IJCTA, 9(21), 2016, pp. 133-136

Missing Numbers in Graceful Graphs

Ajay Kumar* Amit Verma**

Abstract : The impetus of this paper is to point out a technique developed by which a new class of graphs is obtained from cycles. An adequate program is also given for finding missing numbers while giving graceful labeling to the new class of graphs. AMS Subject Classification: 05C78

Keywords : Isomorphic graphs, graceful labeling, graceful cycles.

1. PREAMBLE

We are talking about finite, unlabeled and simple graphs (recall that a graph free from self-loops and multiple edges is called simple graph) $G = (V_G, E_G)$. As the notation tells that the two parameters V_G and E_G are required to specify a graph abstractly. Naming a graph means to give the labels to the vertices or edges or both under certain conditions. The labels are from our real world and came into existence due to human interest in the field of science and technology. Labeling gives a new definition to a given graph and the graph become more useful then its unlabeled structure. The distinct labeling are obtained by applying certain conditions on them. See the new genre of capacious survey of graph labeling in Gallian [3]. Applying a rule $f : V_G \rightarrow N \cup \{0\}$ on the vertices and $\varphi : E_G \rightarrow N$ on the edges of G such that f and φ adhere to the following conditions :

(*i*) f is an injective map.

(*ii*) $f(v_i) = \{0, 1, 2, ..., |E_G|\}$

(*iii*) φ is an injective map.

(*iv*) $\varphi(e_i) = \{1, 2, ..., |E_G|\}$

(*v*) $\varphi(e_i) = |f(u) - f(v)|$, " $\forall\, u, v \in V_G, e_i \in E_G$.

Then the labeling $f\varphi^G$ is said to be graceful labeling with respect to the vertex labeling f and induced edge labeling φ for the graph G. Sometimes f and φ are called vertex weight function (vertex indexer) and edge weight function (edge indexer) respectively. If f is a graceful labeling of a graph G, then the mapping g is defined as :

$$g(u) = |E_G| - f(u), \quad \forall_u \in V_G$$

is also a graceful labeling and g is called complementary labeling. Rosa [5] showed that a cycle C_n has a graceful labeling iff $n \equiv 0$ or 3 (mod 4). A graceful labeling of C_n has n distinct vertex labels and only one vertex label from this set is missing. Bagga, Heinz and Majumder [1] scrutinized for the missing numbers of graceful cycles and several of their properties. Bagga, Heinz and Majumder [2] presented an algorithm that computes all graceful labeling of cycles up to $n = 20$. Truszczynski [6] conjectured that all unicyclic graphs except the cycle C_m, where

$m \equiv 1$ or 2 (mod 4), are graceful. Pradhan and Kumar [4] have proved that $m = 3n/2$ is the missing number for C_n $1K_1$, where $n \equiv 0$ (mod 4). They also proved that if G be a graceful graph obtained by attaching one pendant edge to each pendant vertex of graceful hairy cycle C_n $1K_1$, then the missing number m in the graceful labeling of G is $9n/4$, when $n \equiv 0$ (mod 4).

* Department of Mathematics Uttaranchal University Arcadia Grant, PO – chandanwari Premnagar, Dehradun, Uttarakhand, India Email- ajaygraph.gkv.math@gmail.com,

** Department of Computer Science University of Petroleum and Energy Studies Energy Acres, PO – Bidholi Dehradun – 248007 Uttarakhand, India Email- amit.verma@ddn.upes.ac.in

II. MAIN RESULTS

The impetus of this paper is to introduce a new class of graphs. The graph obtained by attaching an edge between any two vertices of two isomorphic cycles each of length n, where $n \equiv 0 \pmod 4$ and represented by $C_n - C_n$. We extend the results of Bagga, Heinz, Majumder [1], [2], and Pradhan, Kumar [4], for $C_n - C_n$ graphs and prove that these are graceful and have exactly two missing numbers m_1 and m_2. A vertex $v \in V_G$ is said to be high vertex if v has the greatest label among all its neighbors. Similarly, a vertex $w \in V_G$ is said to be low vertex if w has the smallest label among all its neighbors. Let H and L denote the set of labels of high vertices and low vertices respectively. The remaining vertex labels of G (neithier in H nor in L) are called intermidiate vertices and the set of all intermediate vertices is called intermediate set, denoted by I.

