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Multi-Switching Synchronization of Non-Identical Chaotic Systems

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ABSTRACT

In this manuscript, combination-combination synchronization is achieved among different switches of four chaotic systems where Li system and Newton Leipnik system are considered as master systems and Wang system and Bhalekar Gejji system are considered as slave systems. Controllers are designed by using non linear control method and Lyapunov stability theory for combination-combination synchronization among different switches of four chaotic systems. Numerical simulations are performed by using ode45 in MATLAB software. Computational results and theoretical results are in excellent agreement.

Keywords: Multi-switching synchronization, chaotic systems, combination-combination synchronization, nonlinear control, Lyapunov stability theory.

1. INTRODUCTION

Chaos has been an integral part of nonlinear sciences since Lorenz discovered first time the phenomenon of sensitivity to initial conditions for a nonlinear dynamical system which was later known as "Lorenz System". Importance of any subject can be seen by its applications and chaos itself has rich variety of applications in the field of secure communications, physics chemistry, medicine etc. Synchronization is a main branch of chaos which started to fascinate the researchers after the inspirational and excellent work of Pecora and Carroll [1] in 1990. Before this work, it was really difficult to assume that two chaotic systems whether identical or non-identical can follow the same trajectory on applying some suitable control functions because of unpredictability of chaotic systems.

Following the initial work of Pecora and Carroll researchers have suggested different types of synchronization like antisynchronization [2], phase synchronization [3], lag synchronization [4], projective synchronization [5], Q-S synchronization, [6] etc and recently hybrid function projective synchronization [7], compound synchronization [8]-[9], combination synchronization[10], combination-combination synchronization [11], etc have also been suggested. These synchronization phenomenon have been developed by various methods like active control[12], adaptive control [13], sliding mode control [14], time delay control [15], impulsive control[16], active backstepping [17], fuzzy control [18], predictive control, [19] etc.

Ucar [20] introduced the concept of multi-switching synchronization in 2008. He achieved the multi-switching synchronization by active control method. In recent years, multi-switching synchronization has been a hot topic among the researchers because it provides liberty to synchronize arbitrary pairs of state variables which is very useful in secure communications, since it is difficult to predict that which state variables will be synchronized. Except integer order chaotic systems, work on multi-switching synchronization is also being done in the field of fractional order chaotic systems [21].

In recent years, significant works have been done on multi-switching synchronization like complete synchronization between identical systems [22], combination synchronization via non linear control [23], combination synchronization by active backstepping [24], synchronization between non identical chaotic systems with fully unknown parameters

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[25], etc. Multi-switching synchronization may be defined as a phenomenon where any state variable of slave system can be synchronized to arbitrary state variable of master system. We can form many switches in this way and in case of combination- combination synchronization there may be so many combinations of state variables of master systems and slave systems but we have considered only four switches for theoretical analysis, while numerical results are presented just for two switches. In this paper, multi-switching combination-combination synchronization has been achieved among four non identical chaotic systems where Li system [26] and Newton Leipnik system [27] are considered as master systems and Wang system [28] and Bhalekar Gejji system [29] are considered as slave systems.

The rest of the paper is organized as follows, In section (2), methodology for combination-combination synchronization is given. Section (3) contains description of master and slave systems and in the section (4) multiswitching combination-combination synchronization is described for considered master and slave systems. Numerical simulations are given in section (5) and section (6) contains conclusion.

2. COMBINATION-COMBINATION SYNCHRONIZATION METHODOLOGY [11]

First, we explain the methodology of combination-combination synchronization with two master systems and two slave systems. Suppose the master systems are

$$\begin{cases} \dot{v}_1 = X_1(v_1) \\ \dot{v}_2 = X_2(v_2) \end{cases} \tag{1}$$

and corresponding slave systems are

$$\begin{cases} \dot{w}_1 = h_1(w_1) + U \\ \dot{w}_2 = h_2(w_2) + U^* \end{cases} \tag{2}$$

where $v_1 = (v_{11}, v_{21}, ..., v_{n1})', v_2 = (v_{12}, v_{22}, ..., v_{n2})', w_1 = (w_{11}, w_{21}, ..., w_{n1})',$

$w_1 = (w_{12}, w_{22}, ..., w_{n2})'$ are the vectors of state variables of all four systems given above. $X_1, X_2, h_1, h_2 : R^n \to R^n$ are continuous vector functions. Our aim is to design appropriate controllers U, U^* so that combination-combination synchronization will be achieved among different switches of four chaotic systems.

Definition 1 [11] If there exist four constant matrices P, Q, R, S $\in \mathbf{R}^{n\times n}$, where R, S not both zero such that

$$\lim_{t\to\infty} \left\| Pw_2 + Qw_1 - Rv_2 - Sv_1 \right\| = 0$$

where ||.|| is matrix norm, then slave systems (2) will be in the state of combination-combination synchronization with master systems (1).

Remark 1 Suppose scaling matrices P, Q, R, S are chosen as diag(p_1, p_2,…,p_n), diag(q_1, q2, … , q_n),diag(r_1, r_2, …, r_n), diag(s_1, s_2,…,s_n), respectively. If error components are defined as

$$p_a w_{a2} + q_b w_{b1} - r_c v_{c2} - s_d v_{d1},$$

such that at least one of the indices a, b, c, d is different from others, then master and slave systems (1), (2) will achieve multi-switching combination synchronization.

Remark 2. If any of P or Q is 0, then we will achieve multi-switching combination synchronization. If P = S = 0, Q = I or P = R = 0, Q = I or Q = S = 0, P = I or Q = R = 0, P = I, then multi-switching projective synchronization will be achieved for one master and one slave system.

3. MASTER AND SLAVE SYSTEMS

Li system is considered as first master system given by

$$\begin{cases} \dot{v}_{11} = \zeta_1(v_{21} - v_{11}), \\ \dot{v}_{21} = -v_{21} + v_{11}v_{31}, \\ \dot{v}_{31} = \eta_1 - v_{11}v_{21} - \theta_1 v_{31} \end{cases} \tag{3}$$

which shows chaotic behavior for the parameter values $\zeta_1 = 5$, $\eta_1 = 16$, $\theta_1 = 1$, Second master system is considered as Newton Leipnik system which given by

$$\begin{cases} \dot{v}_{12} = \zeta_2 v_{21} - v_{22} + 10v_{22}v_{32} \\ \dot{v}_{22} = -v_{12} - 0.4v_{22} + 5v_{12}v_{32}, \\ \dot{v}_{32} = \eta_2 v_{32} - 5v_{12}v_{22}. \end{cases} \tag{4}$$

which exhibits chaotic behavior for parameter values $\zeta_2 = 0{,}4$, $\eta_2 = 0{,}175$.

