*International Review of Pure and Applied Mathematics*
Volume 14 • Number 2 • July-December 2018

# THE SOLUTION OF BLASIUS EQUATION BY DIAGONAL PADE' APPROXIMANT AND COMPARISON WITH VARINATIONAL ITERATION METHOD

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***Abstract:*** The Blasius differential equation is a well-known ordinary differential equation of 3rd order, which is the mother of all boundary layer equation in fluid dynamics. This equation is discussed in many articles by analytical and numerical ways. The current paper is aimed to propose an approximate method for solving the Blasius boundary layer problem. Approximate solution is derived by Diagonal Pade' [3/3] approximant and compared to the results obtained from Varinational Iteration Method (VIM) & R-K 4th order method (Numerical Method). Results reveal that the proposed method (Diagonal Pade' approximant) leads to a nearly accurate solution which is in agreement with VIM and R-K 4th order method. An efficient algorithm is also programmed using MATLAB which make the solution of Blasius equation relatively simple and quick. The accuracy of the proposed method is higher than other approximation analytical solution, hence suggest that proposed method is efficient and practical.

***Keywords:*** Blasius equation, Boundary layer, Diagonal Pade' approximation method, ordinary differential equation, Varinational Iteration method (VIM).

## I. INTRODUCTION

In Fluid Dynamics, the Blasius equation isof $3^{rd}$ order ordinary differential equation describes by Paul Richard Heinrich Blasius which arises in certain boundary layer problems. Blasius himself solved the equation by using a power series expansion method in 1908 [1]. Falkner and Skan later generalized Blasius's solution to wedge flow (Falkner-Skan boundary layer), i.e. flows in which the plate is not parallel to the flow. We consider the generalized Blasius equation is

$$u'''(t) + \alpha u(t)\, u''(t) = 0, \quad 0 \le t < \infty, \tag{1}$$

Where $\alpha = 1$ or $\alpha = 1/2$, with boundary conditions

$$u(0) = 0,\ u'(0) = \delta,\ u'(\infty) = 1$$

For special case of $\alpha = 1/2$ and $\delta = 0$, the Blasius equation is

$$u'''(t) + \frac{1}{2}u(t)u''(t) = 0 \; with \; u(0) = u'(0) = 0, u'(\infty) = 1 \tag{2}$$

where $\boldsymbol{u(t)}$ is the free stream velocity function, $\boldsymbol{t}$ stands for thickness of boundary layer and $\boldsymbol{u^n}$ $(\boldsymbol{t})$ denotes the nth order derivative.

The Blasius equation have been solved by several analytical and numerical methods. Chen and Yu [2] used the differential transformation method for solution. Lin [3] employed the parameter iteration method for the solution of Blasius equation. Parand and Taghavi [4] used collocation method to solve this equation. Liao [5] solved the Blasius equation using Homotopy Analysis method(HAM).Wang [6] used the Adomian Decomposition Method (ADM) to solve the Blasius equation. Wazwaz [7.] applied the modified ADM to obtained analytical solution of Blasius equation.

## II. DIAGONAL PADE' APPROXIMANTS

A Pade' approximant is the ratio of two polynomials constructed from the coefficients of the Taylor series expansion of a function $u$ $(t)$. The technique was developed around 1890 by Henri Pade'. The Padé approximant often gives better closed form approximation of the function and it may still work where the Taylor series does not converge. For these reasons Padé approximants are used extensively in computer calculation. The [L/M] Pade' approximants to a function $u$ $(t)$ are given by [8,9]. Recently Kaur & Garg [10, 11] investigate the acceleration motion of a single vertically falling non-spherically particle in incompressible Newtonian fluid and Radiation effect on velocity of a vertically falling non-spherical particle in incompressible Newtonian fluid by Diagonal Pade'[3/3]. A Pade' approximant is the ratio of two polynomials constructed from the coefficients of the Taylor series expansion of a function $u$ $(t)$. The [L/M] Pade' approximants to a function $u$ $(t)$ are given by

$$\left[\frac{L}{M}\right]=\frac{P_L(t)}{Q_M(t)} \tag{3}$$

Where $P_L$ $(t)$ is a polynomial of the degree of at most $L$ and $Q_M(t)$ is a polynomial of the degree of at most $M$. The formal power series is given

$$u(t)=\sum_{i=1}^{\infty} a_i\, t^i$$

*i.e.* $u(t) = a_0 + a_1\, t^1 + a_2\, t^2 + a_3\, t^3 + \cdots \ldots \ldots \ldots ..$ (3a)

Find the coefficients $a_0$, $a_1$, $a_2$, $a_3$,…

The given equation is

$$u'''(t) = -\frac{1}{2}uu'', \text{ with conditions } u(0) = u'(0) = 0,\ u'(\infty) = 1 \tag{3b}$$

Solve the above equation by Taylor series about zero is given by

$$u(t)=u_0+tu_0'+\frac{t^2}{2!}u_0''+\frac{t^3}{3!}u_0'''+\frac{t^4}{4!}u^{iv}{}_0+\frac{t^5}{5!}u^{v}{}_0+\frac{t^6}{6!}u^{vi}{}_0+\frac{t^7}{7!}u^{vii}{}_0+\frac{t^8}{8!}u^{viii}{}_0+......$$

