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THE CONTINUITY OF THE SOLUTION OF THE NATURAL EQUATION IN THE ONE - DIMENSIONAL CASE

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Abstract. We consider the so-called $\natural$-model. this model is expressed by a stochastic differential equation called $\natural$-equation, introduced in the article "Random times with given survival probability and their $\mathbb{F}$-martingale decomposition formula" published in *Stochastic Processes And their Applications*. This equation plays an essential role in this article, but its application has been submitted to a hypothesis of continuity. Then it is important to know under what conditions the hypothesis of continuity is satisfied. This is the main motivation of our research, but the proof given in the present paper is different from Song's (which study a more general case).

1. Introduction

Firstly, we will give a description of the natural model called the one-default model determined in [5]. For this, we define a filtered probability space $(\Omega, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, an $\mathbb{F}$-adapted continuous increasing process Λ null at the origin, and a positive $(\mathbb{P}, \mathbb{F})$ local martingale N such that $0 < Z_t = N_t e^{-\Lambda_t}$ satisfies $Z_t \leq 1, t \geq 0$. Precisely, it is proved in [5] that, for any continuous local martingale Y, for any Lipschitz function f on $\mathbb{R}$ null at the origin, there exist a probability measure $\mathbb{Q}$ and a random time $\tau > 0$ on an extension of $(\Omega, \mathbb{F}, \mathbb{P})$, such that the survival probability of τ, i.e., $\mathbb{Q}[\tau > t|\mathcal{F}_t]$ is equal to Z_t for $t \geq 0$. In the same last reference, it is also shown that there exist several solutions and that an increasing family of martingales, combined with a stochastic differential equation, constitutes a natural way to construct these solutions, which means that $X_t^u = \mathbb{Q}[\tau \leq |\mathcal{F}_t], 0 < u, t < \infty$, satisfy the following stochastic differential equation:

$$(\natural_u): \begin{cases} dX_t = X_t \left(-\dfrac{e^{-\Lambda_t}}{1 - Z_t} dN_t + f(X_t - (1 - Z_t)) dY_t \right), & t \in [u, \infty) \\ X_u = x \end{cases}$$

where the initial condition x can be any $\mathcal{F}_u$-measurable random variable. In actuality, this model played an important role in finance mostly in the credit risk modeling. The remarkable property about the $\natural$-model is its rich system of parameters Z, Y, f. The parameter Z determines the default intensity. The parameters Y and f describe the evolution of the market after the default time

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τ. Such a system of parameters sets up a propitious framework for inferring the market behavior and for calibrating the financial data. We believe that the $\natural$-model can be a useful instrument to modeling financial market. In this paper, we want to show the continuity of the process $X_t^u(x)$ such as:

$$X_t^u(x) = x + \int_u^t X_s \left(-\frac{e^{-\Lambda_s}}{1-Z_s}\right) dN_s + \int_u^t X_s f(X_s - (1-Z_s))dY_s \ , \ u \leq s \leq t$$

is the solution of the $(\natural_u)$-equation. Our aim is to look at the regularity of the process $(u,t,x) \longmapsto X_t^u(x)$ with respect to all the variables u, t, x. Our fundamental tools are the theorem of Kolmogorov and the lemma of Gronwall. We should mention that, this main result given in this present paper is less general and different from Song's one [8], which study a more general case where there are jumps and where the coefficients are Markovian. The paper is organized as follows: in the next section, we prove the found results on the continuity of the stochastic flows, i.e. the continuity of the solution of a stochastic differential equation with respect to a parameter (which can be taken to be, of course, the initial condition). Section 3 presents the main result of this paper.

2. The Found Results on the Continuity of the Solutions of SDE

2.1. The case studied by Philipe E. Protter. This subsection is borrowed from ([6], chapter 5). We consider a general system of equations of the form

$$\zeta_t^x = H_t^x + \int_0^t F(\zeta^x)_{s\_} dS_s \tag{2.1}$$

where ζ_t^x and H_t^x are column vectors in $\mathbb{R}^n$, S is a column vector of m semimartingales with $S_0 = 0$, and F is an $n \times m$ matrix with elements (F_α^i). For x fixed, for each y we have that $\overline{\zeta}_t = \zeta_t^y - \zeta_t^x$ is a solution of the equation

$$\overline{\zeta_t} = H_t^y - H_t^x + \int_0^t \overline{F}(\overline{\zeta})_{s-} dS_s \tag{2.2}$$

where $\overline{F}(\grave{\zeta}) = F(\zeta^x + \grave{\zeta}) - F(\zeta^x)$.

Theorem 2.1. *Let H^x be processes in $\mathbb{D}^n$ i.e. the space of processes $H = (H^1, ..., H^n)$ where each H_i is an adapted càdlàg process $(1 \leq i \leq n)$, and let $x \longmapsto H^x : \mathbb{R}^n \longrightarrow \mathbb{D}^n$ be pre-locally Lipschitz continuous process. F be an $n \times m$ matrix of functional Lipschitz operators $(F_\alpha^i), 1 \leq i \leq n, 1 \leq \alpha \leq m$. Then there exists a function $\zeta(t, \omega, x)$ on $\mathbb{R}_+ \times \Omega \times \mathbb{R}^n$ such that*

1. *for each x the process $\zeta_t^x = \zeta(t, \omega, x)$ is a solution of 2.1,and*
2. *for almost all ω, the flow $x \longmapsto \zeta(., \omega, x)$ is continuous in the topology of uniform convergence on compacts.*

Proof. We recall the method of proof used to show the existence and uniqueness of a solution (see [6], chapter 5, Theorem 7). By stopping at a fixed time t_0, we can assume the Lipschitz process is just a random variable μ which is finite a.s. Then by conditioning (see [6], chapter 5, the proofs of Theorems 7, 8, or 15 for this argument) we can assume without loss of generality that this Lipschitz constant is non-random, and we call it $\theta < \infty$ By replacing H_t^x with $\zeta_t^x + \int_0^t F(0)_{s-} dS_s$, and then by replacing F with Q given by $Q(\grave{\zeta})_t = F(\grave{\zeta})_t - F(0)_t$, we can further assume without loss of generality that $F(0) = 0$. Then for $\nu = \lambda_p(\theta)$, by (see [6], chapter 5, Theorem 5) we can find an arbitrarily large stopping time T such that $S^{T-} \in \mathcal{S}(\nu)$ i.e. ν-sliceable, and H^x is Lipschitz continuous on $[0,T)$ Then by Lemma 2 (see [6], chapter 5, preceding Theorem 7) we have that for the solution $\overline{\zeta}$ of 2.2.

$$\|\overline{\zeta}^{T-}\| \leq \lambda_p(\theta, S)\|(H^x - H^y)^{T-}\| \tag{2.3}$$

for any $p \geq 2$ and some (finite) constant $\lambda_p(\theta, M)$. Choose $p > n$, and we have

$$\mathbb{E}\{\sup_{s<T} |\zeta_s^x - \zeta_s^y| \leq \lambda_p(\theta, S)\mu\|x-y\|^p\} \tag{2.4}$$

due to the Lipschitz hypothesis on $x \longmapsto H^x$. By Kolmogorov's theorem (theorem 3.1 in this article) we have the result on $\mathbb{R}^n \times [0,T)$. However since T was arbitrarily large, the result holds as well on $\mathbb{R}^n \times \Omega \times \mathbb{R}_+$. □

2.2. The case studied by H. Kunita. This subsection is borrowed from [4]. Let $\{V_k : \mathbb{R}_+ \times \mathbb{R}^d\}_{k=0,\dots,m}$ is a family of vector fields on $\mathbb{R}^d$, for $s > 0$ et $x \in \mathbb{R}^d$. Let $t \longmapsto \xi_{st}(x)$ the solution of Stochastic differential equation of the form

$$\xi_{st}(x) = x + \sum_{k=0}^{m} \int_0^t V_k(r, \xi_{sr}(x))dB_r^k \tag{2.5}$$

where B^k is a family of standard Brownian motions. Here we are interested in the regularity of the process $(s,t,x) \longmapsto \xi_{st}(x)$ with respect to the parameters s, t, x. We have the The following theorem.

