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COMMON FIXED POINT THEOREMS FOR WEAKLY COMMUTATIVE MAPS IN DISLOCATED METRIC SPACES

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Abstract

In this paper, we prove common fixed point theorems for weakly commutative maps in the context of dislocated metric spaces. Further, we prove existence and uniqueness of fixed points of such mappings.

Keywords: *Weakly commutative maps, Dislocated metric space, common fixed point.*

1. INTRODUCTION

Fixed point theory is one of the most dynamic research subject in non linear analysis and many fruitful results have come into the literature in the last few decades. The most remarkable result was given by Banach [1] in 1922 as Banach contraction principle. Later on, a lot of generalization of Banach contraction came into existence ([2]-[5]).

A major alter in the arena of fixed point theory came in 1976 when Jungck [6], introduced the concept of commutative maps and proved the common fixed point results for such maps. After which, Sessa [7] gave the concept of weakly compatible maps and proved fixed point results for such maps.In 2000, P. Hitzler and A.K. Seda [8] introduced the concept of dislocated metric space and generalized of well known Banach Contraction Principle in this space, which played a key role in the development of logic programming semantics. In this paper, we generalize the concept of common fixed point theorems for weakly commutative maps in the setting of dislocated metric spaces.

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2. PRELIMINARIES

Hitzler and Seda [8] introduced the concept of dislocated metric space (d-metric space) as follows:

Definition 2.1: Let X be a non empty set and let $d : X \times X \to [0, \infty)$ be a function satisfying the following conditions:

(1) $d(x, y) = d(y, x)$

(2) $d(x, y) = d(y, x) = 0$ implies $x = y$

(3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$

Then d is called dislocated metric (or simply d-metric) on X and the pair (X, d) is called dislocated metric space.

Definition 2.2: A sequence $\{x_n\}$ in a d-metric space (X, d) is said to be convergent if for every given $\in\ > 0$ there exist an $n \in N$ and $x \in X$ such that $d(x_n, x) < \in$ for all $n > N$ and it is denoted by $\lim_{n \to \infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

Definition 2.3: A sequence $\{x_n\}$ in a d-metric space (X, d) is said to be Cauchy sequence if for every $\in > 0$ there exist $n_0 \in N$ such that $d(x_n, x_m) < \in$ for all $m, n \in n_0$.

Definition 2.4: A d-metric space (X, d) is called complete if every Cauchy sequence is convergent.

Definition 2.5: Let (X, d) be a dislocated metric space. Then

- a convergent sequence in X is bounded and its limit is unique.
- a convergent sequence in X is a Cauchy sequence.

Lemma 2.6: Let (X, d) be a d-metric space, (x_n) be a sequence in X and $x \in X$. Then $x_n \to x(n \to \infty)$ if and only if $d(x_n, x) \to 0(n \to \infty)$.

Lemma 2.7: Let (X, d) be a d-metric space and let (x_n) be a sequence in X. If the sequence (x_n) is convergent then the limit point is unique.

Theorem 2.8: Let (X, d) and (Y, d) be two d-metric space. $f : X \to Y$ be a mapping and (x_n) be any sequence in X. Then f is continuous at the point $x \in X$ if and only if $f(x_n) \to f(x)$ for every sequence (x_n) with $x_n \to x(n \to \infty)$.

Definition 2.9: Let (X, d) be a complete d-metric space and let $T : X \to X$ be a mapping satisfying the following condition for all $x, y \in X$:

$$d(Tx, Ty) \leq kd(x, y),$$

where, $k \in [0, 1)$. Then T has a unique fixed point.

Sessa [7] introduced the concept of weakly compatible maps as follows.

Definition 2.10: The self maps S and T of a metric space (X, d) are said to be weakly commutative iff $d(S(T(x)), T(S(x))) \le d(S(x), T(x))$ for all $x \in X$.

Main Result

Lemma 3.1: Let $\{y_n\}$ be a sequence of complete d-metric space (X, d). If there exists $\alpha \in (0, 1)$ such that $d(y_{n+1}, y_n) \le \alpha d(y_n, y_{n-1})$ for all n, then $\{y_n\}$ converges to a point in X.

Proof: Direct consequence of Theorem 2.9.

Theorem 3.2: Let T be a continuous mapping of a complete d-metric space (X, d) into itself. Then T has a fixed point in X if there exists non negative constants α_1, α_2, α_3 with $\alpha_1 + \alpha_2 + \alpha_3 < 1$ and a mapping $S : X \to X$ which satisfies following:

(i) S commutes weakly with T;

(ii) $S(X) \subset T(X)$;

(iii) $d(Sx, Sy) \le \alpha_1\, d(Tx, Ty) + \alpha_2\, d(Tx, Sx) + \alpha_3\, d(Ty, Sy)$ for all $x, y \in X$

Then T and S have a unique common fixed point.