Solution is $u(t)=(k^1)\frac{t^2}{2}-(k^2)\frac{t^5}{240}+(k^3)\frac{11t^8}{161280}-(k^4)\frac{5\,t^{11}}{4257792}+\cdots$ (3c)

Where $k=u''_0$

So $a_0=0,\ a_1=0,\ a_2=(k^1)\frac{1}{2},\ a_3=0, a_4=0,\ a_5=-(k)^2\frac{1}{240},\ a_6=0, a_7=0,$

$a_8=(k^3)\frac{11}{161280},\ a_9=0,\ a_{10}=0,\ a_{11}=-(k^4)\frac{5}{4257792}$ and so on

From previous literature, a highly accurate numerical value of

$$k=u''_0=u''(0)=0.332[12,13]$$

$$u(t)-\frac{P_L(t)}{Q_M(t)}=0\ (t^{L+M+1})$$

Determine the coefficient of $p_L(t)$ and $Q_M(t)$ by the help of eq. (3.6.2) and take normalization condition $Q_M(0) = 1$

From, $u(t)-\frac{P_L(t)}{Q_M(t)}=0$

$$P_L(t)=p_0+p_1\,t^1+p_2\,t^2+p_3\,t^3+\ldots\ldots\ldots\ldots..+p_L\,t^L$$
$$Q_M(t)=q_0+q_1\,t^1+q_2\,t^2+q_3\,t^3+\ldots\ldots\ldots\ldots..+q_M\,t^M$$

To obtain a diagonal pade' approximants of a different order [3/3], the symbolic calculus software MATLAB is used.

**For pade' [3/3]**

$$u(t)-\frac{P_3(t)}{Q_3(t)}=0$$

i.e. $u(t)=a_0+a_1t^1+a_2t^2+a_3t^3+.........=\frac{p_0+p_1t^1+p_2t^2+p_3t^3}{q_0+q_1t^1+q_2t^2+q_3t^3}$

$$(a_0+a_1\,t^1+a_2\,t^2+a_3\,t^3+\ldots..)\,(q_0+q_1\,t^1+q_2\,t^2+q_3\,t^3)=p_0+p_1\,t^1+p_2\,t^2+p_3\,t^3 \quad (3d)$$

From (3d), $a_0\, q_0 = p_0$

$$a_1\, q_0 + a_0\, q_1 = p_1$$

$a_2\, q_0 + a_1\, q_1 + a_0\, q_2 = p_2$

$a_3\, q_0 + a_2\, q_1 + a_1\, q_2 + a_0\, q_3 = p_3$

$a_4\, q_0 + a_3\, q_1 + a_2\, q_2 + a_1\, q_3 = 0$

$a_5\, q_0 + a_4\, q_1 + a_3\, q_2 + a_2\, q_3 = 0$

$a_6\, q_0 + a_5\, q_1 + a_4\, q_2 + a_3\, q_3 = 0$

Generally, $p_L = a_{L+} \Sigma_{i=1}^{\min.(L,M)} q_i a_{L-i}$

Solving these non-linear system of equations by Gauss elimination method and using eq

$$p_0 = 0,\ \ p_1 = 0,\ \ p_2 = \frac{k}{2},\ \ p_3 = 0 \text{ and } q_0 = 1,\ q_1 = 0,\ q_2 = 0,\ q_3 = \frac{k}{120}$$

So, Pade' [3/3]

$$u = \frac{\frac{k}{2}t^2}{1 + \frac{k}{120}t^3} = \frac{60kt^2}{120 + kt^3}$$

$$u1 = u' = \frac{120kt}{120 + kt^3} - \frac{180k^2t^4}{(120 + kt^3)^{2'}}$$

$$u2 = u'' = \frac{120k}{120 + kt^3} - \frac{1080k^2t^3}{(120 + kt^3)^2} - \frac{1080k^3t^6}{(120 + kt^3)^{3'}} \qquad \text{(3e)}$$

**Table 1**

**Solution of BLASIUS Equation by Diagonal Pade' [3/3] approximant (k=0.3320)**

| $t$ | $u_{Pade'[3/3]}$ | $u1 = u'_{Pade'[3/3]}$ | $u2 = u''_{Pade'[3/3]}$ |
|---|---|---|---|
| 0.00 | 0.0000 | 0.0000 | 0.3320 |
| 0.25 | 0.0104 | 0.0830 | 0.3319 |
| 0.50 | 0.0415 | 0.1659 | 0.3309 |
| 0.75 | 0.0933 | 0.2483 | 0.3281 |
| 1.00 | 0.1655 | 0.3297 | 0.3229 |
| 1.25 | 0.2580 | 0.4094 | 0.3143 |

*contd. table 1*

| $t$ | $u_{Pade'[3/3]}$ | $u1 = u'_{Pade'[3/3]}$ | $u2=u''_{Pade'[3/3]}$ |
|---|---|---|---|
| 1.50 | 0.3700 | 0.4865 | 0.3018 |
| 1.75 | 0.5009 | 0.5600 | 0.2848 |
| 2.00 | 0.6496 | 0.6285 | 0.2629 |
| 2.25 | 0.8147 | 0.6910 | 0.2361 |
| 2.50 | 0.9945 | 0.7462 | 0.2045 |
| 2.75 | 1.1871 | 0.7929 | 0.1686 |
| 3.00 | 1.3902 | 0.8301 | 0.1291 |