Theorem 2.2. *There exists a random, continuous, hölder's function in s, t, x, with exponents γ, ρ for any $\gamma < \frac{1}{2}$ and $\rho < 1$. Moreover we have a.s. the equation 2.5 is proved for any s, t, x and the property of the flow $\xi_{\varsigma t}(\xi_{s\varsigma})$ is valid for any s, t, x.*

Proof. The proof is a direct consequence of the Kolmogorov's theorem and the following estimation demonstrated in theorem 2.6:

$$\mathbb{E}|\xi_{st}(x) - \xi_{\grave{s}\grave{t}}(\grave{x})|^p \leq |x - \grave{x}|^p + (1 + |x| + |\grave{x}|)(|s - \grave{s}|^{\frac{p}{2}} + |t - \grave{t}|^{\frac{p}{2}}) \tag{2.6}$$

Given a compact $\chi \subseteq \mathbb{R}^d$ and $\mathcal{T} > 0$, the estimation 2.6 is enough to apply the Kolmogorov's theorem, which provides that there is a continuous Version in $(s,t,x) \in [0,\mathcal{T}]^2 \times \chi$ of $\xi_{st}(x)$ for which we have

$$|\xi_{st}(x) - \xi_{\grave{s}\grave{t}}(\grave{x})| \leq \vartheta_{\chi,\mathcal{T},p,\gamma,\rho}(\omega)(|t - \grave{t}|^\gamma + |s - \grave{s}|^\gamma + |x - \grave{x}|^\rho) \tag{2.7}$$

uniformly in s,t,x for any $\gamma < \frac{1}{2}$ and $\rho < 1$. □

Remark 2.3. We may well call the map ξ the stochastic flow.

Lemma 2.4. *For any $p \in \mathbb{R}$, $\mathcal{T} > 0$ and $\varepsilon > 0$, we have the inequality*

$$\mathbb{E}(\varepsilon + |\xi_{st}(x)|^2)^p \leq \vartheta_{\varepsilon,p,\mathcal{T}}(\varepsilon + |x|^2)^p \tag{2.8}$$

$$\mathbb{E}(\varepsilon + |\xi_{st}(x) - \xi_{st}(y)|^2)^p \leq \vartheta_{p,\mathcal{T}}(\varepsilon + |x-y|^2)^p \tag{2.9}$$

for any $0 \leq s \leq t \leq \mathcal{T}$.

Proof. We put $g(x) = (\varepsilon + |x|^2)$ and $F(x) = g(x)^p$. An easy calculation gives

$$\nabla_i F(x) = 2g(x)^{p-1}x$$
$$\nabla^2_{ij} F(x) = 2p\, g(x)^{p-2}(g(x)\delta_{ij} + 2(p-1)x_i x_j), \quad i, j = 1, ..., d$$

and if we denote $L_t = \xi_{st}(x)$ then, by Itô's formula applied to the semi-martingale $F(L_t)$ we have

$$F(L_t) = F(L_s) + \sum_{i=1}^{d} \int_s^t \nabla_i F(L_r) dL_r^i + \frac{1}{2} \sum_{i,j=1}^{d} \int_s^t \nabla^2_{ij} F(L_t) d < L^i, L^j >_r$$

$$dL_r^i = d\xi_{sr}^i(x) = \sum_{k=0}^{m} V_k^i(r, \xi_{sr}(x)) dB_r^k$$

$$\begin{aligned} d < L^i, L^j >_r &= \sum_{k,l=0}^{m} V_k^i(r, \xi_{sr}(x)) V_l^j(r, \xi_{sr}(x)) d < B^k, B^l >_r \\ &= \sum_{k=1}^{m} V_k^i(r, \xi_{sr}(x)) V_k^j(r, \xi_{sr}(x)) dt \end{aligned}$$

Given that $< B^k, B^l >_t = t$ if $k = l = 1, ..., m$ and it is zero otherwise (particularly if $k = 0$ or $l = 0$ because $B_t^0 = t$), therefore

$$\begin{aligned} F(L_t) = F(L_s) &+ \sum_{i=1}^{d} \sum_{k=0}^{m} \int_s^t \nabla_i F(L_r) V_k^i(r, L_r) dB_r^k \\ &+ \frac{1}{2} \sum_{i,j=1}^{d} \sum_{k=1}^{m} \int_s^t \nabla^2_{ij} F(L_t) V_k^i(r, L_r) V_k^j(r, L_r) dr \end{aligned}$$

Now take the expectation of this last quantity, the stochastic integral gives zero and $L_s = \xi_{ss}(x) = x$ a.s, therefore

$$\mathbb{E}F(L_t) = F(x) + \tfrac{1}{2} \sum_{i,j=1}^{d} \sum_{k=1}^{m} \int_s^t \mathbb{E}[\nabla^2_{ij} F(L_t) V_k^i(r, L_r) V_k^j(r, L_r)] dr$$

To estimate the quantity inside the integral, we note that by hypothesis

$$|V_k(r, x)| \leq M(1 + |x|) \leq \vartheta_\varepsilon \sqrt{g(x)}$$

where the constant ϑ_ε depends on ε, then

$$|\nabla^2_{ij} F(L_t) V_k^i(r, L_r) V_k^j(r, L_r)| \leq \vartheta_\varepsilon F(L_r)$$

and

$$\mathbb{E}F(L_t) \leq F(x) + \vartheta_\varepsilon \int_s^t \mathbb{E}F(L_r) dr$$

So we can conclude by Gronwall's lemma (lemma 3.3 in this article) and get the first two inequalities. For the second inequality we proceed in the same way but this time we put $L_t = \xi_{st}(x) - \xi_{st}(y)$. The process L_t is again a semi-martingale and

$$dL_t = \sum_{k=0}^{m}[V_k^i(r,\xi_{sr}(x)) - V_k^i(r,\xi_{sr}(y))]dB_r^k$$

$$d < L^i, L^j >_r = \sum_{k=1}^{m}[V_k^i(r,\xi_{sr}(x)) - V_k^i(r,\xi_{sr}(y))][V_k^j(r,\xi_{sr}(x)) - V_k^j(r,\xi_{sr}(y))]dt$$

This time, it was that

$$|V_k^i(r,\xi_{sr}(x)) - V_k^i(r,\xi_{sr}(y))| \leq M|\xi_{st}(x) - \xi_{st}(y)| \leq M\, g(L_r)^{\frac{1}{2}}$$

independently from ε. Therefore with the same method as before and can be applied Gronwall wrap up. □

