated by using this method

$$f(t) = \frac{\log 2}{t} \sum_{j=1}^{k} D(j, K)\, F\left(\frac{\log 2}{t} j\right)$$

with

$$D(j, K) = (-1)^{j+M} \sum_{n=m}^{\min(j,M)} \frac{n^M (2n)!}{n!(M-n)!\,(j-n)!\,(n-1)!\,(2n-j)!}$$

Where m is an integer part of $(j + 1)/2$ and $M = K/2$, here the value of K (an even integer) depends on the word length of computer used. Gaver-Stehfast algorithm suggests the optimal value of K for the fast convergence of results with desired accuracy.

The Romberg numerical integration technique; Press *et al.* [29], with variable step size was used to evaluate the results involved.

6. PARTICULAR CASES

(i) If we ignore the diffusion effect (i.e. β_2, b, $a = 0$), we obtain the expressions for stress components, components of displacement, chemical potential and temperature change for viscothermoelastic isotropic half space.

(ii) $\tau_p = \tau_\eta = 0$, then the corresponding relation reduces to thermoelastic with dual phase lag model.

(iii) $\tau_q = 0$ and $\tau_p = 0$, then DPLD and DPLT models reduce to single phase-lag diffusion model (SPLD) and single phase-lag heat model (SPLT).

7. NUMERICAL RESULTS AND DISCUSSION

For the purposes of numerical computation, the mathematical model is prepared with copper material. The material constants of copper for the problem have been taken from Dhaliwal and Singh [30]

$K = 386 JK^{-1}\, m^{-1}\, s^{-1}$, $\mu = 3.86 \times 10^{10}\, Nm^{-2}$, $\rho = 8954\, Kgm^{-3}$, $D = 0.88 \times 10^{-8}\, kgs/m^3$

$\lambda = 7.76 \times 10^{10}\, Nm^{-2}$, $\beta_1 = 5.518 \times 10^6\, Nm^{-2}\, deg^{-1}$, $a = 1.2 \times 10^4\, m^2/s^2$ $kb = 0.9 \times 10^6\, m^5/kgs^2$, $T_0 = 293K$, $\beta_2 = 61.38 \times 10^6\, Nm^{-2}\, \deg^{-1}$, $C_E = 383.1\, Jkg^{-1}\, K^{-1}$

To study the effect of viscosity, the graphs have been plotted on the various quantities for the range $0 \le r \le 10$. $g_0 = 1$

For viscoelastic medium we take the values $Q_1 = 0.5$, $Q_2 = 1.0$

(i) Solid line represents the thermoelastic diffusion with viscosity (VS)

(ii) Small dashed line corresponds to thermoelastic diffusion without viscosity (W VS)

Fig. 1 exhibits the variations of displacement component u_r with distance r. Near the loading surface, the values of displacement component u_r for both the cases decrease sharply and follow opposite oscillatory trends afterwards for the rest of region attaining minima at $r = 5.5$. Viscosity effect decreases the value of displacement component u_r.

Fig. 2. shows the variations of normal displacement component u_z with distance r. it is noticed that the values of u_z, with and without viscosity follow similar

oscillatory trends with change of amplitude. Viscosity increases the value of displacement component u_z. It is maximum at $r = 1$, whereas the variation corresponding to without viscosity decreases for range $0 \le r \le 2.5$, thereafter follows oscillatory trend.

Fig. 3 depicts the variations of radial stress component σ_{rr} with distance r. it is evident that the variations corresponding to without viscosity decrease monotonically for the range $0 \le r \le 3$ and oscillate afterwards whereas viscosity increases the value near the boundary surface. The trends are monotonically decreasing for the range $2 \le r \le 6$, thereafter; it shows a sharp increase for $6 \le r \le 8$ and then decreases.

Fig. 4 shows the behaviour and variation of hoop stress component $\sigma_{\theta\theta}$ with distance r which is similar as σ_{rr} with difference in their magnitude value.

Fig. 5. shows the variations of shear stress component σ_{rz} with distance r. Near the loading surface, there is a sharp decrease for the range $0 \le r \le 2$ and as r moves away, the variations are near boundary surface and follow similar oscillatory trends with change of amplitude for both the cases with and without viscosity effect although their magnitude value are different.