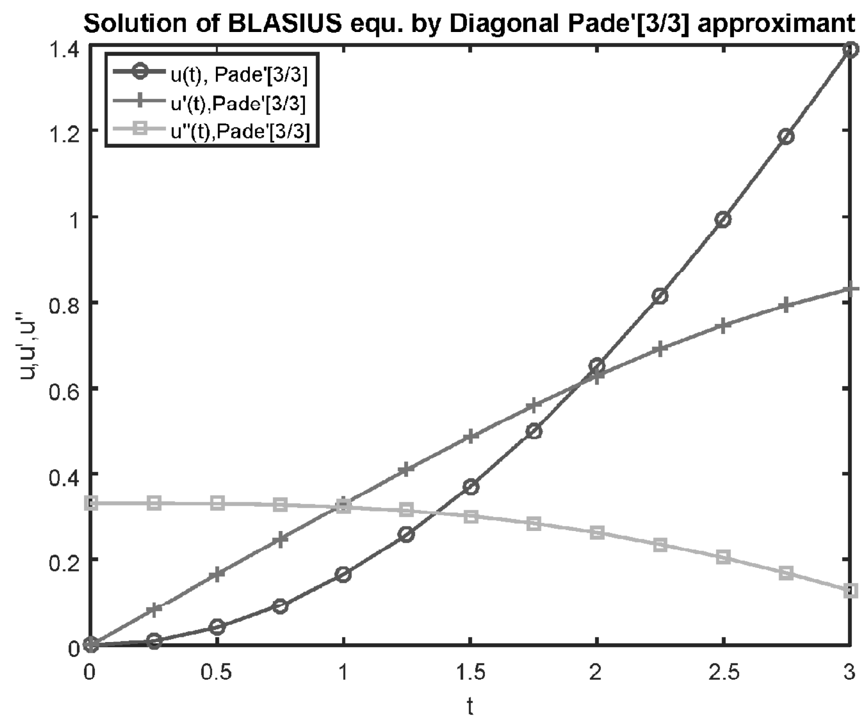


**Figure 1: The solution of BLASIUS equation in boundary layer flow of equ. (2) by Diagonal Pade' approximant**

## III. VARINATIONAL ITERATION METHOD (VIM)

J. He, was introduced Varinational iteration Method (VIM) to solve the several nonlinear ordinary and partial differential equations in 1997 [14]. He's Varinational iteration method (VIM) has been extensively applied as a power tool for solving various kinds of problems. Very recently it was recognized that the VIM can be an effective procedure for solution of various nonlinear problems without usual restrictive assumptions [15,16]. Application of the method have been enlarged due to its flexibility, convenience and accuracy. In this section, the concept of VIM is briefly introduced. Consider the non-linear differential equation given in the form

$$Lu\ (t) + Nu(t) = g(t) \tag{4}$$

where $L$ is a linear operator, $N$ is a nonlinear operator and $g(t)$ is a unknown function. By using the Varinational iteration method, a correction functional can be constructed as

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda\{Lu_n(\zeta) + N\tilde{u}(\zeta) - g(\zeta)\}d\zeta \tag{4a}$$

Where $\lambda$ is a general Lagrange multiplier, which can be determined, the subscript $n$ means the nth approximation; $u_n$ is restricted variation and $\delta u_n = 0$.

Using VIM to solve the Blasius equation

$$Nu(t) = u'''(t) + 1/2u(t)\ u''(t) \text{ and } g(t) = 0 \tag{4b}$$

Equation (4b) can be transformed as $u'''(s) + 1/2u(s)u'' = 0$, with the correction function

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda\ \{u'''_n(s) + \frac{1}{2}u_n(s)u''_n(s)\}ds \tag{4c}$$

Where $\lambda$ is the Lagrange multiplier.

Integrating the above equation by parts

The stationary conditions obtained from equation (4c) are

$$1 + \lambda''\ |\ s = t = 0 \rightarrow \lambda'' = -1\ ,$$

$$\lambda'\ |\ s = t = 0 \rightarrow \lambda' = 0,$$

$$\lambda|s = t = 0 \rightarrow \lambda = 0$$

From the above equations, the Lagrange multiplier $\lambda$ can be derived $\lambda = \frac{-(s-t)^2}{2}$

The correction function can be expressed as

$$u_{n+1}(t) = u_n(t) + \int_0^t \frac{-(s-t)^2}{2}\{u'''_n(s) + \frac{1}{2}u_n(s)u''_n(s)\}ds \tag{4d}$$

In order to start iteration using equ. (4*d*), $u_0(t)$ is needed, which is represented McLaurin series with the first three terms

$$u_0(t) = \Sigma_{m=0}^{2}\frac{t^m}{m!}u^{(m)}(0) = u_0 + tu_0' + \frac{t^2}{2!}u_0'' \tag{4e}$$

And assuming $u''_0 = k = 0.332$ [17], following equation can be obtained from equation (4e)

$$u_0(t) = 0.166t^2 \tag{4f}$$

Solve the equation (4c) by mathematical software MATLAB.