First slave system is taken as Wang system given by following equations

$$\begin{cases} \dot{w}_{11} = \zeta_3(w_{11} - w_{21}) - w_{21}w_{31}, \\ \dot{w}_{21} = -\eta_3 w_{21} + w_{11}w_{31}, \\ \dot{w}_{31} = w_{11}w_{3212} - \theta w_{31} + \vartheta_3 w_{11}. \end{cases} \tag{5}$$

Wang system is chaotic for $\zeta_3 = 1$, $\eta_3 = 5.7$, $\theta_3 = 5$, $\vartheta 3 = 0{,}06$. Second slave system is Bhalekar-Gejji system which shows chaotic behavior for $\zeta_4 = 27.3$, $\eta 4 = 1$, $\theta_4 = -2{,}667$, $\eta_4 = 10$ given by

$$\begin{cases} \dot{w}_{12} = \theta_4 w_{12} - w_{22}^2, \\ \dot{w}_{22} = \vartheta_4(w_{32} + w_{22}), \\ \dot{w}_{32} = w_{12}w_{22} - \zeta_4 w_{22} + \eta_4 w_{32}. \end{cases} \tag{6}$$

4. MULTI-SWITCHING SYNCHRONIZATION OF FOUR DIFFERENT CHAOTIC SYSTEMS

In this section, we will design appropriate controllers for the systems considered in section 3. The first slave system with controller is

$$\begin{cases} \dot{w}_{11} = \zeta_3(w_{11} - w_{21}) - w_{21}w_{31} + U_1, \\ \dot{w}_{21} = -\eta_3 w_{21} + w_{11}w_{31} + U_2 \\ \dot{w}_{31} = w_{11}w_{3212} - \theta w_{31} + \vartheta_3 w_{11} + U_3 \end{cases} \tag{7}$$

The second slave system Bhalekar Gejji system with controller is

$$\begin{cases} \dot{w}_{12} = \theta_4 w_{12} - w_{22}^2 + U_1^* \\ \dot{w}_{22} = \vartheta_4(w_{32} - w_{22}) + U_2^* \\ \dot{w}_{32} = w_{12}w_{22} + \zeta_4 w_{22} - \eta_4 w_{32} + U_3^*. \end{cases} \tag{8}$$

where $U_1, U_2, U_3, U_1^*, U_2^*, U_3^*$ represent different controllers in any switch . Our aim is to design effective controllers so that synchronization will be achieved between master and slave systems.

In case of combination combination multi-switching, suppose error is defined as

$$E_{abcd} = p_a w_{a2} + q_b w_{b1} - r_c v_{c2} - s_d v_{d1}, \tag{9}$$

Hence, the components of state variables of master systems in the form p_1w_{12}, p_2w_{22}, p_3w_{32}, q_1w_{11}, q_2w_{21}, q_3w_{31} and components of state variables of slave systems as r_1v_{12}, r_2v_{22}, r_3v_{32}, s_1v_{11}, s_2v_{21}, s_3v_{31}, will be combined in an arbitrary manner in each switch for different values of p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3, s_1, s_2, s_3, Then the four systems are said to be in the state of multi-switching if at least one of the indices a,b,c,d is different from remaining three and a, b, c, d can take any integer value of 1, 2, 3. We will say that the four systems are in multiswitching state if errors will take the following forms,

For $a \neq b = c = d$, we have E_{2111}, E_{3111}, E_{1333}, E_{2333}, E_{3222}, E_{1222}.

For $b \neq a = c = d$, we have E_{1211}, E_{1311}, E_{3133}, E_{3233}, E_{2322}, E_{2122}.

For $c \neq b = d = a$, we have E_{1121}, E_{1131}, E_{3313}, E_{3323}, E_{2232}, E_{2212}.

For $d \neq b = a = c$, we have E_{1112}, E_{1113}, E_{3331}, E_{3332}, E_{2223}, E_{2221}.

For $a = b \neq c = d$, we have E_{2211}, E_{3311}, E_{3322}, E_{1122}, E_{1133}, E_{2233}.

For $a = c \neq b = d$, we have E_{2121}, E_{3232}, E_{1212}, E_{2323}, E_{3131}, E_{1313}.

For $a = d \neq b = c$, we have E_{2112}, E_{3223}, E_{1331}, E_{3113}, E_{2112}, E_{2332}.

For $a = b \neq c \neq d$, we have E_{1123}, E_{2213}, E_{1132}, E_{2231}, E_{3312}, E_{3321}.

For $a = c \neq b \neq d$, we have E_{2123}, E_{2321}, E_{1312}, E_{1213}, E_{3132}, E_{3231}.

For $a = d \neq b \neq c$, we have E_{2132}, E_{2312}, E_{3123}, E_{3213}, E_{1231}, E_{1321}.

For $b = c \neq a \neq d$, we have E_{2113}, E_{3112}, E_{1223}, E_{3221}, E_{1332}, E_{2332}.

For $b = d \neq a \neq c$, we have E_{1232}, E_{3212}, E_{1323}, E_{2313}, E_{2131}, E_{3121}.

For $c = d \neq b \neq a$, we have E_{2133}, E_{1233}, E_{1322}, E_{3122}, E_{2133}, E_{1233}.