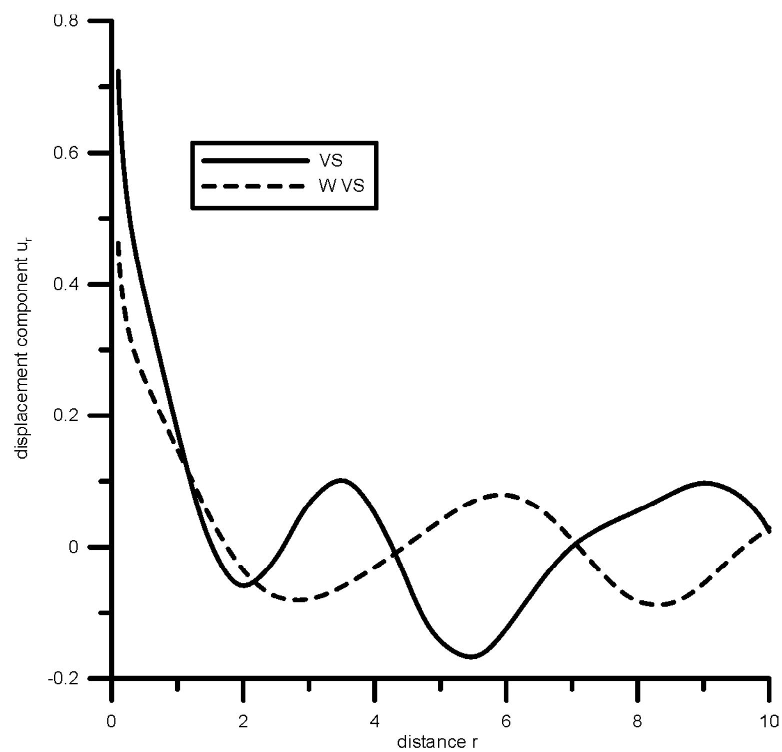

Figure 1: Variations of displacement component u_r with distance r

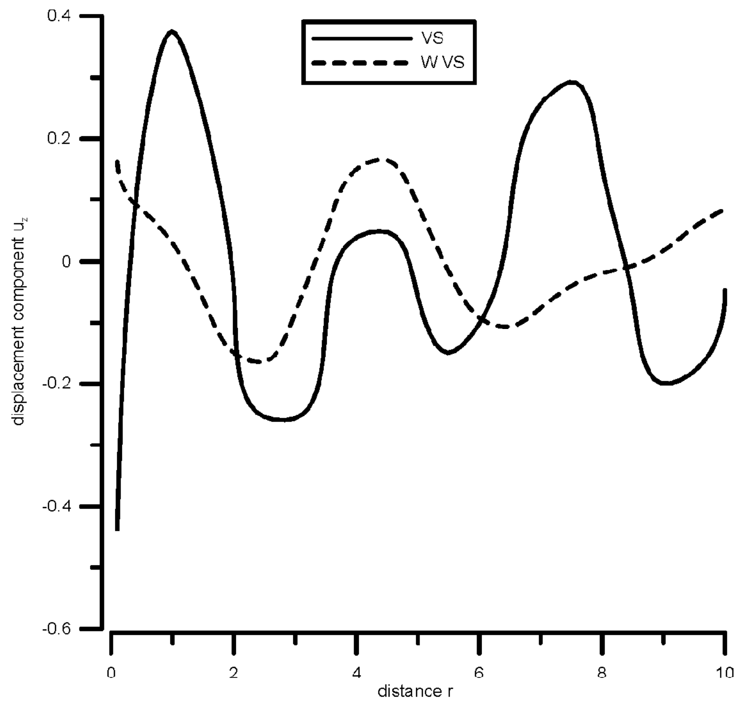

Figure 2: Variations of displacement component u_z with distance *r*

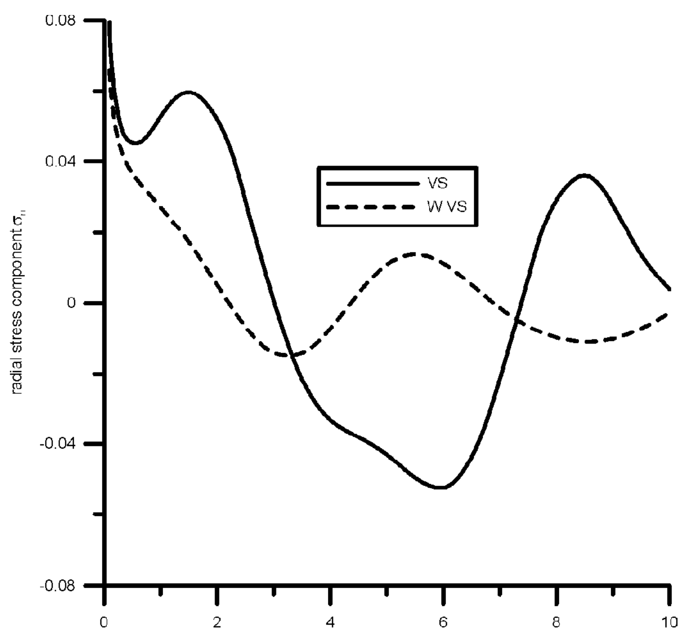

Figure 3: Variations of radial stress component σ_{rr} with distance *r*

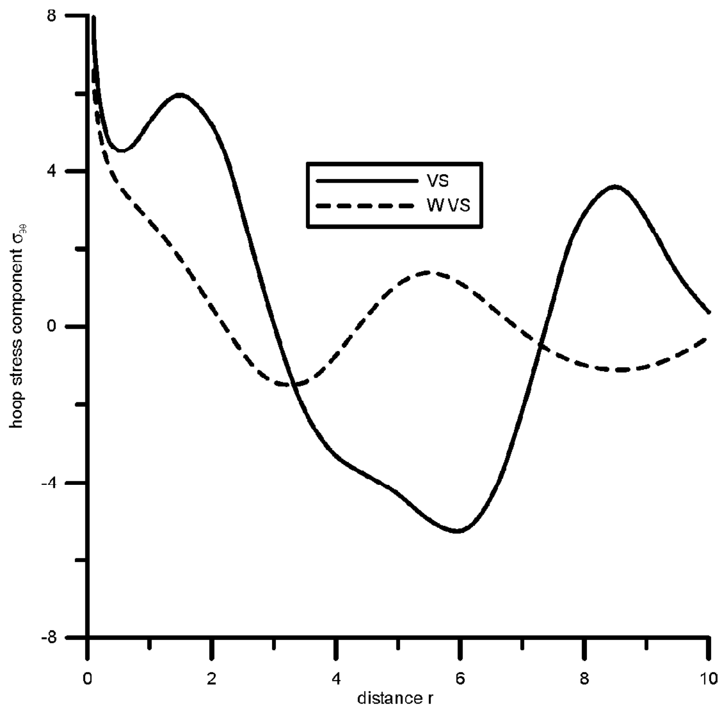

Figure 4: Variations of hoop stress component $\sigma_{\theta\theta}$ with distance *r*