Lemma 2.5. *For any* $0 \leq s \leq \varsigma \leq t \leq \mathcal{T}$ *and* $x \in \mathbb{R}^d$ *we have a.s*

$$\xi_{\varsigma t}(\xi_{s\varsigma}) = \xi_{st}(x)$$

Proof. The previous lemma and Kolmogorov's theorem imply that for all fixed s, t, the application $x \longmapsto \xi_{st}(x)$ is almost surely continuous. Moreover, it is easy to see that we can choose the family of random variables

$$x \longmapsto \sum_{k=0}^{m}\int_s^t V_k(r,\xi_{sr}(x))dB_r^k$$

continuous in x, in fact

$$\begin{aligned}
&\mathbb{E}[\int_s^t V_k(r,\xi_{sr}(x))dB_r^k - \int_s^t V_k(r,\xi_{sr}(y))dB_r^k]^p \\
&\leq \vartheta_p\mathbb{E}[\int_s^t |V_k(r,\xi_{sr}(x))dB_r^k - \int_s^t V_k(r,\xi_{sr}(y))|^2 dr]^{\frac{p}{2}} \\
&\leq \vartheta_p(t-s)^{\frac{p}{2}-1}\mathbb{E}[\int_s^t |V_k(r,\xi_{sr}(x))dB_r^k - \int_s^t V_k(r,\xi_{sr}(y))|^p dr] \quad \text{(byJensen)} \\
&\leq \vartheta_p M(t-s)^{\frac{p}{2}-1}\mathbb{E}[_s^t|\xi_{sr}(x) - \xi_{sr}(y)|^p dr] \quad \text{(byhypothesis)} \\
&\leq \vartheta_p(t-s)^{\frac{p}{2}}|x-y|^2 \quad \text{(byLemma2.4)}
\end{aligned}$$

and therefore can still be used again Kolmogorov to obtain a continuous version in x of the stochastic integral and show that for a fixed $s \leq \varsigma \leq t$, the integral equation

$$\xi_{\varsigma t}(x) = x + \sum_{k=0}^{m}\int_\varsigma^t V_k(r,\xi_{\varsigma r}(x))dB_r^k$$

is true for all $x \in \mathbb{R}^d$ almost surely. So if in this equation replacing x by the random function $\xi_{s\varsigma}(x)$ we get

$$\xi_{\varsigma t}(\xi_{_{s\varsigma}}) = \xi_{_{s\varsigma}} + \sum_{k=0}^{m}\int_\varsigma^t V_k(r,\xi_{\varsigma r}(\xi_{s\varsigma}))dB_r^k$$

Let now $\hat{\xi}_{st}(x) = \xi_{\varsigma t}(\xi_{_{s\varsigma}})$ if $t > \varsigma$ and $\hat{\xi_{st}}(x) = \xi_{st}(x)$ otherwise. the process $\hat{\xi}_{st}(x)$ satisfies the equation

$$\hat{\xi_{st}}(x) = x + \sum_{k=0}^{m} \int_s^t V_k(r, \hat{\xi_{sr}}(x)) dB_r^k$$

for any $t \geq s$ and any $x \in \mathbb{R}^d$ and therefore the uniqueness of the solution, we must have that $\xi_{sr} = \hat{\xi_{st}}(x)$ a.s. and hence the thesis. □

Theorem 2.6. *for all* $p \geq 2$, $0 \leq s \leq t \leq \mathcal{T}$, $0 \leq \acute{s} \leq \acute{t} \leq \mathcal{T}$, $x, \acute{x} \in \mathbb{R}^d$*:*

$$\mathbb{E}|\xi_{st}(x) - \xi_{\acute{s}\acute{t}}(x)|^p \leq \vartheta\{|x - \acute{x}|^p + (1 + |x| + |\acute{x}|)^p(|t - \acute{t}|^{\frac{p}{2}} + |s - \acute{s}|^{\frac{p}{2}})\} \quad (2.10)$$

Proof. For brevity we consider only the case $0 \leq s \leq \acute{s} \leq t \leq \acute{t} \leq \mathcal{T}$, using the previous lemma, we have:

$$\xi_{\acute{s}\acute{t}}(\acute{x}) = \acute{x} + \sum_{k=0}^{m} \int_{\acute{s}}^{t} V_k(r, \xi_{\acute{s}r}(\acute{x})) dB_r^k + \sum_{k=0}^{m} \int_t^{\acute{t}} V_k(r, \xi_{\acute{s}r}(\acute{x})) dB_r^k$$

$$\xi_{st}(x) = \xi_{s\acute{s}}(x) + \sum_{k=0}^{m} \int_{\acute{s}}^{t} V_k(r, \xi_{\acute{s}r}(\xi_{s\acute{s}}(x)) dB_r^k$$

Therefore

$$|\xi_{st}(x) - \xi_{\acute{s}\acute{t}}(x)|^p \leq (2m+3)^{p-1}\{\underbrace{|\xi_{s\acute{s}} - \acute{x}|^p}_{\mathcal{A}}$$

$$+ \sum_{k=0}^{m} \underbrace{|\int_{\acute{s}}^{t} [V_k(r, \xi_{\acute{s}r}(\acute{x})) - V_k(r, \xi_{\acute{s}r}(\xi_{s\acute{s}}))] dB_r^k|^p\}}_{\mathcal{B}} + \sum_{k=0}^{m} \underbrace{|\int_t^{\acute{t}} V_k(r, \xi_{\acute{s}r}) dB_r^k|^p}_{\mathcal{C}}$$

We used the inequality (often very useful): $|\sum_{i=1}^N a_i|^p \leq N^{p-1} \sum_{i=1}^N |a_i|^p$, which follows from Jensen's inequality. We will estimate the expectation of the three terms $\mathcal{A}, \mathcal{B}, \mathcal{C}$ one by one. We start with a small additional result:

$$\mathbb{E}|\xi_{s\acute{s}}(x) - x|^p \leq (m+3)^{p-1} \sum_{k=0}^{m} \mathbb{E}|\int_s^{\acute{s}} V_k(r, \xi_{sr}(x)) dB_r^k|^p$$

$$\leq \vartheta_p M \mathbb{E}(\int_s^{\acute{s}} (1 + |\xi_{sr}(x)|)^2 dr)^{\frac{p}{2}}$$

$$\leq \vartheta_p (\acute{s} - s)^{\frac{p}{2}} (1 + |x|^p)$$

With this estimate we have easily that

$$\mathbb{E}[\mathcal{A}] \leq 2^{p-1}\{|x - \acute{x}|^p + \mathbb{E}|\xi_{s\acute{s}}(x) - x|^p\} \leq \vartheta_p[|x - \acute{x}|^p + (\acute{s} - s)^{\frac{p}{2}}(1 + |x|^p)]$$

By similar calculations we can also infer that

$$\mathbb{E}[\mathcal{B}] \leq \vartheta_p[|x - \acute{x}|^p + (\acute{s} - s)^{\frac{p}{2}}(1 + |x|^p)]$$

$$\mathbb{E}[\mathcal{C}] \leq \vartheta_p (t - \acute{t})^{\frac{p}{2}} (1 + |\acute{x}|)^p$$

All these estimates allow us to conclude the proof. □

2.3. The case studied by G. Barles. This subsection is borrowed from [1]. We consider a general system of equations of the form

$$J_t^{s,x} = x + \int_s^t \theta(r, J_r^{s,x})dr + \int_s^t \sigma(r, J_r^{s,x})dW_r, \quad 0 \le t \le \mathbb{T} \tag{2.11}$$

For any $(s,t) \in [0,\mathbb{T}] \times \mathbb{R}^n$, where $\mathbb{T}$ is a strictly positive real, W a d-dimensional Brownian motion and $\theta : [0,\mathbb{T}] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $\sigma : [0,\mathbb{T}] \times \mathbb{R}^n \longrightarrow \mathbb{R}^{n\times d}$ are two measurable, Lipschitz functions to linear increase. To demonstrate the continuity properties of the flow, we will use the Kolmogorov's theorem (theorem 3.1). To do this, we must make estimates on the moments of $J_t^{s,x}$. The demonstrations are somehow technical but not difficult, it is often used the inequalities of Hölder and Burkholder-Davis-Gundy - BDG in the More - and Gronwall's lemma (lemma 3.3).