**Table 2**
**Solution of BLASIUS Equation by VIM 4th iteration**

| $t$ | $u_{VIM\ 4th\ iteration}$ | $u1 = u'_{VIM\ 4th\ iteration}$ | $u2=u''_{VIM\ 4th\ iteration}$ |
|---|---|---|---|
| 0.00 | 0.0000 | 0.0000 | 0.3320 |
| 0.25 | 0.0104 | 0.0830 | 0.3319 |
| 0.50 | 0.0415 | 0.1659 | 0.3309 |
| 0.75 | 0.0933 | 0.2483 | 0.3281 |
| 1.00 | 0.1655 | 0.3297 | 0.3230 |
| 1.25 | 0.2580 | 0.4095 | 0.3146 |
| 1.50 | 0.3701 | 0.4867 | 0.3025 |
| 1.75 | 0.5011 | 0.5604 | 0.2866 |
| 2.00 | 0.6499 | 0.6297 | 0.2667 |
| 2.25 | 0.8154 | 0.6935 | 0.2434 |
| 2.50 | 0.9961 | 0.7511 | 0.2174 |
| 2.75 | 1.1904 | 0.8021 | 0.1898 |
| 3.00 | 1.3966 | 0.8460 | 0.1619 |

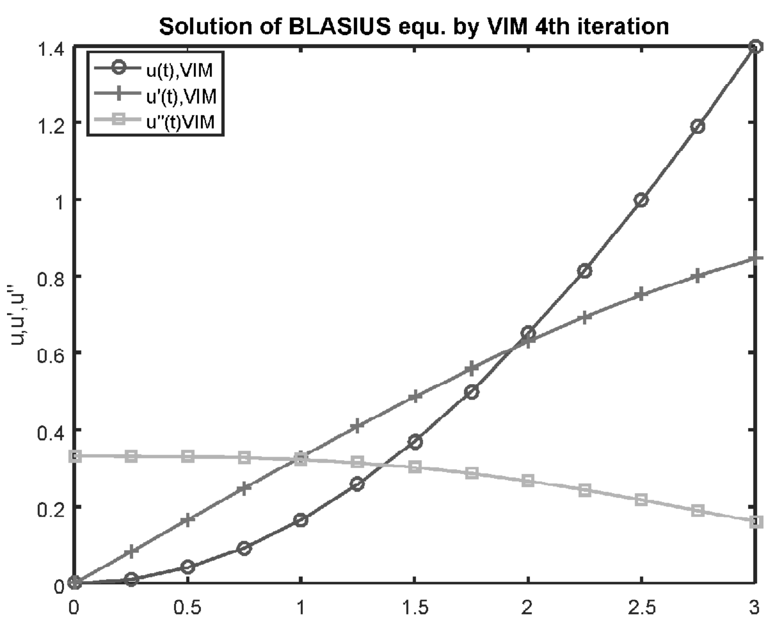


**Figure 2: The solution of BLASIUS equation in boundary layer flow of equ. (2) by VIM**

## VI. R-K 4th ORDER METHOD (NUMERICAL METHOD)

In numerical analysis, the Runge-Kutta methods are a family of implicit and explicit iterative methods. These methods were developed around 1900 by German mathematicians C. Runge and M. W. Kutta.

$$\text{Let } u = F,\ u' = F' = G,\ u'' = F'' = G' = H,\ u''' = H' \tag{5}$$

The equ. (2) becomes as follows

$$u'''(t) = -\frac{1}{2}u(t)u''(t) \ i.e. H' = -\frac{1}{2}FH \ \text{ with } F(0) = 0,\ G(0) = 0\ \&\ H(0) = 0.332 \tag{5a}$$

**Table 3**
**Solution of BLASIUS Equation by R-K 4th order (Numerical Method)**

| $t$ | $u_{R\text{-}K\ 4th\ order}$ | $u1=u'_{R\text{-}K\ 4th\ order}$ | $u2=u''_{R\text{-}K\ 4th\ order}$ |
|---|---|---|---|
| 0.0 | 0.0000 | 0.0000 | 0.3320 |
| 0.5 | 0.0415 | 0.1659 | 0.3309 |
| 1.0 | 0.1656 | 0.3297 | 0.3230 |
| 1.5 | 0.3701 | 0.4867 | 0.3025 |
| 2.0 | 0.6500 | 0.6297 | 0.2667 |
| 2.5 | 0.9962 | 0.7511 | 0.2174 |
| 3.0 | 1.3966 | 0.8459 | 0.1614 |
| 3.5 | 1.8375 | 0.9129 | 0.1078 |
| 4.0 | 2.3054 | 0.9554 | 0.0643 |
| 4.5 | 2.7898 | 0.9793 | 0.0341 |
| 5.0 | 3.2829 | 0.9914 | 0.0160 |
| 5.5 | 3.7801 | 0.9967 | 0.0067 |
| 6.0 | 4.2791 | 0.9988 | 0.0025 |
| 6.5 | 4.7787 | 0.9996 | 0.0008 |
| 7.0 | 5.2786 | 0.9998 | 0.0003 |
| 7.5 | 5.7785 | 0.9999 | 0.0001 |
| 8.0 | 6.2784 | 0.9999 | 0.0000 |
| 8.5 | 6.7784 | 0.9999 | 0.0000 |
| 9.0 | 7.2783 | 0.9999 | 0.0000 |
| 9.5 | 7.7782 | 0.9999 | 0.0000 |
| 10 | 8.2782 | 0.9999 | 0.0000 |
| - | - | - | - |
| - | - | - | - |
| - | - | - | - |
| ∞ | ∞ | 1 | 0 |