Proof: Suppose there is a map S of X into itself which commutes weakly with T and for which (iii) holds. We show that the condition is sufficient to ensure that T and S have a unique fixed point. To this end, let $x_0 \in X$ and let x_1 be such that $T(x_1) = S(x_0)$. In general, choose x_n so that

$$T(xn) = S(x_{n-1}) \tag{1}$$

We can do this since $S(X) \subset T(X)$. The relation (iii) and (1) imply that

$$d(Sx_n, Sx_{n-1}) = d\left(Tx_{n+1}, Tx_n\right) \le \alpha_1 d\left(Tx_n, Tx_{n-1}\right) + \alpha_2 d\left(Tx_n, Sx_n\right) + \alpha_3 d\left(Tx_{n-1}, Sx_{n-1}\right);$$

$$d(Sx_n, Sx_{n-1}) = \alpha_1 d\left(Tx_n, Tx_{n-1}\right) + \alpha_2 d\left(Tx_n, Tx_{n+1}\right) + \alpha_3 d\left(Tx_{n-1}, Tx_n\right);$$

$$= \left(\alpha_1 + \alpha_3\right) d\left(Tx_n, Tx_{n-1}\right) + \alpha_2 d\left(Tx_n, Tx_{n+1}\right);$$

$$\left(1 - \alpha_2\right) d\left(Tx_{n+1}, Tx_n\right) \le \left(\alpha_1 + \alpha_3\right) d\left(Tx_n, Tx_{n-1}\right);$$

$$d(Tx_{n+1}, Tx_n) \le \frac{\alpha_1 + \alpha_3}{1 - \alpha_2} d\left(Tx_n, Tx_{n-1}\right);$$

$$d(Tx_{n+1}, Tx_n) \le \lambda\, d(Tx_n, Tx_{n-1}).$$

where, $\lambda = (\alpha_1 + \alpha_3)/(1 - \alpha_2)$ and $\lambda < 1$.

For all n the lemma 3.1 yields $u \in X$ such that

$$T(x_n) \to u \tag{2}$$

But then (1) implies that

$$S(x_n) \to u \tag{3}$$

Now

$$d(Sx_n, Sx_n) \le \alpha_1 d(Tx_n, Tx_n) + \alpha_2 d(Tx_n, Sx_n) + \alpha_3 d(Tx_n, Sx_n).$$

Taking $n \to \infty$ in equation

$$d(u, u) \le (\alpha_1 + \alpha_2 + \alpha_3) d(u, u) \Rightarrow d(u, u) = 0$$

Now, from the definition of weakly commutative

$$d(T(S(x_n)), S(T(x_n))) \le d(T(x_n), S(x_n))$$

$$\Rightarrow \qquad d(T(u), S(u)) \le d(u, u) \Rightarrow T(u) = S(u)$$

So, u is a coincidence point of S and T. So, $T(Su) = S(Tu) = S(Su)$. We can therefore infer $d(Su, Su) \le \alpha_1\, d(Tu, Tu) + \alpha_2\, d(Tu, Su) + \alpha_3\, d(Tu, Su) = (\alpha_1 + \alpha_2 + \alpha_3)\, d(Su, Su)$;

$$(1 - \alpha_1 - \alpha_2 - \alpha_3)\, d(Su, Su) \le 0 \in d(Su, Su) = 0.$$

Similarly $d(S(Su), S(Su)) = 0$.

Hence

$$d(Su, S(Su)) \le \alpha_1\, d(Tu, T(Su)) + \alpha_2\, d(Tu, Su) + \alpha_3\, d(T(Su), S(Su));$$

$$= \alpha_1\, d(Su, T(Su)) + \alpha_2\, d(Su, Su) + \alpha_3\, d(S(Su), S(Su));$$

$$d(Su, S(Su)) \le \alpha_1\, d(Su, S(Su)).$$

Hence, $d(Su, S(Su))(1 - \alpha_1) \le 0 \le Su = S(Su) = T(Su)$, i.e., $S(u)$ is common fixed point of S and T.

To see that S and T can have only one common fixed point, suppose that $x = T(x) = S(x)$ and $y = T(y) = S(y)$. Then (iii) implies that

$$d(x, y) = d(Sx, Sy) \le \alpha_1\, d(Tx, Ty) + \alpha_2\, d(Tx, Sx) + \alpha_3\, d(Ty, Sy)$$

$$= (\alpha_1 + \alpha_2 + \alpha_3) d(x, y) = \sigma d(x, y)$$

Where $\sigma = \alpha_1 + \alpha_2 + \alpha_3$. So we get $d(x, y)(1 - \sigma) \le 0$.