Proposition 2.7. *Let $p \ge 1$, there is a constant R, depending on $\mathbb{T}$ and p such that $\forall s \in [0,\mathbb{T}]$, $\forall x \in \mathbb{R}^n$*

$$\mathbb{E}[\sup_{0\le t\le \mathbb{T}} |J_t^{s,x}|^p] \le R(1+|x|^p) \tag{2.12}$$

Proof. We demonstrated in the case $n = d = 1$. We start with the case $p \ge 2$. We fix s and x, we note J_t in place of $J_t^{s,x}$ for ease of writing. In the following R is a constant depending on p and $\mathbb{T}$ whose value may change from one line to another but which does not depend on (s,x). We have firstly,

$$\sup_{t\in[0,\mathbb{T}]} |J_t|^p \le \sup_{t\in[0,s]} |J_t|^p + \sup_{t\in[s,\mathbb{T}]} |J_t|^p \le |x|^p + \sup_{t\in[s,\mathbb{T}]} |J_t|^p$$

it suffices to establish the inequality $\mathbb{E}[\sup_{t\in[s,\mathbb{T}]} |J_t|^p]$. As we do not know a priori if this quantity is finite or not, we introduce the stopping time $\varrho_n = \inf\{t \in [0,\mathbb{T}], |J_t| > n\}$ and we take $n > |x|$ such that $\varrho_n > s$. The inequality $(a+b+c)^p \le 3^{p-1}(a^p + b^p + c^p)$ supplies estimates, for any $\ell \in [s,\mathbb{T}]$,

$$\begin{aligned} |J_{\ell\wedge\varrho_n}|^p &\le 3^{p-1}\left(|x|^p + \sup_{s\le\ell\le t}\left|\int_s^{\ell\wedge\varrho_n} \theta(r,J_r)dr\right|^p + \sup_{s\le\ell\le t}\left|\int_s^{\ell\wedge\varrho_n} \sigma(r,J_r)dW_r\right|^p\right) \\ &\le 3^{p-1}\left(|x|^p + \left(\int_s^{t\wedge\varrho_n} |\theta(r,J_n)|dr\right)^p + \sup_{s\le\ell\le t}\left|\int_s^{\ell\wedge\varrho_n} \sigma(r,J_r)dW_r\right|^p\right) \end{aligned}$$

The inequality BDG leads to:

$$\begin{aligned} &\mathbb{E}\left[\sup_{s\le\ell\le t\wedge\varrho_n} |J_\ell|^p\right] \\ &\le R\left(|x|^p + \mathbb{E}\left[\left(\int_s^{t\wedge\varrho_n} |\theta(r,J_r)|dr\right)^p\right] + \mathbb{E}\left[\left(\int_s^{t\wedge\varrho_n} |\sigma(r,J_r)|^2 dr\right)^{\frac{p}{2}}\right]\right) \end{aligned}$$

using the Hölder inequality ($\frac{p}{2} \geq 1$), noting p^* the conjugate of p and q that of $\frac{p}{2}$,

$$\mathbb{E}\left[\sup_{s\leq \ell\leq t\wedge\varrho_n} |J_\ell|^p\right]$$
$$\leq R\left(|x|^p + \mathbb{T}^{\frac{p}{p^*}}\mathbb{E}\left[\int_s^{t\wedge\varrho_n} |\theta(r,J_r)|^p dr\right] + \mathbb{T}^{\frac{p}{2q}}\mathbb{E}\left[\int_s^{t\wedge\varrho_n} |\sigma(r,J_r)|^p dr\right]\right)$$

Furthermore, as θ and σ are linear increase, we have:

$$\begin{aligned}\mathbb{E}\left[\int_s^{t\wedge\varrho_n} |\theta(r,J_r)|^p dr\right] &\leq \Im^p\mathbb{E}\left[\int_s^{t\wedge\varrho_n} (1+|J_r|)^p dr\right]\\ &\leq R\left(1+\mathbb{E}\left[\int_s^{t\wedge\varrho_n} |J_r|^p dr\right]\right)\end{aligned}$$

and therefore

$$\mathbb{E}\left[\int_s^{t\wedge\varrho_n} |\theta(r,J_r)|^p dr\right] \leq R\left(1+\mathbb{E}\left[\int_s^t \sup_{s\leq\ell\leq r\leq\varrho_n} |J_\ell|^p dr\right]\right)$$

and the same inequality is valid for the term σ. As a result, we obtain:

$$\mathbb{E}\left[\sup_{s\leq \ell\leq t\wedge\varrho_n} |J_\ell|^p\right] \leq R\left(1+|x|^p + \int_s^t \mathbb{E}\left[\sup_{s\leq\ell\leq r\leq\varrho_n} |J_\ell|^p\right] dr\right)$$

where R does not depend on n. Gronwall's lemma then gives for all n,

$$\mathbb{E}\left[\sup_{s\leq \ell\leq t\wedge\varrho_n} |J_\ell|^p\right] \leq R(1+|x|^p)$$

We fact tender n to infinity and apply Fatou's lemma to get:

$$\mathbb{E}\left[\sup_{s\leq \ell\leq \mathbb{T}} |J_\ell|^p\right] \leq R(1+|x|^p)$$

which completed the proof in the case $p \geq 2$. If now $1 \leq p \leq 2$ then $2p \geq 2$ and Hölder inequality given

$$\mathbb{E}\left[\sup_{s\leq \ell\leq \mathbb{T}} |J_\ell|^p\right] \leq \left(\mathbb{E}\left[\sup_{s\leq \ell\leq \mathbb{T}} |J_\ell|^{2p}\right]\right)^{\frac{1}{2}} \leq R^{\frac{1}{2}}(1+|x|^{2p})^{\frac{1}{2}}$$

this leads to,

$$\mathbb{E}\left[\sup_{s\leq \ell\leq \mathbb{T}} |J_\ell|^p\right] \leq R^{\frac{1}{2}}(1+|x|^p)$$

This last inequality completes the proof of this proposition. □

Now, we know that the solution of a stochastic differential equation has moments of any order, we show a similar estimate for the moments of the increments of J.