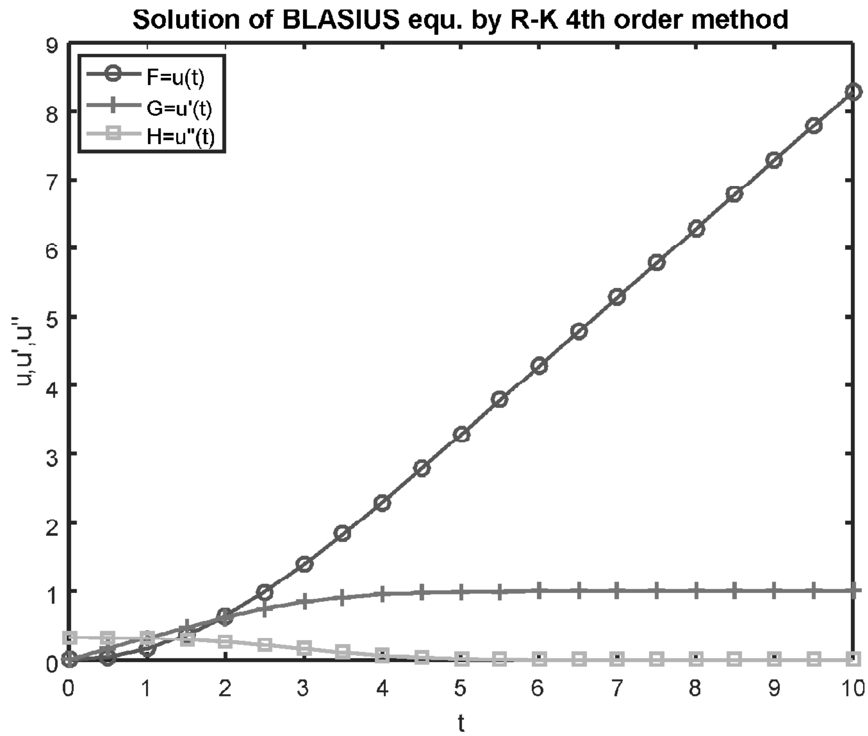


**Figure 3: The solution of BLASIUS equation in boundary layer flow of equ. (2) by R-K 4th order method (Numerical Method)**

Figs.1, 2, & 3, $u(t)$, $u'(t)$ and $u''(t)$ (vertically) denotes the velocity functions of Blasius equation w.r.t. t (Horizontally) stands for the boundary-layer thickness. These figs. shows the results of Blasius equation by different methods which satisfy the initial and boundary conditions.

## V. RESULTS AND DISCUSSION

The approximate solution of Blasius equation is obtained by proposed method (Diagonal Pade'[3/3]). Also VIM and R-K 4th order methods are employed to examine the effect of approximating of the nonlinear term on accuracy of the solution. The results of proposed approach which are depicted in figs 4 and 5 in comparison with VIM and R-K 4th order method.

**Table 4**
**Solution of BLASIUS Equation by Pade' [3/3] & VIM 4th iteration**

| $t$ | $u_{Pade'[3/3]}$ | $u1=u'_{Pade'[3/3]}$ | $u2=u''_{Pade'[3/3]}$ | $u_{VIM}$ 4th iteration | $u1=u'_{VIM}$ 4th iteration | $u2=u''_{VIM}$ 4th iteration |
|---|---|---|---|---|---|---|
| 0.00 | 0.0000 | 0.0000 | 0.3320 | 0.0000 | 0.0000 | 0.3320 |
| 0.25 | 0.0104 | 0.0830 | 0.3319 | 0.0104 | 0.0830 | 0.3319 |
| 0.50 | 0.0415 | 0.1659 | 0.3309 | 0.0415 | 0.1659 | 0.3309 |
| 0.75 | 0.0933 | 0.2483 | 0.3281 | 0.0933 | 0.2483 | 0.3281 |
| 1.00 | 0.1655 | 0.3297 | 0.3229 | 0.1655 | 0.3297 | 0.3230 |
| 1.25 | 0.2580 | 0.4094 | 0.3143 | 0.2580 | 0.4095 | 0.3146 |
| 1.50 | 0.3700 | 0.4865 | 0.3018 | 0.3701 | 0.4867 | 0.3025 |
| 1.75 | 0.5009 | 0.5600 | 0.2848 | 0.5011 | 0.5604 | 0.2866 |
| 2.00 | 0.6496 | 0.6285 | 0.2629 | 0.6499 | 0.6297 | 0.2667 |
| 2.25 | 0.8147 | 0.6910 | 0.2361 | 0.8154 | 0.6935 | 0.2434 |
| 2.50 | 0.9945 | 0.7462 | 0.2045 | 0.9961 | 0.7511 | 0.2174 |
| 2.75 | 1.1871 | 0.7929 | 0.1686 | 1.1904 | 0.8021 | 0.1898 |
| 3.00 | 1.3902 | 0.8301 | 0.1291 | 1.3966 | 0.8460 | 0.1619 |