There may be several possibilities for choice of switches but we have just discussed four switches defined in the following way,

Switch one

$$\begin{cases} E_{1332} = p_1 w_{12} + q_3 w_{31} - r_3 v_{32} - s_2 v_{21} \\ E_{2121} = p_2 w_{22} + q_1 w_{11} - r_2 v_{22} - s_1 v_{11} \\ E_{3213} = p_3 w_{32} + q_2 w_{21} - r_1 v_{12} - s_3 v_{31} \end{cases} \tag{10}$$

Switch two

$$\begin{cases} E_{1221} = p_1 w_{12} + q_2 w_{3211} - r_2 v_{22} - s_1 v_{11} \\ E_{2332} = p_2 w_{22} + q_3 w_{31} - r_3 v_{32} - s_2 v_{21} \\ E_{3113} = p_3 w_{32} + q_1 w_{11} - r_1 v_{12} - s_3 v_{31} \end{cases} \tag{11}$$

Switch Three

$$\begin{cases} E_{1123} = p_1 w_{12} + q_1 w_{11} - r_2 v_{22} - s_3 v_{31} \\ E_{2231} = p_2 w_{22} + q_2 w_{21} - r_3 v_{32} - s_1 v_{11} \\ E_{3312} = p_3 w_{32} + q_3 w_{31} - r_1 v_{12} - s_2 v_{21} \end{cases} \quad (12)$$

Switch Four

$$\begin{cases} E_{1112} = p_1 w_{12} + q_1 w_{11} - r_1 v_{12} - s_2 v_{21} \\ E_{2323} = p_2 w_{22} + q_3 w_{31} - r_2 v_{22} - s_3 v_{31} \\ E_{3231} = p_3 w_{32} + q_2 w_{21} - r_3 v_{32} - s_1 v_{11} \end{cases} \quad (12)$$

4.1 Switch one

Now, we design the controllers for switch one. For this switch by equation (10), the error dynamical system can be written as

$$\begin{cases} \dot{E}_{1332} = p_1 \dot{w}_{12} + q_3 \dot{w}_{31} - r_3 \dot{v}_{32} - s_2 \dot{v}_{21} \\ \dot{E}_{2121} = p_2 \dot{w}_{22} + q_1 \dot{w}_{11} - r_2 \dot{v}_{22} - s_1 \dot{v}_{11} \\ \dot{E}_{3213} = p_3 \dot{w}_{32} + q_2 \dot{w}_{21} - r_1 \dot{v}_{12} - s_3 \dot{v}_{31} \end{cases} \quad (14)$$

By using the equations of master and slave systems the error dynamical system can be written as

$$\begin{cases} \dot{E}_{1332} = p_1[(\theta_4 w_{12} - w_{22}^2 + U_1^*) + q_3(w_{11}w_{21} - \theta_3 w_{31} + \vartheta_3 w_{11} + U_3) \\ \qquad -r_3(\eta_2 v_{32} - 5 v_{12} v_{22}) - s_2(-v_{21} + v_{11} v_{31})] \\ \dot{E}_{2121} = p_2(\vartheta_4(w_{32} - w_{22}) + U_2^*) + q_1(\zeta_3(w_{11} - w_{21}) - w_{21} w_{31} + U_1) \\ \qquad -r_2(v_{22} - 0.4 v_{22} + 5 v_{12} v_{32}) - s_1(\zeta_1(v_{21} - v_{11})) \\ \dot{E}_{3213} = p_3(w_{12} w_{22} + \zeta_4 w_{32} - \eta_4 w_{32}) + U_3^*) + q_2(-\eta_3 w_{21} + w_{11} w_{31} + U_2) \\ \qquad -r_1(-\zeta_2 v_{12} + v_{22} + 10 v_{22} v_{32}) - s_3(\eta_1 - v_{11} v_{21} - \theta_1 v_{31}). \end{cases} \quad (15)$$

Suppose

$$\begin{cases} U_{13} = p_1 U_1^* + q_3 U_3, \\ U_{21} = p_2 U_2^* + q_1 U_1, \\ U_{32} - p_3 U_3^* + q_2 U_2. \end{cases} \quad (16)$$

Now, we have the following results,

Theorem 1 Wang system (7) and Bhalekar Gejji (8) system will be in the state of multi-switching combination–combination synchronization with Li system (3) and Newton Leipnik system (4) for the following controllers,

$$U_{13} = -(p_1 w_{12} + q_3 w_{31} - r_3 v_{32} - s_2 v_{21}) + \zeta_1(p_2 w_{22} + q_1 w_{11} - r_2 v_{22} - s_1 v_{11})$$
$$-(p_3 w_{32} + q_2 w_{21} - r_1 v_{12} - s_3 v_{31}) - p_1(\theta_4 w_{12} - w_{22}^2)$$

$$- q_3(w_{11}w_{21} - \theta_3 w_{31} + \vartheta_3 w_{11}) + r_3(\eta_2 v_{32} - 5v_{12}v_{22}) + s_2(-v_{21} + v_{11}v_{31}),$$

$$U_{21} = -(p_2 w_{22} + q_1 w_{11} - r_2 v_{22} - s_1 v_{11}) - \zeta_1(p_1 w_{12} + q_3 w_{31} - r_3 v_{32} - s_2 v_{21})$$

$$-p_2(\vartheta_4(w_{32} - w_{22})) - q_1(\zeta_3(w_{11} - w_{21}) - w_{21}w_{31})$$

$$+ r_2(-v_{12} - 0.4v_{22} + 5v_{12}v_{32}) + s_1(\zeta_1(v_{21} - v_{11})),$$

$$U_{32} = (p_1 w_{12} + q_3 w_{31} - r_3 v_{32} - s_2 v_{21}) - (p_3 w_{32} + q_2 w_{21} - r_1 v_{12} - s_3 v_{31})$$

$$- p_3(w_{12}w_{22} + \zeta_4 w_{22} - \eta_4 w_{32}) - q_2(-\eta_3 w_{21} + w_{11}w_{31})$$

$$+ r_1(-\zeta_2 v_{12} + v_{22} + 10v_{22}v_{32}) + s_3(\eta_1 - v_{11}v_{21} - \theta_1 v_{31}), \tag{17}$$

Proof. Defining the Lyapunov function in the following manner

$$K(E_{1332}, E_{2121}, E_{3213}) = 0.5(E_{1332}^2 + E_{2121}^2 + E_{3213}^2) \tag{18}$$