Theorem 1. If C_n is graceful and $n \equiv 0 \pmod 4$, the graph $C_n - C_n$ obtained by joining the vertices of zero labels of two isomorphic copies of C_n by an edge with

$$\theta(v) = \begin{cases} f(v) & \text{if } v \in X_1 \cup Y_2 \\ f(v) + n + 1 & \text{if } v \in X_2 \cup Y_1 \end{cases}$$

is also graceful.

Proof. Let f be the graceful labeling of the cycles C_{n1} and C_{n2}, where $n_1 = n_2 \equiv 0 \pmod 4$. Let V_1 and V_2 be the set of labels of vertices C_{n1} and C_{n2} respectively. Suppose partitions $V_1 = X_1 \cup Y_1$ and $V_2 = X_2 \cup Y_2$ exist. Let n be the common length of each cycle *i.e* $n = n_1 = n_2$. Each of the two cycles are gracefuly labeled by f. An outsider vertex v is joined at zero label of one of the cycle C_n. Then a new unicyclic graph $C_n + v$ formed. Merging this outsider vertex to the zero label of the other graceful cycle. Consider the following labeling θ of $C_n - C_n$

$$\theta(v) = \begin{cases} f(v) & \text{if } v \in X_1 \cup Y_2 \\ f(v) + n + 1 & \text{if } v \in X_2 \cup Y_1 \end{cases}$$

We claim that θ is a graceful labeling of $C_n - C_n$. Consider a weight k of an edge in C_n. The weight k is also in $C_n - C_n$ and increased from $k + 1$ to $2n + 1$.

Hence, θ is a graceful labeling.

Further, we check that infact whether θ is a graceful labeling.

- Let $a \in$ L and a is in $f(v)$. The vertex labels b and c are adjacent to a.
- Infact, a, b, c have distinct labels such tha $b > a$, $c > a$ with either $b > c$ or $b < c$.
- a is a label in $X_1 \cup Y_2$, then $a + n + 1$ is a label in $X_2 \cup Y_1$.
- Clearly, all the vertex labels are distinct and all edge labels are distinct in $C_n - C_n$.

Hence θ is a graceful labeling.

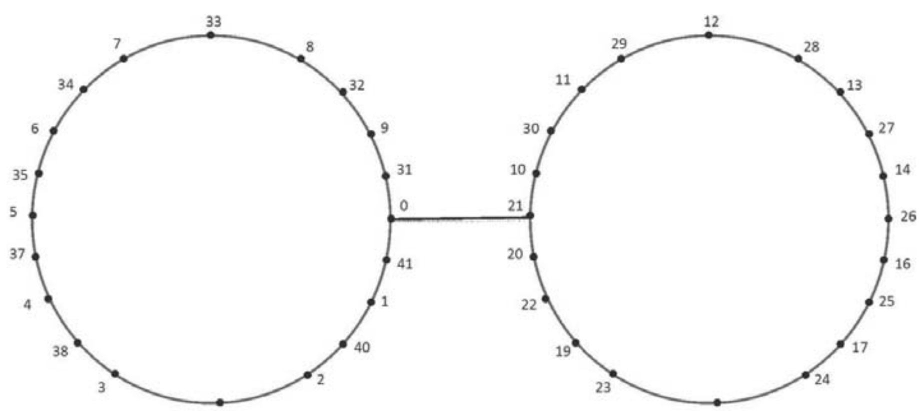

Fig 1. Graceful C20 – C20 graph

Theorem 2. The graph C_n–C_n obtained from Theorem 1. has only two missing numbers.

Proof. While giving the graceful labeling of a cycle C_n, vertices are labeled from $\{0, 1, 2, ..., n\}$ and the edge are labeled from $\{1, 2, ..., n\}$. Since, the order and size of a cycle C_n are same. Thus, we can say that one label from the set $\{0, 1, 2, ..., n\}$ remains missing. When a graceful $C_n – C_n$ graph is obtained from Theorem 1., then the vertices are labeled from $\{0, 1, 2, ..., 2n + 1\}$ and edges are labeled from $\{1, 2, ..., 2n + 1\}$. Since there are $2n + 2$ labels which may be provide for $2n$ vertices of $C_n – C_n$. Thus, two labels from the set $\{0, 1, 2, ..., 2n + 1\}$ will be miss.