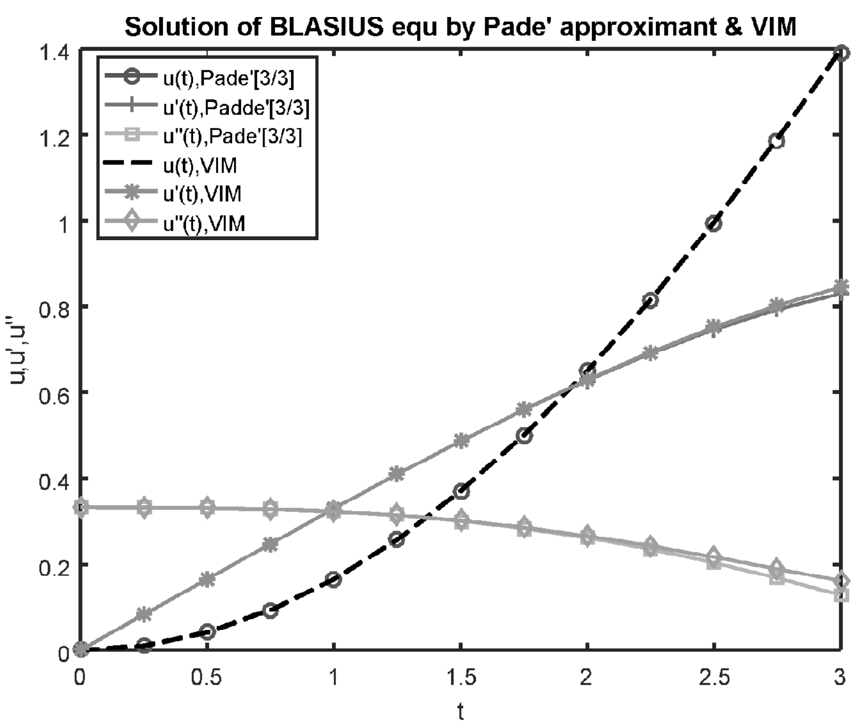


**Figure 4: The results of Blasius equation: Comparison between which are obtained from proposed method& VIM**

**Table 5**
**Solution of BLASIUS Equation for by Pade' [3/3] & R-K 4th order**

| $t$ | $u_{Pade'[3/3]}$ | $u1=u'$ $_{Pade'[3/3]}$ | $u2=u''$ $_{Pade'[3/3]}$ | $u_{R\text{-}K\ 4th\ order}$ | $u1=u'$ $_{R\text{-}K\ 4th\ order}$ | $u2=u''_{R\text{-}K}$ $_{4th\ order}$ |
|---|---|---|---|---|---|---|
| 0.00 | 0.0000 | 0.0000 | 0.3320 | 0.0000 | 0.0000 | 0.3320 |
| 0.25 | 0.0104 | 0.0830 | 0.3319 | 0.0104 | 0.0830 | 0.3319 |
| 0.50 | 0.0415 | 0.1659 | 0.3309 | 0.0415 | 0.1659 | 0.3309 |
| 0.75 | 0.0933 | 0.2483 | 0.3281 | 0.0933 | 0.2483 | 0.3281 |
| 1.00 | 0.1655 | 0.3297 | 0.3229 | 0.1655 | 0.3297 | 0.3230 |
| 1.25 | 0.2580 | 0.4094 | 0.3143 | 0.2580 | 0.4095 | 0.3146 |
| 1.50 | 0.3700 | 0.4865 | 0.3018 | 0.3701 | 0.4867 | 0.3025 |
| 1.75 | 0.5009 | 0.5600 | 0.2848 | 0.5011 | 0.5604 | 0.2866 |
| 2.00 | 0.6496 | 0.6285 | 0.2629 | 0.6499 | 0.6297 | 0.2667 |
| 2.25 | 0.8147 | 0.6910 | 0.2361 | 0.8154 | 0.6935 | 0.2434 |
| 2.50 | 0.9945 | 0.7462 | 0.2045 | 0.9961 | 0.7511 | 0.2174 |
| 2.75 | 1.1871 | 0.7929 | 0.1686 | 1.1904 | 0.8020 | 0.1896 |
| 3.00 | 1.3902 | 0.8301 | 0.1291 | 1.3966 | 0.8459 | 0.1614 |