Then derivative of the Lyapunov function is

$$\dot{K}(E_{1332}, E_{2121}, E_{3213})$$

$$= E_{1332}\dot{E}_{1332} + E_{2121}\dot{E}_{2121} + E_{3213}\dot{E}_{3213}$$

$$= E_{1332}[p_1(\theta_4 w_{12} - w_{22}^2 + U_1^*) + q_3(w_{11}w_{21} - \theta_3 w_{31} + \vartheta_3 w_{11} + U_3)$$

$$-r_3(\eta_2 v_{32} - 5v_{12}v_{22}) - s_2(-v_{21} + v_{11}v_{31})] + E_{2121}[p_2(\vartheta_4(w_{32} - w_{22}) + U_2^*)$$

$$+q_1(\zeta_3(w_{11} - w_{21}) - w_{21}w_{31} + U_1) - r_2(-v_{12} - 0.4v_{22} + 5v_{12}v_{32})$$

$$-s_1(\zeta_1(v_{21} - v_{11}))] + E_{3213}[p_3(w_{12}w_{22} + \zeta_4 w_{22} - \eta_4 w_{32} + U_3^*)$$

$$+q_2(-\eta_3 w_{21} + w_{11}w_{31} + U_2) - r_1(-\zeta_2 v_{12} + v_{22} + 10v_{22}v_{32})$$

$$-s_3(\eta_1 - v_{11}v_{21} - \theta_1 v_{31})]$$

$$= E_{1332}[p_1(\theta_4 w_{12} - w_{22}^2) + q_3(w_{11}w_{21} - \gamma_3 w_{31} + \delta_3 w_{11}) - r_3(\eta_2 v_{32} - 5v_{12}v_{22})$$

$$= -s_2(-v_{21} + v_{11}v_{31}) + p_1 U_1^* + q_3 U_3] + E_{2121}[p_2(\vartheta_4(w_{32} - w_{22}))$$

$$+q_1(\zeta_3(w_{11} - w_{21}) - w_{21}w_{31}) - r_2(-v_{12} - 0.4v_{22} + 5v_{12}v_{32}) - s_1(\zeta_1(v_{21} - v_{11}))$$

$$+p_2 U_2^* + q_1 U_1] + E_{3213}[p_3(w_{12}w_{22} + \zeta_4 w_{22} - \eta_4 w_{22})$$

$$+q_2(-\eta_3 w_{21} + w_{11}w_{31}) - r_1(-\zeta_2 v_{12} + v_{22} + 10v_{22}v_{32}) - s_3(\eta_1 - v_{11}v_{21} - \theta_1 v_{31})$$

$$+p_3 U_3^* + q_2 U_2]. \tag{19}$$

Substituting (17) into (20), we obtain

$$\dot{K}(E_{1332}, E_{2121}, E_{3213})$$

$$= E_{1332}\dot{E}_{1332} + E_{2121}\dot{E}_{2121} + E_{3213}\dot{E}_{3213}$$

$$= E_{1332}[p_1(\theta_4 w_{12} - w_{22}^2) + q_3(w_{11}w_{21} - \theta_3 w_{31} + \vartheta_3 w_{11}) - r_3(\eta_2 v_{32} - 5v_{12}v_{22})$$

$$-s_2(-v_{21}+v_{11}v_{31})+\{-(p_1w_{12}+q_3w_{31}-r_3v_{32}-s_2v_{21})+\zeta_1(p_2w_{22}+q_1w_{11}-r_2v_{22}$$
$$-s_1v_{11})-(p_3w_{32}+q_2w_{21}-r_1v_{12}-s_3v_{31})-p_1(\theta_4w_{12}-w_{22}^2)-q_3(w_{11}w_{21}-\theta_3w_{31}$$
$$+\vartheta_3w_{11})+r_3(\eta_2v_{32}-5v_{12}v_{22})+s_2(-v_{21}+v_{11}v_{31})\}]+E_{2121}[p_2(\vartheta_4(w_{32}-w_{22}))$$
$$+q_1(\zeta_3(w_{11}-w_{21})-w_{21}w_{31})-r_2(-v_{12}-0.4v_{22}+5v_{12}v_{32})-s_1(\zeta_1(v_{21}-v_{11}))$$
$$+\{-(p_2w_{22}+q_1w_{11}-r_2v_{22}-s_1v_{11})-\zeta_1(p_1w_{12}+q_3w_{31}-r_3v_{32}-s_2v_{21})$$
$$-p_2(\vartheta_4(w_{32}-w_{22})-q_1(\zeta_3(w_{11}-w_{21})-w_{21}w_{31})+r_2(-v_{12}-0.4v_{22}+5v_{12}v_{32})$$
$$+s_1(\zeta_1(v_{21}-v_{11}))\}]+E_{3213}[p_3(w_{12}w_{22}+\zeta_4w_{22}-\eta_4w_{22})+q_2(-\eta_3w_{21}+w_{11}w_{31})$$
$$-r_1(-\zeta_2v_{12}+v_{22}+10v_{22}v_{32})-s_3(\eta_1-v_{11}v_{21}-\theta_1v_{31})+\{(p_1w_{12}+q_3w_{31}-r_3v_{32}$$
$$-s_2v_{21})-(p_3w_{32}+q_2w_{21}-r_1v_{12}-s_3v_{31})-p_3(w_{12}w_{22}+\zeta_4w_{22}-\eta_4w_{32})$$
$$-q_2(-\eta_3w_{21}+w_{11}w_{31})+r_1(-\zeta_2v_{12}+v_{22}+10v_{22}v_{32})+s_3(\eta_1-v_{11}v_{21}-\theta_1v_{31})\}]$$
$$=E_{1332}(-E_{1332}+\zeta_1E_{2121}-E_{3213})+E_{2121}(-E_{2121}-\zeta_1E_{1332})+E_{3213}(-E_{3213}+E_{1332})$$
$$=-E_{1332}^2-E_{2121}^2-E_{3213}^2 \qquad (20)$$

which is negative definite. Hence the error dynamical system will be asymptotically stable and all the errors will converge to zero. Thus slave systems will achieve combination-combination synchronization with master systems.

We have the following corollaries from the above theorem and these can be proved similarly, therefore the proofs are omitted.