Since $\sigma < 1$, $x = y$.

Theorem 3.3: Let T be a continuous mapping of a complete d-metric space (X, d)

into itself. Then T has a fixed point in X if there exists non negative constants α_1, α_2, α_3 with $\alpha_1 + \alpha_2 + \alpha_3 < 1$ and a mapping $S : X \to X$ which satisfies following:

(i) S commutes weakly with T;

(ii) $S(X) \subset T(X)$;

(iii) $d(Sx, Sy) \leq \alpha \max\{d(Tx, Ty), d(Tx, Sx), d(Ty, Sy)\}$ for all $x, y \in X$

Then T and S have a unique common fixed point.

Proof: Suppose there is a map S of X into itself which commutes weakly with T and for which (iii) holds. We show that the condition is sufficient to ensure that T and S have a unique fixed point. To this end, let $x_0 \in X$ and let x_1 be such that $T(x_1) = S(x_0)$. In general, choose x_n so that

$$T(x_n) = S(x_{n-1}) \tag{4}$$

We can do this since $S(X) \in T(X)$. The relation (iii) and (4) imply that

$$\begin{aligned} d(Sx_n, Sx_{n-1}) = d(Tx_{n+1}, Tx_n) &\leq \alpha \max\{d(Tx_n, Tx_{n-1}), d(Tx_n, Sx_n), \\ &\quad d(Tx_{n-1}, Sx_{n-1})\}; \\ &= \alpha \max\{d(Tx_n, Tx_{n-1}), d(Tx_n, Tx_{n+1}), d(Tx_{n-1}, Tx_n)\}; \\ &= \alpha \max\{d(Tx_n, Tx_{n-)}, d(Tx_n, Tx_{n+1})\}; \end{aligned}$$

If $\max\{d(Tx_n, Tx_{n-1}), d(Tx_n, Tx_{n+1})\} = d(Tx_n, Tx_{n+1}) \Rightarrow x_n = x_{n+1}$

Which is contradiction.

Hence $\max\{d(Tx_n, Tx_{n-1}), d(Tx_n, Tx_{n+1})\} = d(Tx_{n-1}, Tx_n)$

$d(Tx_{n+1}, Tx_n) \leq \alpha d(Tx_n, Tx_{n-1})$ and $\alpha < 1$, for all n. So by lemma 3.1 yields $u \in X$ such that

$$T(x_n) \to u \tag{5}$$

But then (4) implies that

$$S(x_n) \to u \tag{6}$$

Now

$$d(Sx_n, Sx_n) \leq \alpha \max\{d(Tx_n, Tx_n), d(Tx_n, Sx_n), d(Tx_n, Sx_n)\}.$$

Taking $n \to \infty$ in equation

$$d(u, u) \leq \alpha d(u, u) \in d(u, u) = 0$$

Now, from the definition of weakly commutative

$$d(\mathrm{T}(\mathrm{S}(x_n)), \mathrm{S}(\mathrm{T}(x_n))) \leq d(\mathrm{T}(x_n), \mathrm{S}(x_n))$$

$$\Rightarrow \quad d(\mathrm{T}(u), \mathrm{S}(u)) \leq d(u, u) \in \mathrm{T}(u) = \mathrm{S}(u)$$

So, u is a coincidence point of S and T. So, $\mathrm{T}(\mathrm{S}u) = \mathrm{S}(\mathrm{T}u) = \mathrm{S}(\mathrm{S}u)$. We can therefore infer

$$d(\mathrm{S}u, \mathrm{S}u) \leq \alpha \max\{d(\mathrm{T}u, \mathrm{T}u), d(\mathrm{T}u, \mathrm{S}u), d(\mathrm{T}u, \mathrm{S}u)\} \leq \alpha d(\mathrm{S}u, \mathrm{S}u)$$

$(1 - \alpha)d(\mathrm{S}u, \mathrm{S}u) \leq 0$. This implies $d(\mathrm{S}u, \mathrm{S}u) = 0$.

Similarly, we can show that $d(\mathrm{S}(\mathrm{S}u), \mathrm{S}(\mathrm{S}u)) = 0$.