Proposition 2.8. *Let $2 \leq p < \infty$. There exists a constant R such that, for any (s,x),$(\acute{s},\acute{x})$ belonging to $[0,\mathbb{T}] \times \mathbb{R}^n$,*

$$\mathbb{E}\left[\sup_{0\leq t\leq \mathbb{T}} |J_t^{s,x} - J_t^{\acute{s},\acute{x}}|^p\right] \leq R\left(|x-\acute{x}|^p + |s-\acute{s}|^{\frac{p}{2}}(1+|\acute{x}|^p)\right) \tag{2.13}$$

Proof. We fix (s,x) and $(\acute{s},\acute{x})$. Trivially,

$$|J_t^{s,x} - J_t^{\acute{s}\acute{x}}|^p \leq 2^{p-1}(|J_t^{s,x} - J_t^{s,\acute{x}}|^p + |J_t^{s,\acute{x}} - J_t^{\acute{s},\acute{x}}|^p)$$

so that we show the inequality to each of the previous two terms. Start with the first $|J_t^{s,x} - J_t^{s,\acute{x}}|^p$. There is no need to take a stopping time because the previous proposition tells us that the expectation of the sup in t is finite. We have

$$\sup_{t\in[0,\mathbb{T}]} |J_t^{s,x} - J_t^{s,\acute{x}}|^p \leq \sup_{t\in[0,s]} |J_t^{s,x} - J_t^{s,\acute{x}}|^p + \sup_{t\in[s,\mathbb{T}]} |J_t^{s,x} - J_t^{s,\acute{x}}|^p$$

so that

$$\sup_{t\in[0,\mathbb{T}]} |J_t^{s,x} - J_t^{s,\acute{x}}|^p \leq |x-\acute{x}|^p + \sup_{t\in[s,\mathbb{T}]} |J_t^{s,x} - J_t^{s,\acute{x}}|^p$$

therefore, we are only interested in the second member of this inequality. For all $\ell \in [s,\mathbb{T}]$, we have

$$\begin{aligned}|J_\ell^{s,x} - J_\ell^{s,\acute{x}}|^p \leq 3^{p-1}&\left(|x-\acute{x}|^p + \left(\int_s^t |\theta(r,J_r^{s,x}) - \theta(r,J_r^{s,\acute{x}})|dr\right)^p\right. \\ &\left. + \sup_{\ell\in[s,t]}\left|\int_s^u (\sigma(r,J_r^{s,x}) - \sigma(r,J_r^{s,\acute{x}}))dW_r\right|^p\right)\end{aligned}$$

BDG and Hölder inequalities lead to the inequality, noting p^* the conjugate of p,

$$\begin{aligned}\mathbb{E}\left[\sup_{\ell\in[s,t]} |J_\ell^{s,x} - J_\ell^{s,\acute{x}}|^p\right] &\leq R\left(|x-\acute{x}|^p + \mathbb{T}^{\frac{p}{p^*}}\mathbb{E}\left[\int_s^t |\theta(r,J_r^{s,x}) - \theta(r,J_r^{s,\acute{x}})|^p dr\right]\right. \\ &\left. + \mathbb{E}\left[\left(\int_s^t |\sigma(r,J_r^{s,x}) - \sigma(r,J_r^{s,\acute{x}})|^2 dr\right)^{\frac{p}{2}}\right]\right)\end{aligned}$$

Using again the Hölder inequality, we obtain, noting q the conjugate of $\frac{p}{2}$,

$$\mathbb{E}\left[\left(\int_s^t |\sigma(r,J_r^{s,x}) - \sigma(r,J_r^{s,\acute{x}})|^2 dr\right)^{\frac{p}{2}}\right] \leq \mathbb{T}^{\frac{p}{2q}}\mathbb{E}\left[\int_s^t |\sigma(r,J_r^{s,x}) - \sigma(r,J_r^{s,\acute{x}})|^p dr\right]$$

θ and σ are Lipschitz, the previous inequality gives

$$\mathbb{E}\left[\sup_{\ell\in[s,t]} |J_\ell^{s,x} - J_\ell^{s,\acute{x}}|^p\right] \leq R\left(|x-\acute{x}|^p + \int_s^t \mathbb{E}\left[\sup_{\ell\in[s,r]} |J_\ell^{s,x} - J_\ell^{s,\acute{x}}|^p\right] dr\right)$$

Gronwall's lemma then gives-changing R

$$\mathbb{E}\left[\sup_{\ell\in[s,\mathbb{T}]} |J_\ell^{s,x} - J_\ell^{s,\acute{x}}|^p\right] \leq R|x-\acute{x}|^p$$

It remains to study the term $\mathbb{E}[\sup_t |J_t^{s,\acute{x}} - J_t^{\acute{s},\acute{x}}|^p]$. We assume without loss of generality that $s \leq \acute{s}$ and cutting into three parts,

$$\sup_{t\in[0,\mathbb{T}]} |J_t^{s,\acute{x}} - J_t^{\acute{s},\acute{x}}|^p \leq \sup_{t\in[0,s]} |J_t^{s,\acute{x}} - J_t^{\acute{s},\acute{x}}|^p + \sup_{t\in[s,\acute{s}]} |J_t^{s,\acute{x}} - J_t^{\acute{s},\acute{x}}|^p + \sup_{t\in[\acute{s},\mathbb{T}]} |J_t^{s,\acute{x}} - J_t^{\acute{s},\acute{x}}|^p$$

from which we deduce that

$$\sup_{t\in[0,\mathbb{T}]} |J_t^{s,\acute{x}} - J_t^{\acute{s},\acute{x}}|^p \leq \sup_{t\in[s,\acute{s}]} |J_t^{s,\acute{x}} - \acute{x}|^p + \sup_{t\in[\acute{s},\mathbb{T}]} |J_t^{s,\acute{x}} - J_t^{\acute{s},\acute{x}}|^p$$

For the first term of the right side of the previous inequality, we have

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[s,\acute{s}]} |J_t^{s,\acute{x}} - \acute{x}|^p\right] \leq 2^{p-1}\Bigg(&\mathbb{E}\left[\left(\int_s^{\acute{s}} |\theta(r, J_r^{s,\acute{x}})|dr\right)^p\right] \\ &+ \mathbb{E}\left[\sup_{t\in[s,\acute{s}]} \left|\int_s^t \sigma(r, J_r^{s,\acute{x}})dW_r\right|^p\right]\Bigg)\end{aligned}$$

The Hölder inequality and the mark 2.12 give, using the linear increase of θ,

$$\mathbb{E}\left[\left(\int_s^{s} |\theta(r, J_r^{s,\acute{x}})|dr\right)^p\right] \leq (\acute{s}-s)^p \mathbb{E}[\sup_{\ell\in[s,\acute{s}]} |\theta(\ell, J_\ell^{s,\acute{x}})|^p] \leq R\mathbb{T}^{\frac{p}{2}}|s-\acute{s}|^{\frac{p}{2}}(1+|\acute{x}|^p)$$

On the other hand, inequality BDG gives

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[s,\acute{s}]} \left|\int_s^t \sigma(r, J_r^{s,\acute{x}})dW_r\right|^p\right] &\leq \mathbb{E}\left[\left(\int_s^{\acute{s}} |\sigma(r, J_r^{s,\acute{x}})|^2 dr\right)^{\frac{p}{2}}\right] \\ &\leq (s-\acute{s})^{\frac{p}{2}} \mathbb{E}\left[\sup_{\ell\in[s,\acute{s}]} \left|\sigma(\ell, J_\ell^{s,\acute{x}})\right|^p\right]\end{aligned}$$

and because of the increase of σ and the estimate (1), we obtain

$$\mathbb{E}\left[\sup_{t\in[s,\acute{s}]} \left|\int_s^t \sigma(r, J_r^{s,\acute{x}})dW_r\right|^p\right] \leq R|s-\acute{s}|^{\frac{p}{2}}(1+|\acute{x}|^p)$$