Corollary 1. If we take $p_1 = p_2 = p_3 = 0$, then for the following control laws,

$$U_{13}=-(q_3w_{31}-r_3v_{32}-s_2v_{21})+\zeta_1(q_1w_{11}-r_2v_{22}-s_1v_{11})$$
$$-(q_2w_{21}-r_1v_{12}-s_3v_{31})+q_3(w_{11}w_{21}-q_3w_{31}+\vartheta_3w_{11})$$
$$+r_3(\eta_2v_{32}-5v_{12}v_{22})+s_2(-v_{21}+v_{11}v_{31})$$
$$U_{21}=-(q_1w_{11}-r_2v_{22}-s_1v_11)-\zeta_1(q_3w_{31}-r_3v_{32}-s_2v_{21})-q_1(\zeta_3(w_{11}-w_{21})-w_{21}w_{31})$$
$$+r_2(-v_{12}-0.4v_{22}+5v_{12}v_{32})+s_1(\zeta_1(v_{21}-v_{11})),$$
$$U_{32}=(q_3w_{31}-r_3v_{32}-s_2v_{21})-(q_2w_{21}-r_1v_{12}-s_3v_{31})-q_2(-\eta_3w_{21}+w_{11}w_{31})$$
$$+r_1(-\zeta_2v_{12}+v_{22}+10v_{22}v_{32})+s_3(\eta_1-v_{11}v_{21}-q_1v_{31}) \qquad (21)$$

Wang system (7) will be in state of multi-switching combination synchronization with Li system (3) and Newton Leipnik system (4).

Corollary 2. If we take $q_1 = q_2 = q_3 = 0$, then for the following control laws:

$$U_{13} = -(p_1w_{12}-r_3v_{32}-s_2v_{21}) + \zeta_1(p_2w_{22}-r_2v_{22}-s_1v_{11}) - (p_3w_{32}-r_1v_{12}-s_3v_{31})$$
$$- p_1(\gamma_4w_{12}-w_{22}^2) - \theta_3w_{31} + d_3w_{11}) + r_3(\eta_2v_{32}-5v_{12}v_{22}) + s_2(-v_{21}+v_{11}v_{31})$$
$$U_{21} = -(p_2w_{22}-r_2v_{22}-s_1v_{11}) - \zeta_1(p_1w_{12}-r_3v_{32}-s_2v_{21}) - p_2(\vartheta_4(w_{32}-w_{22}),$$
$$+r_2(-v_{12}-0.4v_{22}+5v_{12}v_{32})+s_1(z_1(v_{21}-v_{11}))$$
$$U_{32} = (p_1w_{12}-r_3v_{32}-s_2v_{21}) - (p_3w_{32}-r_1v_{12}-s_3v_{31}) - p_3(w_{12}w_{22}+\zeta_4w_{22}-\eta_4w_{32})$$

$$+r_1(-\zeta_2 v_{12}+v_{22}+10v_{22}v_{32}) + s_3(\eta_1-v_{11}v_{21}-q_1v_{31}) \tag{22}$$

Bhalekar Gejji systems (6) will be in state of multi-switching combination synchronization with master systems Li (3) and Newton Leipnik system (4).

Corollary 3. If we take $p_1 = p_2 = p_3 = 0$, and $q_1 = q_2 = q_3 = 1$, and $r_1 = r_2 = r_3 = 0$, then for the following control laws,

$$\begin{cases} U_{13} = -(w_{31} - s_2 v_{21}) + \zeta_1(w_{11} - s_1 v_{11}) - (w_{21} - s_3 v_{31}) + s_2(-v_{21} + v_{11}v_{31}), \\ U_{21} = -(w_{11} - s_1 v_{11}) - \zeta_1(w_{31} - s_2 v_{21}) - (\zeta_3(w_{11} - w_{21}) - w_{21}w_{31}) + s_1(z_1(v_{21} - v_{11})), \\ U_{32} = (w_{31} - s_2 v_{21}) - (w_{21} - s_3 v_{31}) - (-\eta_3 w_{21} + w_{11}w_{31}) + s_3(\eta_1 - v_{11}v_{21} - \theta_1 v_{31}), \end{cases} \tag{23}$$

Wang system (7) will be in state of multi-switching projective synchronization with Li system (3) and if all $q_a's$ are not equal to 1, then modified projective synchronization will be achieved.

Corollary 4. If we take $p_1 = p_2 = p_3 = 0$, and $q_1 = q_2 = q_3 = 1$, and $s_1 = s_2 = s_3 = 0$, then for the following control laws,

$$\begin{cases} U_{13} = -(w_{31} - r_3 v_{32}) + \zeta_1(w_{11} - r_2 v_{22}) - (w_{21} - r_1 v_{12}) - (w_{11}w_{21} - \theta_3 w_{31} + \vartheta_3 w_{11}) \\ \qquad + r_3(\eta_2 v_{32} - 5v_{12}v_{22}), \\ U_{21} = -(w_{11} - r_2 v_{22}) - \zeta_1(w_{31} - r_3 v_{32}) - (\zeta_3(w_{11} - w_{21}) - w_{21}w_{31}) \\ \qquad + r_2(-v_{12} - 0.4v_{22} + 5v_{12}v_{32}), \\ U_{32} = (w_{31} - r_3 v_{32}) - (w_{21} - r_1 v_{12}) - (-\eta_3 w_{21} + w_{11}w_{31}) + r_1(-\zeta_2 v_{12} + v_{22} + 10v_{22}v_{32}), \end{cases} \tag{24}$$

Wang system (7) will be achieve multi-switching projective synchronization with Newton Leipnik system (4) and if all $q_a's$ are not equal to 1, this it will become a case of modified projective synchronization.

Corollary 5. If we take $p_1 = p_2 = p_3 = 1$, $q_1 = q_2 = q_3 = 0$, and $r_1 = r_2 = r_3 = 0$, then for the following control laws,

$$\begin{cases} U_{13} = -(w_{12} - s_2 v_{21}) + \zeta_1(w_{22} - s_1 v_{11}) - (w_{32} - s_3 v_{31}) - (\theta_4 w_{12} - w_{22}^2) + s_2(-v_{21} + v_{11}v_{31}), \\ U_{21} = -(w_{22} - s_1 v_{11}) - \zeta_1(w_{12} - s_2 v_{21}) - \vartheta_4(w_{32} - w_{22}) + s_1(\zeta_1(v_{21} - v_{11}), \\ U_{32} = (w_{12} - s_2 v_{21}) - (w_{32} - s_3 v_{31}) - (w_{12}w_{22} + \zeta_4 w_{22} - \eta_4 w_{32}) + s_3(\eta_1 - v_{11}v_{21} - \theta_1 v_{31}), \end{cases} \tag{25}$$

Bhalekar Gejji system (8) will be in the state of multi-switching projective synchronization with Li system (3) and if all $p_a's$ are not equal to 1, then the systems will be in the state of modified projective synchronization.