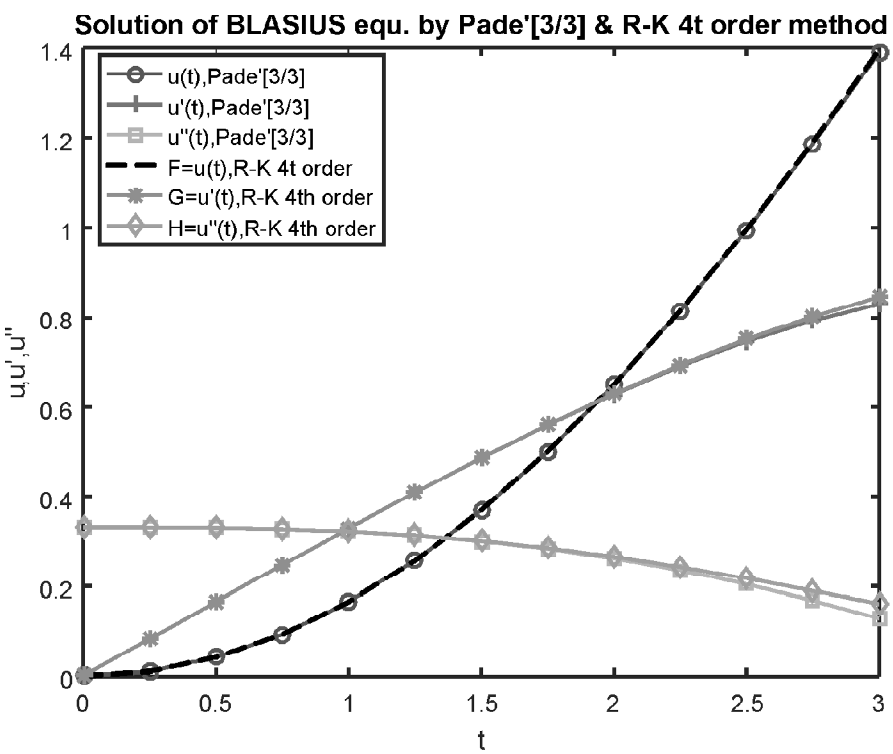


**Figure 5: The results of Blasius equation: Comparison between which are obtained from proposed method & R-K 4th order method**

Also the results of the present analytical scheme as well as the VIM and numerical results are plotted and compared in above figures. According to this, the results obtained by applying the Diagonal Pade' accurately approximate while the results achieved from the VIM and r-k 4th order method.

**Table 6**
**Relative Error % of Pade' [3/3] with VIM 4th iteration and R-K 4thorder**

| *t* | *Relative Error %* (Pade'[3/3] &VIM) | | | *Relative Error %* (Pade'[3/3] & R-K 4th order method) | | | *Relative Error %* (VIM & R-K 4th order method) | | |
|---|---|---|---|---|---|---|---|---|---|
| | *u(t)* | *u1= u'(t)* | *u2= u''(t)* | *u(t)* | *u1= u'(t)* | *u2= u''(t)* | *u(t)* | *u1= u'(t)* | *u2= u''(t)* |
| 0.00 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.25 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.50 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 0.75 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| 1.00 | 0.0000 | 0.0000 | 0.0310 | 0.0000 | 0.0000 | 0.0310 | 0.0000 | 0.0000 | 0.0000 |
| 1.25 | 0.0000 | 0.0244 | 0.0953 | 0.0000 | 0.0244 | 0.0953 | 0.0000 | 0.0000 | 0.0000 |
| 1.50 | 0.0270 | 0.0411 | 0.2314 | 0.0270 | 0.0411 | 0.2314 | 0.0000 | 0.0000 | 0.0000 |
| 1.75 | 0.0399 | 0.0713 | 0.6280 | 0.0399 | 0.0713 | 0.6280 | 0.0000 | 0.0000 | 0.0000 |
| 2.00 | 0.0461 | 0.1906 | 1.4248 | 0.0461 | 0.1906 | 1.4248 | 0.0000 | 0.0000 | 0.0000 |
| 2.25 | 0.0858 | 0.3604 | 2.9991 | 0.0858 | 0.3604 | 2.9991 | 0.0000 | 0.0000 | 0.0000 |
| 2.50 | 0.1606 | 0.6524 | 5.9338 | 0.1606 | 0.6524 | 5.9338 | 0.0000 | 0.0000 | 0.0000 |
| 2.75 | 0.2772 | 1.1470 | 11.1696 | 0.2772 | 1.1345 | 11.0759 | 0.0000 | 0.0125 | 0.1055 |
| 3.00 | 0.4582 | 1.8794 | 20.2594 | 0.4582 | 1.8678 | 20.0124 | 0.0000 | 0.0118 | 0.3098 |

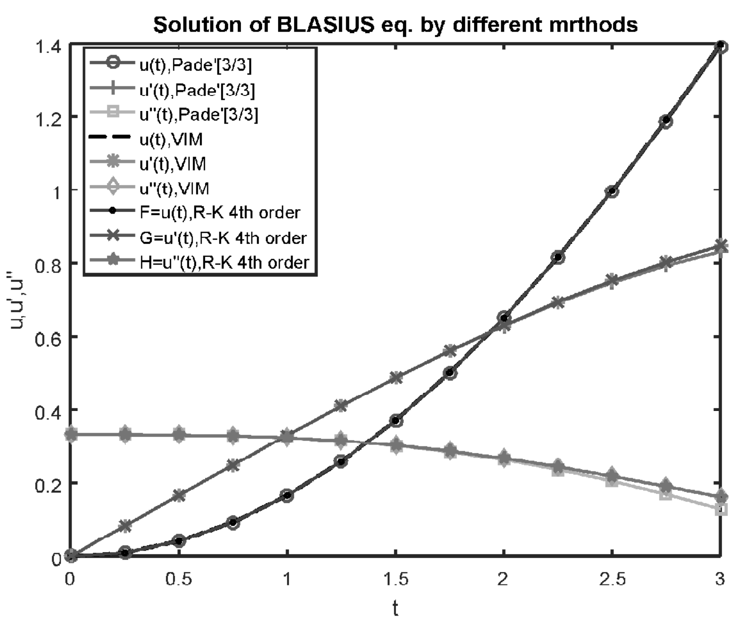


**Figure 6: Comparison between different methods**

To gain a more sensible comparison among the proposed method, Table. 6 exhibits the comparisons of $u(t)$, $u'(t)$ and $u''(t)$ obtained from the present method and those via VIM and R-K method. In Fig. 6, as can be seen, the Diagonal Pade approximant method leads to more accurate solution which gently follows the results of the numerical solutions of Blasius equation. It is apparent from the comparison that the largest relative errors are 20.2594% and 20.0124% which belong to u''(t) of proposed method and other methods.