Finally,

$$\mathbb{E}\left[\sup_{t\in[s,\acute{s}]} |J_t^{s,\acute{x}} - J_t^{\acute{s},\acute{x}}|^p\right] \leq R|s-\acute{s}|^{\frac{p}{2}}(1+|\acute{x}|^p)$$

Study to finish the term $\mathbb{E}[\sup_{t\in[\acute{s},\mathbb{T}]} |J_t^{s,\acute{x}} - J_t^{\acute{s},\acute{x}}|^p]$. Note that, for $t \in [\acute{s}, \mathbb{T}]$,

$$J_t^{s,\acute{x}} = J_{\acute{s}}^{s,\acute{x}} + \int_{\acute{s}}^t \theta(r, J_r^{s,\acute{x}})dr + \int_{\acute{s}}^t \sigma(r, J_r^{s,\acute{x}})dW_r$$

$$J_t^{\acute{s},\acute{x}} = \acute{x} + \int_{\acute{s}}^t \theta(r, J_r^{\acute{s},\acute{x}})dr + \int_{\acute{s}}^t \sigma(r, J_r^{\acute{s},\acute{x}})dW_r$$

We have therefore, for any $\ell \in [\acute{s}, t]$

$$|J_\ell^{s,\acute{x}} - J_u^{\acute{s},\acute{x}}|^p \leq 3^{p-1}\left(|J_{\acute{s}}^{s,\acute{x}} - \acute{x}|^p + \left(\int_{\acute{s}}^t |\theta(r, J_r^{s,\acute{x}}) - \theta(r, J_r^{\acute{s}\acute{x}})|dr\right)^p\right.$$
$$\left. + \sup_{\ell\in[\acute{s},t]}\left|\int_{\acute{s}}^t (\sigma(r, J_r^{s,\acute{x}}) - \sigma(r, J_r^{\acute{s},\acute{x}}))dW_r\right|^p\right)$$

Using inequalities hӧlder and BDG, and the bound (3),and the fact that θ and σ are Lipschitz,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{\ell\in[\acute{s},t]} |J_\ell^{s,\acute{x}} - J_\ell^{\acute{s},\acute{x}}|^p\right] \\
&\leq R\left(|s-\acute{s}|^{\frac{p}{2}}(1+|\acute{x}|^p) + \mathbb{E}\left[\int_{\acute{s}}^t |J_r^{s,\acute{x}} - J_r^{\acute{s},\acute{x}}|^p dr\right]\right) \\
&\leq R\left(|s-\acute{s}|^{\frac{p}{2}}(1+|\acute{x}|^p) + \mathbb{E}\left[\int_{\acute{s}}^t \sup_{\ell\in[\acute{s},r]} |J_\ell^{s,\acute{x}} - J_\ell^{\acute{s},\acute{x}}|^p dr\right]\right)
\end{aligned}$$

Gronwall's lemma applied to $r \longmapsto \sup_{\ell\in[\acute{s},r]} |J_\ell^{s,\acute{x}} - J_\ell^{\acute{s},\acute{x}}|^p$ then gives

$$\mathbb{E}\left[\sup_{\ell\in[\acute{s},t]} |J_\ell^{s,\acute{x}} - J_\ell^{\acute{s},\acute{x}}|^p\right] \leq R|s-\acute{s}|^{\frac{p}{2}}(1+|\acute{x}|^p)$$

which completed the proof. □

Remark 2.9. A direct application of the previous estimate and kolmogorov's theorem, shows that there is a modification of process J such that the application $(s, x, t) \longmapsto J_t^{s,x}$ is continuous.

3. The Continuity of the Solution of the $\natural$-equation

This section contains two Subsections. In the first, we present the theorem of Kolmogorov and its demonstration. In the second, we present our main result.

3.1. Kolmogorov's theorem and its demonstration. This subsection is borrowed from [6]. There are several versions of Kolmogorov's theorem, we give here a quite general one.

Theorem 3.1. [6]. *Let (E, d) be a complete metric space, and let U^x be an E-valued random variable for all x dyadic rational in $\mathbb{R}^n$. Suppose that for all x, y, we have $d(U^x, U^y)$ is a random variable and that there exist strictly positive constants ε, C, β such that*

$$\mathbb{E}\{d(U^x, U^y)^\varepsilon\} \leq C\|x-y\|^{n+\beta} \tag{3.1}$$

Then for almost all ω the function $x \longmapsto U^x$ can be extended uniquely to a continuous function from $\mathbb{R}^n$ to E.

Proof. We prove the theorem for the unit cube $[0,1]^n$. Before the statement of the theorem we establish some notations. Let Δ denote the dyadic rational points of the unit cube $[0,1]^n$ in $\mathbb{R}^n$, and let Δ_m denote all $x \in \Delta$ whose coordinates are of the form $k2^{-m}, 0 \leq k \leq 2^m$. Two points x and y in Δ_m are neighbors if $\sup_i |x^i y^i| = 2^{-m}$. We use Chebyshev's inequality on the inequality hypothesized to get

$$\mathbb{P}\{d(U^x, U^y) \geq 2^{-\alpha m}\} \leq C 2^{\alpha \varepsilon m} 2^{-m(n+\beta)}$$

Let

$$\Lambda_m = \{\omega : \exists \, neighbors \; x, y \in \Delta_m \, with \; d(U^x(\omega), U^y(\omega)) \geq 2^{-\alpha m}\}$$

Since each $x \in \Delta_m$ has at most 3^n neighbors, and the cardinality of Δ_m is 2^{mn}, we have

$$\mathbb{P}(\Lambda_m) \leq c 2^{m(\alpha\varepsilon - \beta)}$$

Where the the constant $c = 3^n C$ .Take α a sufficiently small so that $\alpha\varepsilon < \beta$. Then

$$\mathbb{P}(\Lambda_m) \leq c 2^{-m\delta}$$

Where $\delta = \beta - \alpha\varepsilon > 0$. The Borel-Cantelli Lemma then implies $\mathbb{P}(\Lambda_m \, infinitely \; often) = 0$. That is, there exists an m_0 such that for $m \geq m_0$ and every pair (u, v) of points of Δ_m that are neighbors,

$$d(U^u, U^v) \leq 2^{-\alpha m}$$

We now use the preceding to show that $x \longmapsto U^x$ is uniformly continuous on Δ and hence extendable uniquely to a continuous function on $[0,1]^n$. To this end, let $x, y \in \Delta$ be such that $\|x - y\| \leq 2^{-k-1}$. We will show that $d(U^x, U^y) \leq c2^{-\alpha k}$ for a constant c, and this will complete the proof. Without loss of generality assume $k \geq m_0$. Then $x = (x^1, ..., x^n)$ and $y = (y^1, ..., y^n)$ in Δ with $\|x - y\| \leq 2^{-k-1}$ have dyadic expansions of the form