Corollary 6. If we take $p_1 = p_2 = p_3 = 1$, $q_1 = q_2 = q_3 = 0$, and $s_1 = s_2 = s_3 = 0$, then for the following control laws,

$$\begin{aligned} U_{13} &= -(w_{12} - r_3 v_{32}) + \zeta_1(w_{22} - r_2 v_{22}) - (w_{32} - r_1 v_{12}) - (\theta_4 w_{12} - w_{22}^2) + r_3(\eta_2 v_{32} - 5v_{12}v_{22}) \\ U_{21} &= -(w_{22} - r_2 v_{22}) - \zeta_1(w_{12} - r_3 v_{32}) - \vartheta_4(w_{32} - w_{22}) + r_2(-v_{12} - 0.4v_{22} + 5v_{12}v_{32}), \\ U_{32} &= (w_{12} - r_3 v_{32}) - (w_{32} - r_1 v_{12}) - (w_{12}w_{22} + \zeta_4 w_{22} - \eta_4 w_{32}) + r_1(-\zeta_2 v_{12} + v_{22} + 10v_{22}v_{32}). \end{aligned} \tag{26}$$

Bhalekar Gejji (8) system will be in the state of multi-switching projective synchronization with Newton Leipnik system (4) and if all $p_a's$ are not equal to 1, this will be the case of modified projective synchronization.

4.1. Switch 2, Switch 3, Switch 4

Theorem 2. Wang (7) and Bhalekar Gejji (8) systems will be in the state of multi-switching combination–combination synchronization with chaotic Li (3) and Newton Leipnik (4) systems, for the errors defined by (11) if the controllers are defined in the following manner,

$$\begin{cases} U_{12} = -(p_1w_{12} + q_2w_{21} - r_2v_{22} - s_1v_{11}) + (p_2w_{22} + q_3w_{31} - r_3v_{32} - s_2v_{21}) - p_1(\theta_4w_{12} - w_{22}^2) \\ \qquad -q_2(-\eta_3w_{21} + w_{11}w_{31}) + r_2(-v_{12} - 0.4v_{22} + 5v_{12}v_{32}) + s_1(\zeta_1(v_{21} - v_{11})), \\ U_{23} = -(p_2w_{22} + q_3w_{31} - r_3v_{32} - s_2v_{21}) + (p_1w_{12} + q_2w_{21} - r_2v_{22} - s_1v_{11}) \\ \qquad +(p_3w_{32} + q_1w_{11} - r_1v_{12} - s_3v_{31}) - p_2(\vartheta_4(w_{32} - w_{22}) \\ \qquad -q_3(w_{11}w_{21} - q_3w_{31} + \vartheta_3w_{11}) + r_3(b_2v_{32} - 5v_{12}v_{22}) + s_2(-v_{21} + v_{11}v_{31}), \\ -q_1(\zeta_3(w_{11} - w_{21} - w_{21}w_{31}) + r_1(-\zeta_2v_{12} + v_{22} + 10v_{22}v_{32}) + s_3(\beta_1 - v_{11}v_{21} - \theta_1v_{31}), \end{cases} \tag{27}$$

Theorem 3 Wang (7) and Bhalekar Gejji (8) systems are in the state of multi-switching combination-combination synchronization with the systems, Li (3) and Newton Leipnik (4) systems for the errors defined by (12) if the controllers are defined in the following manner,

$$\begin{aligned} U_{11} &= -(p_1w_{12}+q_1w_{11}-r_2v_{22}-s_3v_{31}) + (p_3w_{32}+q_3w_{31}-r_1v_{12}-s_3v_{21}) \\ &\quad - p_1(\theta_4w_{12}-w_{22}^2) - q_1(z_3(w_{11}-w_{21})-w_{21}w_{31}) + r_2(-v_{12}-0.4v_{22}+5v_{12}v_{32}) \\ &\quad +s_3(\eta_1-v_{11}v_{21}-\theta_1v_{31}), \\ U_{22} &= -(p_2w_{22}+q_2w_{21}-r_3v_{32}-s_1v_{11}) + (p_3w_{32}+q_3w_{31}-r_1v_{12}-s_3v_{21}) - p_2(\vartheta_4(w_{32}-w_{22}) \\ &\quad -q_2(-\eta_3w_{21}+w_{11}w_{31}) + r_3(\eta_2v_{32}-5v_{12}v_{22}) + s_1(\zeta_1(v_{21}-v_{11})), \\ U_{33} &= -(p_1w_{12}+q_1w_{11}-r_2v_{22}-s_3v_{31}) - (p_2w_{22}+q_2w_{21}-r_3v_{32}-s_1v_{11}) \\ &\quad - (p_3w_{32}+q_3w_{31}-r_1v_{12}-s3v21) - p_3(w_{12}w_{22}+\zeta_4w_{22}-\eta_4w_{32}) \\ &\quad - q_3(w_{11}w_{21}-q_3w_{31}+\vartheta_3w_{11})+r_1(-\zeta_2v_{12}+v_{22}+10v_{22}v_{32}) + s_2(-v_{21}+v_{11}v_{31}), \end{aligned} \tag{28}$$