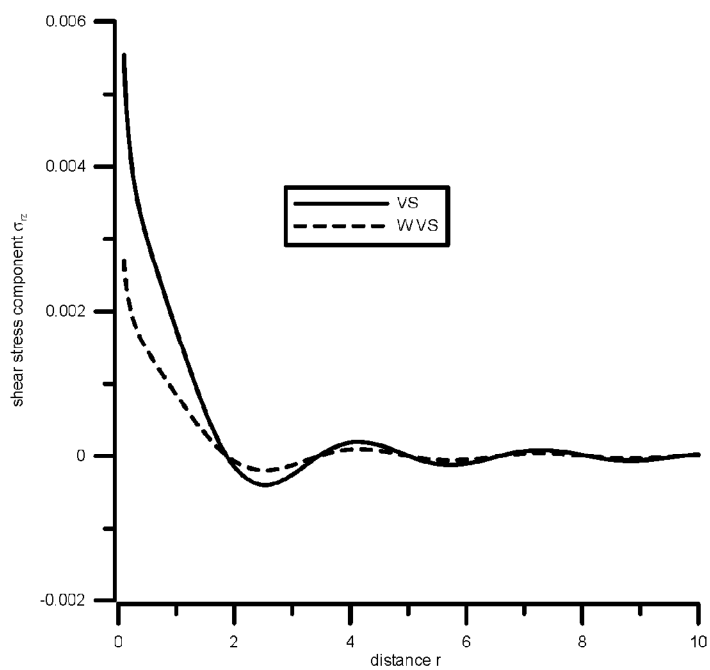

Figure 5: Variations of shear stress component σ_{rz} with distance *r*

Fig. 6 explains the variations of vertical stress component σ_{zz} with distance r. It is evident that viscosity decreases the value of normal stress for the range $0 \leq r \leq 2$, the variations decrease sharply corresponding to both the cases and follow opposite oscillatory trends afterwards. As $r \geq 2$, there is up and down in comparison to without viscosity effect.