Hence

$$d(\mathrm{S}u, \mathrm{S}(\mathrm{S}u)) \leq \alpha \max\{d(\mathrm{T}u, \mathrm{T}(\mathrm{T}u)), d(\mathrm{T}u, \mathrm{S}u), d(\mathrm{S}(\mathrm{T}u), \mathrm{S}(\mathrm{S}u))\}$$

$\Rightarrow \; d(\mathrm{S}u, \mathrm{S}(\mathrm{S}u)) \leq \alpha d(\mathrm{S}u, \mathrm{S}(\mathrm{S}u))$.

Hence, $d(\mathrm{S}u, \mathrm{S}(\mathrm{S}u))(1 - \alpha) \leq 0 \in \mathrm{S}u = \mathrm{S}(\mathrm{S}u)$.

Since $\alpha < 1$, $\mathrm{S}u = \mathrm{S}(\mathrm{S}u) = \mathrm{T}(\mathrm{S}u)$; i.e. $\mathrm{S}(u)$ is common fixed point of T and S.

To see that T and S can have only one common fixed point, suppose that

$$x = \mathrm{T}(x) = \mathrm{S}(x) \text{ and } y = \mathrm{T}(y) = \mathrm{S}(y).$$

Then (iii) implies that

$\mathrm{D}(x, y) = d(\mathrm{S}x, \mathrm{S}y) \leq \alpha \max\{d(\mathrm{T}x, \mathrm{T}y), d(\mathrm{T}x, \mathrm{S}x), d(\mathrm{T}y, \mathrm{S}y)\} \leq \alpha d(x, y)$.

So we get $d(x, y)(1 - \alpha) \leq 0$.

Since $\alpha < 1$, $x = y$.

Theorem 3.4: Let T be a continuous mapping of a complete d-metric space (X, d) into itself. Then T has a fixed point in X if there exists non negative constants α, β, μ with $\alpha + \beta + \mu < 1$ and a mapping $\mathrm{S} : \mathrm{X} \to \mathrm{X}$ which satisfies following:

(i) S commutes weakly with T;

(ii) $\mathrm{S}(\mathrm{X}) \subset \mathrm{T}(\mathrm{X})$;

(iii) $$d(\mathrm{S}x, \mathrm{S}y) \leq \alpha \max\left\{\begin{matrix} d(\mathrm{T}x, \mathrm{T}y), d(\mathrm{T}x, \mathrm{S}x), \\ d(\mathrm{T}y, \mathrm{S}y) \end{matrix}\right\} + \beta \max\left\{\begin{matrix} d(\mathrm{T}x, \mathrm{S}y), \\ d(\mathrm{T}x, \mathrm{T}y) \end{matrix}\right\} + \mu d(\mathrm{T}x, \mathrm{T}y) \text{ for all } x, y \in \mathrm{X}$$

Then T and S have a unique common fixed point.

Proof: Suppose there is a map S of X into itself which commutes weakly with T and for which (iii) holds. We show that the condition is sufficient to ensure that T and S have a unique fixed point. To this end, let $x_0 \in X$ and let x_1 be such that $T(x_1) = S(x_0)$. In general, choose x_n so that

$$T(x_n) = S(x_{n-1}) \tag{7}$$

We can do this since $S(X) \subset T(X)$. The relation (iii) and (7) imply that

$$\begin{aligned}
d(Sx_n, Sx_{n-1}) &= d(Tx_{n+1}, Tx_n) \\
&\leq \alpha \max \begin{Bmatrix} d(Tx_n, Tx_{n-1}), d(Tx_n, Sx_n), \\ d(Tx_{n-1}, Sx_{n-1}) \end{Bmatrix} \\
&\quad + \beta \max \begin{Bmatrix} d(Tx_n, Sx_{n-1}), \\ d(Tx_n, Tx_{n-1}) \end{Bmatrix} + \mu d(Tx_n, Tx_{n-1}) \\
&= \alpha \max \begin{Bmatrix} d(Tx_n, Tx_{n-1}), d(Tx_n, Tx_{n+1}), \\ d(Tx_{n-1}, Tx_n) \end{Bmatrix} \\
&\quad + \beta \max \begin{Bmatrix} d(Tx_n, Tx_n), \\ d(Tx_n, Tx_{n-1}) \end{Bmatrix} + \mu d(Tx_n, Tx_{n-1}) \\
&= \alpha \max\{d(Tx_n, Tx_{n-1}), d(Tx_n, Tx_{n+1})\} \\
&\quad + (\beta + \mu)\{d(Tx_n, Tx_{n-1})\};
\end{aligned}$$

Case I:

If $\max\{d(Tx_n, Tx_{n-1}), d(Tx_n, Tx_{n+1})\} = d(Tx_n, Tx_{n+1})$

Then $d(Tx_n, Tx_{n+1}) \leq \alpha d(Tx_n, Tx_{n+1}) + (\beta + \mu) d(Tx_n, Tx_{n-1})$

$$d(Tx_n, Tx_{n+1}) \leq \frac{\beta + \mu}{1 - \alpha} d(Tx_n, Tx_{n-1})$$

So we get $d(Tx_n, Tx_{n+1}) \leq \tau d(Tx_n, Tx_{n-1})$. Where $\tau = \dfrac{\beta + \mu}{1 - \alpha}$ and by given $\tau < 1$.