Theorem 4. Wang system (7) and Bhalekar Gejji (8) system will achieve multi-switching combination-combination synchronization with Li system 3 and Newton Leipnik (4) system for the errors defined by (13) if the controllers are defined in the following manner

$$\begin{aligned} U_{11} &= -(p_1w_{12}+q_1w_{11}-r_1v_{12}-s_2v_{21}) + (p_2w_{22}+q_3w_{31}-r_2v_{22}-s_3v_{31}) \\ &\quad + (p_3w_{32}+q_2w_{21}-r_3v_{32}-s_1v_{11}) - p_1(\theta_4w_{12}-w_{22}^2) - q_1(\zeta_3(w_{11}-w_{21}) \\ &\quad -w_{21}w_{31}) + r_1(-\zeta_2v_{12}+v_{22}+10v_{22}v_{32}) + s_2(-v_{21}+v_{11}v_{31}), \\ U_{23} &= -(p_1w_{12}+q_1w_{11}-r_1v_{12}-s_2v_{21}) - (p_2w_{22}+q_3w_{31}-r_2v_{22}-s_3v_{31}) \\ &\quad - p_2(\vartheta_4(w_{32}-w_{22})) - q_3(w_{11}w_{21}-\theta_3w_{31}+\vartheta_3w_{11}) + r_2(-v_{12}-0.4v_{22}+5v_{12}v_{32}) \\ &\quad - s_3(\eta_1-v_{11}v_{21}-\theta_1v_{31}), \\ U_{32} &= -(p_1w_{12}+q_1w_{11}-r_1v_{12}-s_2v_{21}) - (p_3w_{32}+q_2w_{21}-r_3v_{32}-s_1v_{11}) \\ &\quad - p_3(w_{12}w_{22}+\zeta_4w_{22}-\eta_4w_{32}) - q_2(-\eta_3w_{21}+w_{11}w_{31}) \\ &\quad +r_3(\eta_2v_{32}-5v_{12}v_{22}) + s_1(\zeta_1(v_{21}-v_{11})). \end{aligned} \tag{29}$$

Proofs of the above theorems are not given here since these theorems can be proved in similar manner as the Theorem (1) has been proved. Also same corollaries can be obtained from these results by assigning some particular values to p_a, q_b, r_c, s_d.

5. NUMERICAL SIMULATION

For numerical simulation parameters values are chosen as $\zeta_1 = 5$, $\eta_1 = 16$, $\theta_1 = 1$, $\zeta_2 = 0.4$, $\eta_2 = 0.175$, $\zeta_3 = 1$, $\eta3 = 5.7$, $\theta_3 = 5$, $\vartheta_3 = 0.06$, $\zeta_4 = 27.3$, $\eta_4 = 1$, $\theta_4 = -2.667$, $\vartheta_4 = 10$ initial conditions for Li system and Newton Leipnik system are (12,8,20) and (0,349,0,–0,16) respectively, and initial conditions for Wang system and Bhalekar Gejji systems are (24,7,18) and (17,22,9) respectively which have been fixed throughout the discussion.

Initial conditions for the error system in switch one are (11.32, 10,–44.698) as we have chosen $p_1 = p_2 = p_3 = 1$, $q_1 = q_2 = q_3 = 1$, $r_1 = r_2 = r_3 = 2$, $s_1 = s_2 = s_3 = 3$. In this switch $w_{12} + w_{31}$, $w_{22} + w_{11}$, $w_{32} + w_{21}$ are synchronized with $2v_{32} + 3v_{21}$, $2v_{22} + 3v_{11}$, $2v_{12} + 3v_{31}$. Synchronization of combination of these variables and errors converging to zero are shown in figures (1) and (2).

Initial conditions for the error system in switch two are (36, 47.84, 53.349) as we have chosen $p_1 = p_2 = p_3 = 1$, $q_1 = q_2 = q_3 =1$, $r_1 = r_2 = r_3 = -1$, $s_1 = s_2 = s_3 = -1$. In this switch we can say that $w_{12} + w_{21}$, $w_{22} + w_{31}$, $w_{32} + w_{11}$ are in anti-synchronized state with $v_{22} + v_{11}$, $v_{32} + v_{21}$, $v_{12} + v_{31}$. Anti-synchronization of combination of these variables and errors converging to zero are shown in figures (3) and (4).

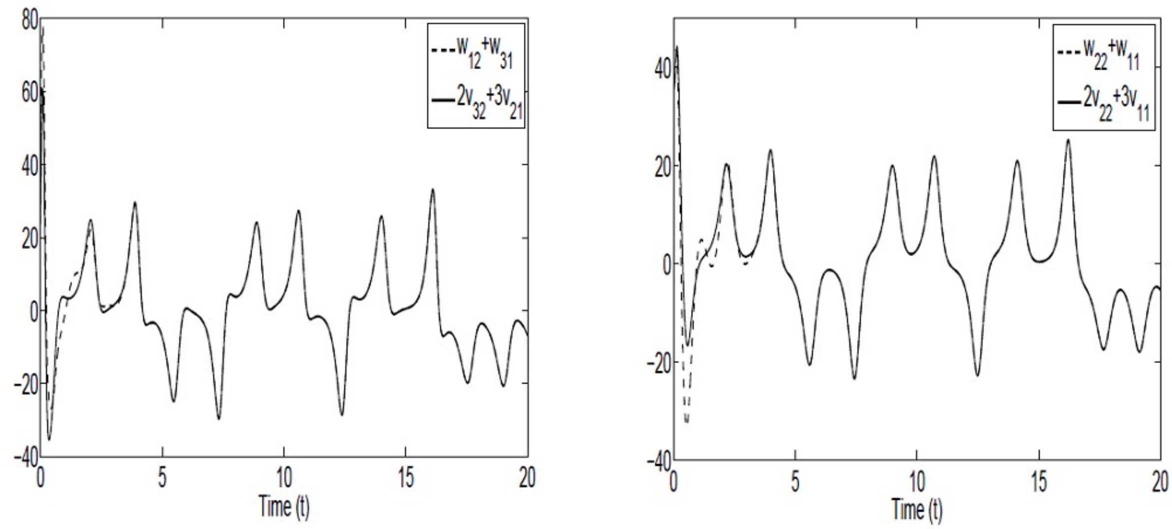

Figure 1. Synchronization between variables $w_{12} + w_{31}$, $2v_{32} + 3v_{21}$ and $w_{22} + w_{11}$, $2v_{22} + 3v_{11}$