$$x^i = u^i + \sum_{j>k} a^i_j 2^{-j}$$

$$y^i = v^i + \sum_{j>k} b^i_j 2^{-j}$$

where a^i_j, b^i_j are each 0 or 1 and u,v are points of Δ_k which are either equal or neighbors. Next set $u_0 = u, u_1 = u_0 + a_{k+1} 2^{-k-1}, u_2 = u_1 + a_{k+2} 2^{-k-2}, \cdots$. We also make analogous definitions for $v_0, v_1, v_2, ...$ Then u_{i-1} and u_i are equal or neighbors in Δ_{k+i} each i, and analogously for v_{i-1} and v_i. Hence

$$d(U^x(\omega), U^u(\omega)) \leq \sum_{j=k}^{\infty} 2^{-\alpha j}$$

$$d(U^y(\omega), U^v(\omega)) \leq \sum_{j=k}^{\infty} 2^{-\alpha j}$$

and moreover

$$d(U^u(\omega), U^v(\omega)) \leq 2^{-\alpha k}$$

The result now follows by the triangle inequality. $\square$

The following subsection is the heart of our article. To show our main result, we need the following lemmas:

Lemma 3.2. [7]. *Let $a(t)$ be a non-negative right-continuous increasing (extended real-valued) function on $\mathbb{R}_+$. Set*

$$C(t) = \inf\{s : a(s) > t\},\ t \in \mathbb{R}_+$$

Then $C(t)$ is a non-negative right-continuous increasing function on $\mathbb{R}_+$, and is called the right-inverse function of $a(t)$. For $t \in \mathbb{R}_+, C(t) < +\infty$ if and only if $t < a(\infty) = \lim\limits_{t \longrightarrow \infty} a(t)$. Set

$$a_-(t) = a(t-) = \lim_{s \uparrow\uparrow t} a(s), t > 0 \text{ (such that s} \uparrow\uparrow \text{t means s} \longrightarrow \text{t, s} < \text{t)},$$

$$C_-(t) = C(t-) = \lim_{s \uparrow\uparrow t} C(s) = \inf\{s : a(s) \geq t\} = \sup\{s : a(s) < t\}, t > 0$$

$$a(0-) = a(0), C(0-) = C(0).$$

Then we have

$$a_-(C_-(t)) \leq a_-(C(t)) \leq t\ ,\ t \in \mathbb{R}_+$$

and

$$a(C(t)) \geq a(C_-(t)) \geq t\ , t < a(\infty)$$

Lemma 3.3. [2]. *Let $(a, b) \in \mathbb{R}^2$ with $a < b$, φ and $\psi : [a, b] \longrightarrow \mathbb{R}$ non-negative continuous functions, such that $\exists \rho \in \mathbb{R}^+, \forall t \in [a, b], \varphi(t) \leq \rho + \int_a^t \varphi(s)\psi(s)ds$. Then*

$$\forall t \in [a, b],\ \varphi(t) \leq \rho \exp\left(\int_a^t \psi(s)ds\right)$$

3.2. The main result. This subsection is the heart of our article. In our model, we show the continuity of the solution of the $\natural$-equation by applying the theorem of Kolmogorov presented in the previous subsection and lemma of Gronwall such that we take $\varepsilon = p$ and $\beta = p - n$ with $p > 0$. We have for $u \leq s \leq t$:

$$X_t^u(x) = \ x\ + \int_u^t X_s \left(-\frac{e^{-\Lambda_s}}{1 - Z_s}\right) dN_s + \int_u^t X_s f(X_s - (1 - Z_s))dY_s$$

We know that the quantity $f(X_s - (1 - Z_s))$ is bounded because f is a Lipschitz function, but as we do not know a priori if the quantity $\left(-\frac{e^{-\Lambda_s}}{1-Z_s}\right)$ is finite or not, we introduce the stopping time $\tau_n = \inf\{t, 1 - Z_t < \frac{1}{n}\}$. Therefore, we assume the process $\tilde{X}$ instead of X:

$$d\tilde{X}_t = \tilde{X}_t(-\frac{e^{-\Lambda_t}}{1 - Z_{t \wedge \tau_n}} dN_t + f(\tilde{X}_t - (1 - Z_t))dY_t)$$

such as $\tilde{X}_t = X_t$, $\forall t \leq \tau_n$, $n \in \mathbb{N}$.

We denote $A_t = \tilde{X}_t^u(x) - \tilde{X}_t^u(y)$ and we apply Itô's formula to the process $|A_t|^p$, we find:

$$A = \tilde{X}^x - \tilde{X}^y$$
$$dA_t = d(\tilde{X}_t^x - \tilde{X}_t^y)$$
$$d|A_t|^p = p|A_t|^{p-1}dA_t + \tfrac{|A_t|^{p-2}}{2}p(p-1)[d < A_t, A_t >]$$

Such as

$$dA_t = d(\tilde{X}_t^x - \tilde{X}_t^y)$$
$$dA_t = (\tilde{X}_t^x - \tilde{X}_t^y)\left(-\frac{e^{-\Lambda_t}}{1 - Z_{t\wedge\tau_n}}\right)dN_t + [\tilde{X}_t^x f(\tilde{X}_t^x - (1 - Z_t)) - \tilde{X}_t^y f(\tilde{X}_t^y - (1 - Z_t))]dY_t$$

Noting

$$\mathcal{V}_t(\tilde{X}_t^x) = \tilde{X}_t^x f(\tilde{X}_t^x - (1 - Z_t))$$
$$\mathcal{V}_t(\tilde{X}_t^y) = \tilde{X}_t^y f(\tilde{X}_t^y - (1 - Z_t))$$

So

$$d|A_t|^p = p|A_t|^{p-1}dA_t + \frac{|A_t|^{p-2}}{2}p(p-1)d < A_t, A_t >$$

and

$$\begin{aligned} &d|A_t|^p \\ &= p|A_t|^{p-1}dA_t + \frac{|A_t|^{p-2}}{2}p(p-1)\Bigg[(\tilde{X}_t^x - \tilde{X}_t^y)^2\left(-\frac{e^{-\Lambda_t}}{1 - Z_{t\wedge\tau_n}}\right)^2 d < N, N >_t \\ &\quad + (\mathcal{V}_t(\tilde{X}_t^x) - \mathcal{V}_t(\tilde{X}_t^y))^2 d < Y, Y >_t + 2(\tilde{X}_t^x - \tilde{X}_t^y)\left(-\frac{e^{-\Lambda_t}}{1 - Z_{t\wedge\tau_n}}\right) \\ &\quad \times (\mathcal{V}_t(\tilde{X}_t^x) - \mathcal{V}_t(\tilde{X}_t^y))d < N, Y >_t \Bigg] \end{aligned}$$

By lemma of Jacod (see [3], page 128,129), there always exists a process G, such that: $C_{11}dG = d < N, N >$, $C_{22}dG = d < Y, Y >$ and $C_{12}dG = d < N, Y >$ with