Case II:

If $\max\{d(Tx_n, Tx_{n-1}), d(Tx_n, Tx_{n+1})\} = d(Tx_n, Tx_{n-1})$

Then $d(Tx_n, Tx_{n+1}) \leq (\alpha + \beta + \mu) d(Tx_n, Tx_{n-1})$

So we get $d(Tx_n, Tx_{n+1}) \leq \delta d(Tx_n, Tx_{n-1})$. Where $\delta = \alpha + \beta + \mu$ and by given $\delta < 1$.

For all n. So by lemma 3.1 yields $u \in X$ such that

$$T(x_n) \to u \tag{8}$$

But then (7) implies that

$$S(x_n) \to u \tag{9}$$

Now $$d(Sx_n, Sx_n) \le \alpha \max\begin{Bmatrix} d(Tx_n, Tx_n), d(Tx_n, Sx_n), \\ d(Tx_n, Sx_n) \end{Bmatrix}$$

$$+ \beta \max\begin{Bmatrix} d(Tx_n, Sx_n), \\ d(Tx_n, Tx_n) \end{Bmatrix} + \mu d(Tx_n, Tx_n)$$

Taking $n \to \infty$ in equation

$$d(u, u) \le (\alpha + \beta + \mu)(u, u) \Rightarrow d(u, u) = 0$$

Now, from the definition of weakly commutative

$$d(T(S(x_n)), S(T(x_n))) \le d(T(x_n), S(x_n))$$

$$\Rightarrow \quad d(T(u), S(u)) \le d(u, u) \Rightarrow T(u) = S(u)$$

So, u is a coincidence point of S and T. So, T(Su) = S(Tu) = S(Su). We can therefore infer

$$d(Su, Su) \le \alpha \max\begin{Bmatrix} d(Tu, Tu), \\ d(Tu, Su), \\ d(Tu, Su) \end{Bmatrix} + \beta \max\begin{Bmatrix} d(Tu, Su), \\ d(Tu, Tu) \end{Bmatrix} + \mu d(Tu, Tu)$$

$$(1 - \alpha - \beta - \mu)d(Su, Su) \le 0. \text{ This implies } d(Su, Su) = 0.$$

Similarly, we can show that $d(S(Su), S(Su)) = 0$.

Hence

$$d(Su, S(Su)) \le \alpha \max\begin{Bmatrix} d(Tu, T(Tu)), \\ d(Tu, Su), \\ d(T(Tu), S(Su)) \end{Bmatrix} + \beta \max\begin{Bmatrix} d(Tu, S(Su)), \\ d(Tu, T(Tu)) \end{Bmatrix}$$

$$+ \mu d(Tu, T(Tu)).$$

$$d(Su, S(Su)) \le \alpha \max\begin{Bmatrix} d(Su, S(Su)), \\ d(Su, Su), \\ d(S(Su), S(Su)) \end{Bmatrix} + \beta \max\begin{Bmatrix} d(Su, S(Su)), \\ d(Su, S(Su)) \end{Bmatrix}$$

$$+ \mu d(Su, T(Su)).$$

$$d(Su, S(Su)) \le (\alpha + \beta + \mu)d(Su, S(Su)).$$

$$d(Su, S(Su)) \le \delta d(Su, S(Su))$$

where, $\delta = \alpha + \beta + \mu$ and by given $\delta < 1$

Hence, $d(Su, S(Su))(1 - \delta) \le 0 \Rightarrow Su = S(Su)$.

Since $\delta < 1$, $Su = S(Su) = T(Su)$; i.e. $S(u)$ is common fixed point of T and S.

To see that T and S can have only one common fixed point, suppose that

$$x = T(x) = S(x) \text{ and } y = T(y) = S(y).$$

Then (iii) implies that

$$d(x, y) = d(Sx, Sy) \le \alpha \max \begin{Bmatrix} d(Tx, Ty), d(Tx, Sx), \\ d(Ty, Sy) \end{Bmatrix} + \beta \max \begin{Bmatrix} d(Tx, Sy), \\ d(Tx, Ty) \end{Bmatrix}$$

$$+ \mu d(Tx, Ty)$$

$$d(x, y) \le (\alpha + \beta + \mu)d(x, y) = \delta d(x, y).$$

Where $\delta = \alpha + \beta + \mu$ and by given $\delta < 1$. So we get $d(x, y)(1 - \delta) \le 0$.