Theorem 3. If m_1 and m_2 are the two missing numbers in graceful $C_n – C_n$ obtained from Theorem 1., then the bounds for m_1 is

$$\left\lceil \frac{n}{4} \right\rceil \leq m_1 < \left\lceil \frac{3n}{4} \right\rceil$$

and bounds for m_2 is

$$\left\lceil \frac{3n}{4} \right\rceil \leq m_2 < 2n$$

Proof. The proof for the first missing number m_1 is given by Bagga, Heinz, Majumder [1]. We are interested to find the second missing number m_2. Since, m_1 and m_2 lies in different cycles. The upper bound for m_1 is the lower bound for m_2. This $C_n – C_n$ graph has $2n$ vertices and $2n$ cannot be equal to the second missing number. Therefore, $2n$ is the upper bound for m_2. This completes the proof.

We also develop computer programs which find the missing numbers only by giving the value of n for the graph $C_n – C_n$ where $n \equiv 0 \pmod 4$. Suppose, if we put a number k not congruent to 0 (mod 4), then the program show on the screen invalid number.

Program 4. Program that computes the missing numbers in the graceful labeling of $C_n – C_n$ is given in Fig.2. Table 1 for missing numbers m_1 and m_2 corresponding to the values of n is formed with the help of a computer program (Fig. 2).

Corollory 5. If m_1 and m_2 are the missing numbers in the graceful $C_n – C_n$ obtained from Theorem 1. and H, L are the sets of higher vertices and lower vertices respectively, then

1. $|H| = |L| = n/2$
2. $|H| + |L| = n$
3. If $m_1 < m_2$, then m_1 is the missed label from L and m_2 is the missed label from H.

Table 1. Table for Missing Numbers

n	m_1	m_2
4	3	8
8	6	15
12	9	22
16	12	29
20	15	29
24	18	43
28	21	50
32	24	57
36	27	64
40	30	71

Observation : $\max(L) = n$ and $\max(H) = 2n + 1$.

```
void missing_vertex()
  {
  int n,m1;
  int temp=0;
  static int fact=0;

  printf("\nExecuted %d time",++fact);
  printf("\nEnter the number of vertices must be the multiple of 4\n");
  scanf("%d",&n);
  a:
  if(n % 4!=0)
  {
        printf("\ninvalid input enter another number\n");
        scanf("%d",&n);
        goto a; // this statement will take the controller back to a:
        }
        else
        {
          temp=n/4;
          m1=n-temp;
          printf("\nM1 = %d\n",m1);
          }
          printf("M2 = %d",(n+m1)+1);
        }
    }
 main()
   {
   int a=1;
   while(1) // infinite loop
   {
     printf("\n1.Find missing vertex\n2.Exit\n");
     printf("\nEnter choice = ");
     scanf("%d",&a);
     if(a==1)
       missing_vertex();
     else
       exit(1);
         }
   }
```

Fig. 2. Image of the program that computes the values of missing numbers

3. REFERENCES

1. J. Bagga, A. Heinz and M. M. Majumder, "Properties of graceful labeling of cycles", *Congressus Numerantium*,vol 186 *pp.* 109-115,2007.
2. J. Bagga, A. Heinz and M. M. Majumder, "*An Algorithm for Graceful Labelings of Cycles*", *Congressus Numerantium*, vol 186 *pp.* 57-63, 2007.
3. J. A. Gallian, "A dynamics survey of graph labeling", *The Electronic Journal of Combinatorics*, #DS6, 2015.
4. P. Pradhan and A. Kumar, "Graceful hairy cycles with pendent edges and some properties of cycles and cycle related graphs", *Bull. Cal. Math. Soc.*, vol 104(1) , *pp*. 61-76, 2012.
5. A. Rosa, "On certain valuations of the vertices of a graph, Theory of Graphs"*(Internat. Symposium, Rome, July 1966)*,*Gordon and Breach, N. Y. and Dunod Paris*, *pp.* 349-355, 1967.
6. M. Truszczy*N*´ki, "Graceful unicyclic graphs", *Demonstratio Math.*,vol 17, *pp.* 377-387, 1984.