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

being a symmetric nonnegative matrix, and the choice of the latter is arbitrary, then

$$\begin{aligned} d|A_t|^p = p|A_t|^{p-1}dA_t + \frac{A_t^{p-2}}{2}p(p-1)\Big[\Big((\tilde{X}^x - \tilde{X}^y), \mathcal{V}_t(\tilde{X}_t^x) - \mathcal{V}_t(\tilde{X}_t^y)\Big) \\ \times \begin{pmatrix} -\dfrac{e^{-\Lambda_t}}{1 - Z_{t\wedge\tau_n}} & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \times \begin{pmatrix} -\dfrac{e^{-\Lambda_t}}{1 - Z_{t\wedge\tau_n}} & 0 \\ 0 & 1 \end{pmatrix} \\ \times \begin{pmatrix} \tilde{X}_t^x - \tilde{X}_t^y \\ \mathcal{V}_t(\tilde{X}_t^x) - V_t(\tilde{X}_t^y) \end{pmatrix}\Big] dG_t \end{aligned}$$

We denote

$$W_t^T = \Big((\tilde{X}^x - \tilde{X}^y), \mathcal{V}_t(\tilde{X}_t^x) - \mathcal{V}_t(\tilde{X}_t^y)\Big)$$
$$M = \begin{pmatrix} -\dfrac{e^{-\Lambda_t}}{1 - Z_{t\wedge\tau_n}} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}\begin{pmatrix} -\dfrac{e^{-\Lambda_t}}{1 - Z_{t\wedge\tau_n}} & 0 \\ 0 & 1 \end{pmatrix}$$

$$W_t = \begin{pmatrix} \tilde{X}_t^x - \tilde{X}_t^y \\ \mathcal{V}_t(\tilde{X}_t^x) - \mathcal{V}_t(\tilde{X}_t^y) \end{pmatrix}$$

So

$$d|A_t|^p = p|A_t|^{p-1}dA_t + \tfrac{A_t^{p-2}}{2}p(p-1)[W_t^T M W_t]dG_t$$

$$|A_t|^p = |x-y|^p + \left[\int_u^t p|A_s|^{p-1}dA_s + \int_u^t \frac{p(p-1)}{2}A_s^{p-2}W_s^T M W_s dG_s\right]$$

$$\mathbb{E}[|A_t|^p] = |x-y|^p + \mathbb{E}[\int_u^t p|A_s|^{p-1}dA_s] + \mathbb{E}[\int_u^t \frac{p(p-1)}{2}A_s^{p-2}W_s^T M W_s dG_s]$$

$$\mathbb{E}[|A_t|^p] \leq |x-y|^p + \mathbb{E}[\int_u^t \frac{p(p-1)}{2}A_s^{p-2}W_s^T M W_s dG_s]$$

$$\mathbb{E}[|A_t|^p] \leq |x-y|^p + \mathbb{E}[\int_u^t \frac{p(p-1)}{2}A_s^{p-2}m_s|W_s|^2 dG_s]$$

such that

$$M = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \text{ and } m = |b_{11}| + |b_{12}| + |b_{21}| + |b_{22}|$$

So

$$\mathbb{E}[|A_t|^p] \leq |x-y|^p + \mathbb{E}[\int_u^t \frac{p(p-1)}{2}A_s^{p-2}m_s((\tilde{X}_s^x - \tilde{X}_s^y)^2 + (V_s(\tilde{X}_s^x) - V_s(\tilde{X}_s^y))^2)dG_s].$$

But f is a lipschitz function, then there exists a real positive constant K, so

$$|V_s(\tilde{X}_s^x) - V_s(\tilde{X}_s^y)|^2 \leq K|\tilde{X}_s^x - \tilde{X}_s^y|^2$$

Therefore

$$\mathbb{E}[|A_t|^p] \leq |x-y|^p + \mathbb{E}[\int_u^t \frac{p(p-1)}{2}A_s^{p-2}m_s(A_s^2 + K|A_s|^2)dG_s]$$

$$\mathbb{E}[|A_t|^p] \leq |x-y|^p + \mathbb{E}[\int_u^t \frac{p(p-1)}{2}|A_s|^p m_s(1+K)dG_s]$$

$$\mathbb{E}[|A_t|^p] \leq |x-y|^p + \frac{p(p-1)}{2}(1+K)\mathbb{E}[\int_u^t |A_s|^p m_s dG_s]$$

We denote

$$a = |x-y|^p$$

$$b = \frac{p(p-1)}{2}(1+K)$$

Then

$$\mathbb{E}[|A_t|^p] \leq a + b\mathbb{E}[\int_u^t |A_s|^p m_s dG_s]$$

to apply Gronwall's lemma (lemma 3.3) we must use the technique of change of time to eliminate the process G; so for this we will use the lemma 3.2 (see [7]). In our case, putting $G(s) = a(s)$ and we consider the stopping time

$$C(t) = \inf\{s,\, G(s) > t\}$$

Such that, for $t \in \mathbb{R}_+$, $C(t) < \infty$ if and only if $t < G(\infty) = \lim_{t \longrightarrow \infty} G(t)$ and

$$G(C(t)) \geq G(C_-(t)) \geq t,\ t \in \mathbb{R}_+$$

In fact

$$\mathbb{E}[|A_t|^p] \leq a + b\,\mathbb{E}\left[\int_u^t |A_s|^p\, m_s dG_s\right]$$

For $s, t, \alpha \in \mathbb{R}_+$ such that $s < \alpha$

$$\mathbb{E}[\sup_{t\leq C(\alpha)} |A_t|^p] \leq a + b\,\mathbb{E}\left[\sup_{t\leq C(\alpha)} \int_u^{C(\alpha)} |A_s|^p\, m_s dG_s\right]$$

We denote

$$B_t = \sup_{t\leq C(\alpha)} |A_t|$$

Then

$$\mathbb{E}[B_t^p] \leq a + b\ \mathbb{E}\left[\int_u^\alpha B_{C(s)}^p\, m_{C(s)} d\, G_{C(s)}\right]$$

$$\mathbb{E}[B_t^p] \leq a + b\ \mathbb{E}\left[\int_u^\alpha B_{C(s)}^p\, m_{C(s)} ds\right]$$

$$\mathbb{E}[B_t^p] \leq a + b\ \mathbb{E}\left[\int_u^{C(\alpha)} B_s^p\, m_s ds\right]$$

Now, we can apply the lemma of Gronwall to this last expression, we have

$$\mathbb{E}[B_t^p] \leq a + b\mathbb{E}\left[\int_u^{C(\alpha)} B_s^p m_s ds\right]$$

We take

$$\varphi(t) = \mathbb{E}[B_t^p]$$
$$\psi(s) = m_s$$
$$a = \rho$$

So, we find

$$\mathbb{E}[B_t^p] \leq a \exp\left(b \int_u^{C(\alpha)} m_s ds\right)$$

Eventually, if $\left(\int_u^{C(\alpha)} m_s ds\right)$ is finite, there exist the constant C which satisfy the condition of Kolmogorov's lemma that is to say that C= $\exp\left(b\int_u^{C(\alpha)} m_s ds\right)$.

4. Conclusion

This document contains a new and original methodological approach to the subject in question and could therefore be a good contribution to the theory of stochastic processes, based on a very interesting lemma of Kolomogorov. Some difficulties have been encountered because the subject deals with a difficult area "the stochastic differential equations". As prospects, we try to prove the same result of the paper, but in a vectorial case; moreover, we also think of demonstrating that the stochastic flow associated with our model will be a diffeomorphism with multidimensional parameters on the same space, and we will investigate whether it is possible to have the same work on manifolds.

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