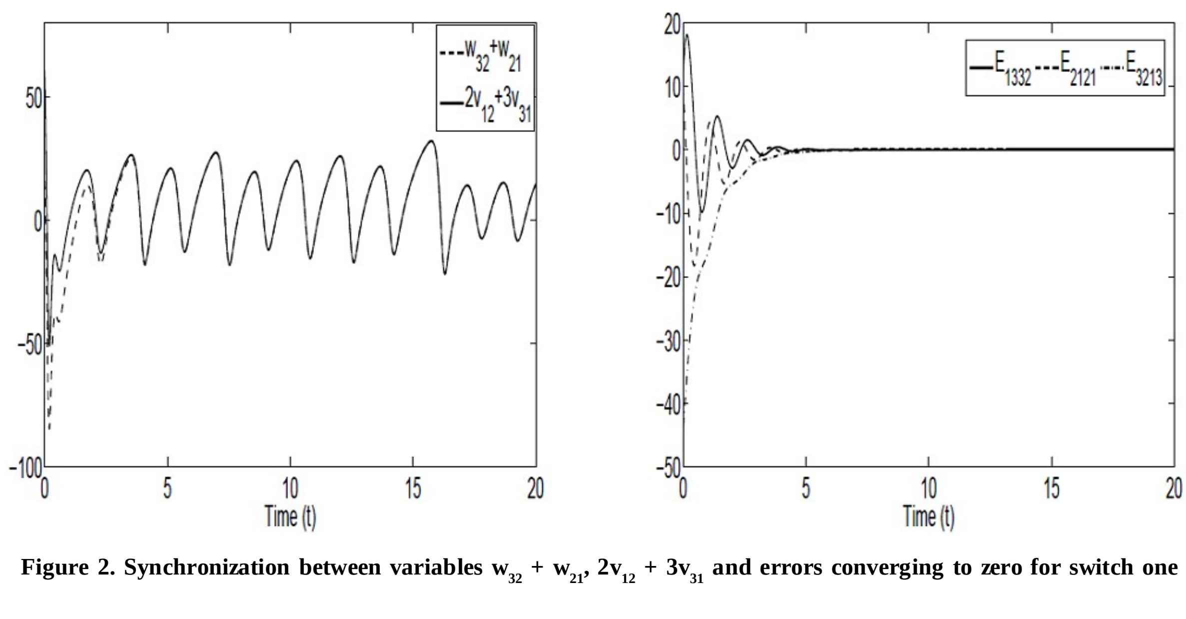

Figure 2. Synchronization between variables $w_{32} + w_{21}$, $2v_{12} + 3v_{31}$ and errors converging to zero for switch one

6. CONCLUSION

In this paper, we have investigated multi switching combination-combination synchronization between master system Li and Newton-Leipnik systems and slave systems Wang and Bhalekar Gejji system by using non linear control. A useful and simple approach is described to construct suitable controllers and fruitful results are obtained.

Both theoretical and numerical results are in agreement. There are so many other ways to extend this work, like combination-combination synchronization with unknown parameter can also be investigated or this synchronization methodology can be applied on hyperchaotic systems so that more states of switching may be considered which will be more beneficial for secure communication.

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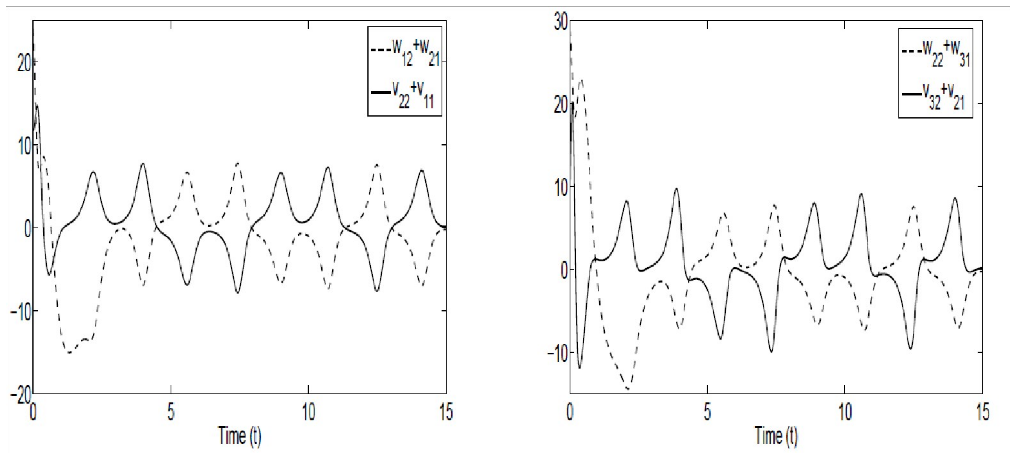

Figure 3. Anti-synchronization between variables $w_{12} + w_{21}$, $v_{22} + v_{11}$ and $w_{22} + w_{31}$, $v_{32} + v_{21}$

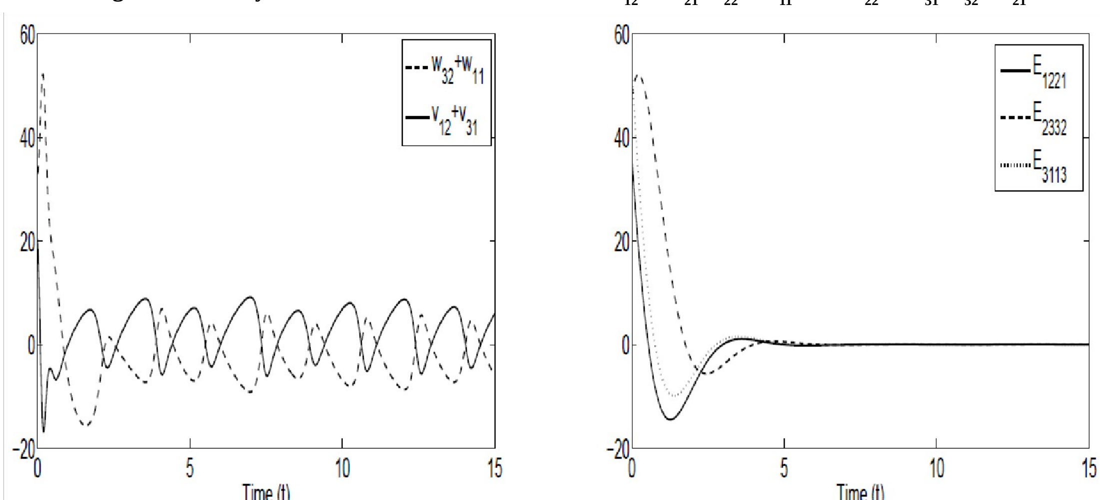

Figure 4. Anti-synchronization between variables $w_{32} + w_{11}$, $v_{12} + v_{31}$ and errors converging to zero for switch two

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