So we get, $x = y$.

Theorem 3.5: Let T be a continuous mapping of a complete d-metric space (X, d) into itself. Then T has a fixed point in X if there exists non negative constants α_1, α_2, α_3, α_4, α_5 with $\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 < 1$ and a mapping $S : X \to X$ which satisfies following:

(i) S commutes weakly with T;

(ii) $S(X) \subset T(X)$;

(iii) $d(Sx, Sy) \le \alpha_1 d(Tx, Ty) + \alpha_2 d(Tx, Sx) + \alpha_3 d(Ty, Sy) + \alpha_4 [d(Tx, Sx) + d(Ty, Sy)] + \alpha_5 [d(Tx, Sy) + d(Ty, Sx)]$ for all $x, y \in X$

Then T and S have a unique common fixed point.

Proof: Suppose there is a map S of X into itself which commutes weakly with T and for which (iii) holds. We show that the condition is sufficient to ensure that T and S have a unique fixed point. To this end, let $x_0 \in X$ and let x_1 be such that $T(x_1) = S(x_0)$. In general, choose x_n so that

$$T(x_n) = S(x_{n-1}) \tag{10}$$

We can do this since $S(X) \subset T(X)$. The relation (iii) and (10) imply that

$$d(Sx_n, Sx_{n-1}) = d(Tx_{n+1}, Tx_n), \le \alpha_1 d(Tx_n, Tx_{n-1}) + \alpha_2 d(Tx_n, Sx_n)$$

$$+ \alpha_3\, d(Tx_{n-1}, Sx_{n-1}) + \alpha_4[d(Tx_n, Sx_n) + d(Tx_{n-1}, Sx_{n-1})]$$
$$+ \alpha_5[d(Tx_n, Sx_{n-1}) + d(Tx_{n-1}, Sx_n)]$$
$$\leq (\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5)d(Tx_n, Tx_{n-1})$$
$$+ (\alpha_2 + \alpha_4 + \alpha_5)d(Tx_n, Tx_{n+1});$$
$$(1 - \alpha_2 - \alpha_4 - \alpha_5)d(Tx_{n+1}, Tx_n) \leq (\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5)d(Tx_n, Tx_{n-1});$$
$$d(Tx_{n+1}, Tx_n) \leq \frac{\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5}{1 - \alpha_2 - \alpha_4 - \alpha_5}\, d(Tx_n, Tx_{n-1});$$
$$d(Tx_{n+1}, Tx_n) \leq \lambda\, d(Tx_n, Tx_{n-1}).$$

where, $\lambda = \dfrac{\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5}{1 - \alpha_2 - \alpha_4 - \alpha_5}$ and $\lambda < 1$.

For all n. So by lemma 3.1 yields $u \in X$ such that

$$T(x_n) \to u \tag{11}$$

But then (10) implies that

$$S(x_n) \to u \tag{12}$$

Now

$$d(Sx_n, Sx_n) \leq \alpha_1\, d(Tx_n, Tx_n) + \alpha_2\, d(Tx_n, Sx_n) + \alpha_3\, d(Tx_n, Sx_n) + \alpha_4\, [d(Tx_n, Sx_n)$$
$$+ d(Tx_n, Sx_n)] + \alpha_5\, [d(Tx_n, Sx_n) + d(Tx_n, Sx_n)].$$

Taking $n \to \infty$ in equation

$$d(u, u) \leq (\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5)(u, u) \Rightarrow d(u, u) = 0$$

Now, from the definition of weakly commutative

$$d(T(S(x_n)), S(T(x_n))) \leq d(T(x_n), S(x_n))$$
$$\Rightarrow \quad d(T(u), S(u)) \leq d(u, u) \Rightarrow T(u) = S(u)$$

So, u is a coincidence point of S and T. So, $T(Su) = S(Tu) = S(Su)$. . We can therefore infer

$$d(S(u), S(u)) \leq \alpha_1\, d(T(u), T(u)) + \alpha_2\, d(T(u), S(u)) + \alpha_3\, d(T(u), S(u))$$
$$+ \alpha_4[d(T(u), S(u)) + d(T(u), S(u))] + \alpha_5[d(T(u), S(u))$$
$$+ d(T(u), S(u))];$$
$$(1 - \alpha_1 - \alpha_2 - \alpha_3 - 2\alpha_4 - 2\alpha_5)d(S(u), S(u)) \leq 0.$$

This implies $d(S(u), S(u)) = 0$.