## VII. CONCLUSION

The achievement of this work is to apply the current method (Diagonal Pade' [3/3] approximant) in order to study the solution of 3rd order Blasius equation with initial and boundary conditions and results are compared with others methods. The current method is applied without using any linearization, discretization, restrictions or transformations. In comparison, the results of applying the Diagonal Pade'[3/3] method for approximating the nonlinear term, lead to a nearly accurate solution which is capable to effectively approximate u and u'. On the other hand, as fig 6 depicted, there is considerable difference between the results of Diagonal Pade' and the solution achieved via VIM and R-K 4th. In addition, this method does not require many iterations like other methods to reach accurate results. The flexibility and adaptation provided by the method have made the method a strong candidate for approximate analytical solution. It can be claimed that here an innovative diagonal Pade approach is developed to approximate the Blasius equation which can be more examined for a wide range of similar flows in the boundary layer theory such as wedge flows.

## ACKNOWLEDGMENT

The authors would like to thank to the Department of Mathematics at IK Gujral Punjab Technical University, Kapurthala (Punjab) where the author worked as a visiting scholar.

## APPENDIX - MATLAB CODE

*Matlab code for Table. 1 and Fig. 1*

```
t=0:0.25:3;
n=length (t)
k=0.332;
for i=1:n;
u(i)=(((60*k*t(i).^2)/(120+k*t(i).^3)));
 u1(i)=(((120*k*t(i))/(120+k*t(i).^3))-((180*k.^2*t(i).^4)/(120+k*t(i).^3).^2));
 u2(i)=((120*k)/(120+k*t(i).^3))-((1080*k.^2*t(i).^3)/(120+k*t(i).^3).^2)+((1080*k.^3*t(i).^6)/
(120+k*t(i).^3).^3);
end
u(:)
u1(:)
u2(:)
plot(t,u,'-o',t,u1,'-+',t,u2,'-s')
```

## REFERENCES

[1] H. Blasius, Grenzschichten in flüssigkeiten mit kleinerreibung. Z Angew Math Phys, 56, (1908) pp. 1-37.

[2] Y. Lien-Tsai , C .Chao-Kaung, The solution of the Blasius equation by the differential transformation method. Mathematical and Computer Modelling, 28, (1998) pp.101-111.

[3] J. Lin. A new approximate iteration solution of Blasius equation Communications in Nonlinear Science and Numerical Simulation, 4(2), (1999), pp. 91-94.

[4] K. Parand, A. Taghavi, Rational scaled generalized Laguerre function collocation method for solving the Blasius equation, Journal of Computational and Applied Mathematics, 233(4), (2009), pp. 980-989.

[5] J. S .Liao,An explicit, totally analytic approximate solution for Blasius' viscous flow problems, International Journal of Non-Linear Mechanics, 34, (1999) pp. 759-778.

[6] L. Wang, A new algorithm for solving classical Blasius equation, Applied Mathematics and Computation, 157, (2004) pp. 1-9.

[7] A. M. Wazwaz, The modified decomposition method and Padé approximants for a boundary layer equation in unbounded domain, Applied Mathematics and Computation, 177, (2006) pp. 737-744.

[8] Pade' approximation-https://en.wikipedia.org/wiki/Pad%C3%A9_approximant

[9] H. Yaghoobi, M. Torabi, Novel solution for acceleration motion of vertically falling non-spherical particle by VIM-Pade' approximant, Powder Technology, (2012) pp. 215-216.

[10] H. Kaur , B.P. Garg, Acceleration motion of a single vertically falling non-spherical particle in incompressible Newtonian Fluid by Different Methods, International Journal of Mathematics Trends and Technology,52(7)(2017),pp.

[11] H. Kaur , B.P. Garg, Radiation Effect on Terminal Velocity of a Vertically Falling Non-Spherical Particle by Diagonal Pade' [3/3] Approximants, International Journal of Statics and Applied Mathematics, 3(2)(2018), pp. 31-36.

[12] J. He, Varinational iteration method - a kind of non-linear analytical technique: some examples", International Journal of Non-Linear Mechanics, 34(4), (1999), pp.699-708.

[13] J. He, Varinational iteration method for autonomous ordinary differential systems, Applied Mathematics and Computation, 114(2/3) ,(2000) pp. 115-123.

[14] L. Howarth, On the solution of the laminar boundary layer equations, Proceedings of The Royal Society A: Mathematical, Physical &Engineering Sciences, 164, (1938) pp. 547-579.

[15] J. He, Varinational iteration method for delay differential equations, Communications in Nonlinear Science and Numerical Simulation, 2(4), (1997) pp. 235-236.

[16] J. He, A new approach to nonlinear partial differential equations, Communications in Nonlinear Science and Numerical Simulation, 2 , (1997) pp. 437-440.

[17] Y. Liu , S. N. Kurra, Solution of Blasius Equation by Varinational Iteration Method, Applied Mathematics 1(1)(2011) pp 24-27.

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