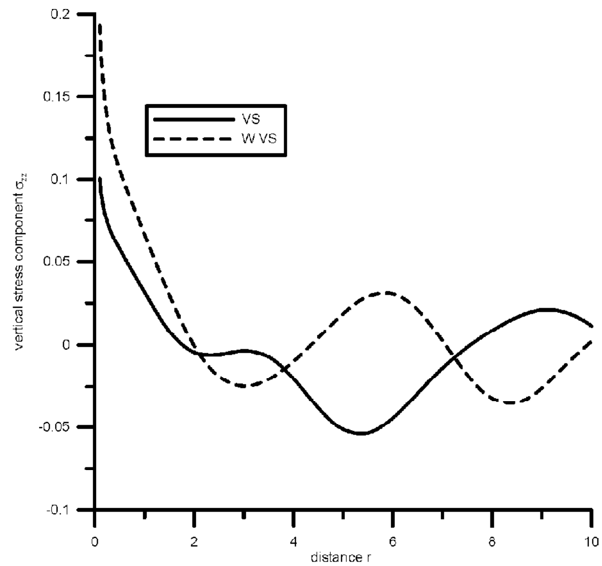

Figure 6: Variations of vertical stress component σ_{zz} with distance r

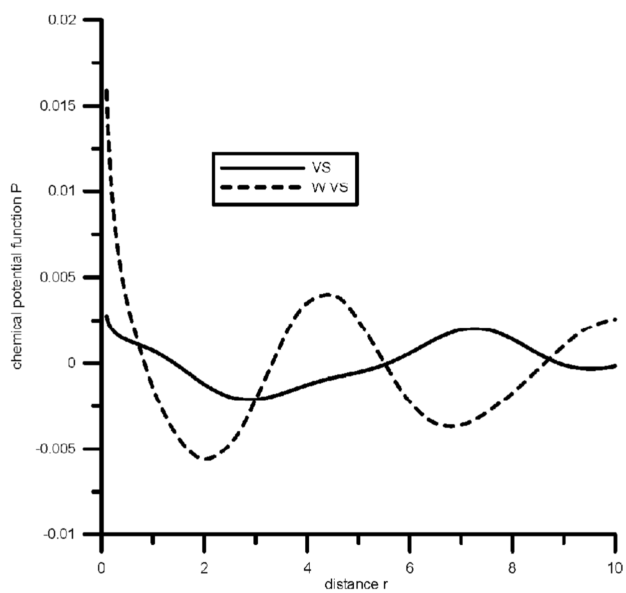

Figure 7: Variations of chemical potential function P with distance r

Fig. 7 shows the variations of chemical potential function P with distance r. Here, it is noticed that the trends corresponding to with and without viscosity are also oscillatory with less amplitude. There is a sharp decrease for range $0 \leq r \leq 2$ without viscosity, thereafter it shows oscillatory behaviour. Viscosity effect decreases the value of chemical potential near the boundary surface and shows oscillatory behaviour thereafter.

8. CONCLUSION

The present investigation is focused on the behaviour of thick circular plate in viscothermoelastic diffusion with and without dual phase lag due to thermal and chemical potential source. The basic equations for the isotropic thermoelastic diffusion medium in the context of dual-phase-lag heat transfer (DPLT) and dual-phase-lag diffusion (DPLD) models in axisymmetric form are presented. Laplace and Hankel transform are used to solve the problem.

Near and away from the application of the source values of displacement components u_r, u_z, radial stress σ_{rr}, hoop stress $\sigma_{\theta\theta}$ shear stress σ_{rz} get higher due to viscosity and for the intermediate range their values are oscillatory in nature. Near and away from the application of the source, the values of vertical stress σ_{zz} and chemical potential P are small due to viscosity and for the intermediate range oscillate, although the values of σ_{zz} for away from the source are increasing due to viscosity effect.

A more realistic model of viscothermoelastic diffusion media is obtained by Using of diffusion phase-lags in mass diffusion equation as it allows a delayed response between the potential gradient and the relative mass flux vector.

For two dimensional problem of dynamic response, the results of the problem are of great use. Due to various sources in viscothermo diffusion it has various industrial and geophysical applications. A sound impact of viscosity on the various components of displacements, stresses, and chemical potential function in the thick circular plate has been observed.

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