Similarly $d(S(S(u)), S(S(u))) = 0$.

Hence,

$$d(Su, S(Su)) \le \alpha_1 d(Tu, T(Su)) + \alpha_2 d(Tu, Su) + \alpha_3 d(T(Tu), S(Su)) + \alpha_4 [d(Tu, Su) + d(T(Tu), S(Su))] + \alpha_5 [d(Tu, S(Su)) + d(T(Tu), Su)].$$

$$d(Su, S(Su)) \le \alpha_1 d(Su, S(Su)) + \alpha_5 [d(Su, S(Su)) + d(Su, S(Su)).$$

$$d(Su, S(Su)) \le (\alpha_1 + 2\alpha_5) d(Su, S(Su)).$$

Hence, $d(Su, S(Su))(1 - \alpha_1 - 2\alpha_5) \le 0 \Rightarrow Su = S(Su)$.

So we have, $Su = S(Su) = T(Su)$; i.e. $S(u)$ is common fixed point of T and S.

To see that T and S can have only one common fixed point, suppose that

$$x = T(x) = S(x) \text{ and } y = T(y) = S(y).$$

Then (iii) implies that

$$\mathcal{D}(x, y) = d(Sx, Sy) \le \alpha_1 d(Tx, Ty) + \alpha_2 d(Tx, Sx) + \alpha_3 d(Ty, Sy) + \alpha_4 [d(Tx, Sx) + d(Ty, Sy)] + \alpha_5 [d(Tx, Sy) + d(Ty, Sx)]$$

$$\mathcal{D}(x, y) = d(Sx, Sy) \le \{\alpha_1 + 2(\alpha_5)\} d(x, y) = \sigma d(x, y).$$

where, $\sigma = \alpha_1 + 2(\alpha_5)$. So we get $d(x, y)(1 - \sigma) \le 0$.

Since, $\sigma < 1$, $x = y$.

Theorem 3.6: Let T be a continuous mapping of a complete d-metric space (X, d) into itself. Then T has a fixed point in X if there exists non negative constants α_1, α_2, α_3 with $\alpha_1 + \alpha_2 + \alpha_3 < 1$ and a mapping $S : X \to X$ which satisfies following:

(i) S commutes weakly with T;

(ii) $S(X) \subset T(X)$;

(iii) $$d(Sx, Sy) \le \alpha_1 \frac{d(Ty, Sy)[1 + d(Tx, Sx)]}{1 + d(Tx, Ty)} + \alpha_2 d(Tx, Ty) + \alpha_3 \frac{d(Ty, Sy) + d(Ty, Sx)}{1 + d(Ty, Sy) d(Ty, Sx)} \text{ for all } x, y \in X$$

Then T and S have a unique common fixed point.

Proof: Suppose there is a map S of X into itself which commutes weakly with T and for which (iii) holds. We show that the condition is sufficient to ensure that T and S have a unique fixed point. To this end, let $x_0 \in X$ and let x_1 be such that $T(x_1) = S(x_0)$. In general, choose x_n so that

$$T(x_n) = S(x_{n-1}) \tag{13}$$

We can do this since $S(X) \subset T(X)$. The relation (iii) and (1) imply that

$$d(Sx_{n-1}, Sx_n) = d(Tx_n, Tx_{n+1})$$

$$\leq \alpha_1 \frac{d(Tx_n, Sx_n)[1 + d(Tx_{n-1}, Sx_{n-1})]}{1 + d(Tx_n, Tx_{n-1})} + \alpha_2\, d(Tx_n, Tx_{n-1})$$

$$+ \alpha_3 \frac{d(Tx_n, Sx_n) + d(Tx_n, Sx_{n-1})}{1 + d(Tx_n, Sx_n)\, d(Tx_n, Sx_{n-1})}$$

$$= \alpha_1 \frac{d(Tx_n, Tx_{n+1})[1 + d(Tx_n, Tx_{n-1})]}{1 + d(Tx_n, Tx_{n-1})} + \alpha_2\, d(Tx_n, Tx_{n-1})$$

$$+ \alpha_3 \frac{d(Tx_n, Tx_{n+1}) + d(Tx_n, Tx_n)}{1 + d(Tx_n, Tx_{n+1})\, d(Tx_n, Tx_n)}$$

$$d(Tx_n, Tx_{n+1})(\alpha_1 + \alpha_3) d(Tx_n, Tx_{n+1}) + \alpha_2\, d(Tx_{n-1}, Tx_n);$$

$$(1 - \alpha_1 - \alpha_3) d(Tx_{n+1}, Tx_n) \leq \alpha_2\, d(Tx_n, Tx_{n-1});$$

$$d(Tx_{n+1}, Tx_n) \leq \frac{\alpha_2}{1 - \alpha_1 - \alpha_3}\, d(Tx_n, Tx_{n-1});$$

$$d(Tx_{n+1}, Tx_n) \leq \lambda\, d(Tx_n, Tx_{n-1}).$$

Where $\lambda = \alpha_2/(1 - \alpha_1 - \alpha_3)$. So by given condition $\lambda < 1$.

For all n. The lemma 3.1 yields $u \in X$ such that

$$T(x_n) \to u \tag{14}$$

But then (13) implies that

$$S(x_n) \to u \tag{15}$$

Now since T is continuous, (13) implies that both S and T are continuous. Hence, (14) and (15) demand that $S(T(x_n)) \to S(u)$. But S and T commute so that $S(T(x_n)) = T(S(x_n))$ for all n. Thus, $S(u) = T(u)$, and consequently $T(T(u)) = T(S(u)) = S(S(u))$ by commutativity. We can therefore infer

$$d(Sx_n, Sx_n) \leq \alpha_1 \frac{d(Tx_n, Sx_n)[1 + d(Tx_n, Sx_n)]}{1 + d(Tx_n, Tx_n)} + \alpha_2\, d(Tx_n, Tx_n)$$

$$+ \frac{d(Tx_n, Sx_n) + d(Tx_n, Sx_n)}{1 + d(Tx_n, Sx_n)\, d(Tx_n, Sx_n)}$$

Taking $n \to \infty$ in equation

$$d(u, u) \leq (\alpha_1 + \alpha_2 + \alpha_3)d(u, u) \Rightarrow d(u, u) = 0$$

Now, from the definition of weakly commutative

$$d(T(S(x_n)), S(T(x_n))) \leq d(T(x_n), S(x_n))$$

$$\Rightarrow \qquad d(T(u), S(u)) \leq d(u, u) \Rightarrow T(u) = S(u)$$

So, u is a coincidence point of S and T. So, T(Su) = S(Tu) = S(Su). We can therefore infer

$$d(Su, Su) \leq \frac{d(Tu, Su)\left[1 + d(Tu, Su)\right]}{1 + d(Tu, Tu)} + \alpha_2\, d(Tu, Tu)$$

$$+ \frac{d(Tu, Su) + d(Tu, Su)}{1 + d(Tu, Su)\, d(S(Tu), Su)}$$

This implies $d(Su, Su) = 0$.

Similarly, $d(S(Su), S(Su)) = 0$.

Hence

$$d(Su, S(Su)) \leq \frac{d(T(Su), S(Su))\left[1 + d(Tu, Su)\right]}{1 + d(Tu, T(Su))} + \alpha_2\, d(Tu, T(Su))$$

$$+ \frac{d(T(Su), S(Su)) + d(T(Tu), Su)}{1 + d(T(Su), S(Su))\, d(Tu, Su)}$$

$$d(Su, S(Su)) \leq \alpha_2\, d(Su, S(Su)) + \alpha_3\, d(Su, S(Su));$$

$$d(Su, S(Su)) \leq (\alpha_2 + \alpha_3) d(Su, S(Su)).$$

Hence, $d(Su, S(Su))(1 - \alpha_2 - \alpha_3) \leq 0 \Rightarrow S(u) = S(S(u))$.

Since $\alpha_1 < 1$, $S(u) = S(S(u)) = T(Su)$; i.e. $S(u)$ is common fixed point of T and S.

To see that T and S can have only one common fixed point, suppose that

$$x = T(x) = S(x) \text{ and } y = T(y) = S(y).$$

Then (iii) implies that

$$\mathcal{D}(x, y) = d(Sx, Sy) \leq \frac{d(Ty, Sy)\left[1 + d(Tx, Sx)\right]}{1 + d(Tx, Ty)} + \alpha_2\, d(Tx, Ty)$$

$$+ \frac{d(\mathrm{T}y, \mathrm{S}y) + d(\mathrm{T}y, \mathrm{S}x)}{1 + d(\mathrm{T}y, \mathrm{S}y)d(\mathrm{T}y, \mathrm{S}x)}$$

$$\mathcal{D}(x, y) \leq (\alpha_1 + \alpha_3)d(x, y) = \sigma d(x, y).$$

Where $\sigma = \alpha_1 + \alpha_3$. So we get $d(x, y)(1 - \sigma) \leq 0$.

Since $\sigma < 1$, $x = y$.

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