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A CONVERSE COMPARISON THEOREM FOR DISCRETE-TIME FINITE-STATE BSDES AND RISK MEASURES USING g-EXPECTATION

ROBERT ELLIOTT, YIN LIN, AND HAILIANG YANG

Abstract. This paper studies properties of non-linear expectations defined using the discrete-time finite-state Backward Stochastic Difference Equations (BSDE) proposed by Cohen and Elliott [6]. We also establish a converse comparison theorem. Properties of risk measures defined by non-linear expectations, especially the representation theorems, will be given. Finally we apply the theory of BSDEs to optimal design of dynamic risk measures.

1. Introduction

The study of BSDEs has been developing rapidly recently. Linear BSDEs were first introduced by Bismut [3]. Then the concept was generalized by Pardoux and Peng [10] by considering equations of the form:

$$Y_t - \int_t^T g(\omega, s, Y_{s-}, Z_s)ds + \int_t^T Z_s dW_s = Q.$$

Here g is the driver, Q is a square-integrable terminal condition and the process W is a d-dimensional Brownian motion. Then g-expectations defined using such BSDEs were proposed by Peng [12]. Cohen and Elliott [5] also considered BSDEs related to continuous-time finite-state Markov chains. However, the assumptions of the work in the continuous-time setting are quite strong and complicated. Thus Cohen and Elliott [6] introduced BSDEs on spaces related to discrete-time finite-state processes and explored the corresponding theory under weaker assumptions.

In this paper, we follow the idea of Cohen and Elliott [6] by considering discrete-time finite-state BSDEs. We first recall the results of Cohen and Elliott [6] in Section 2. Properties of related g-expectations are discussed in Section 3. Based on these results, we prove a converse comparison theorem for a general case, together with the independent case and the deterministic case in Section 4. Then we apply the results of Section 3 to obtain properties of risk measures using g-expectations especially the representation theorem in Section 5. In Section 6, applications to optimal design of dynamic risk measures are explored, including optimal solutions and characterization of inf-convolution of dynamic entropic risk measures and associated drivers. We summarize this paper in Section 7.

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2. The Discrete-time Finite-state Model

We follow Cohen and Elliott [6] by considering a discrete-time finite-state process X as the underlying stochastic process. Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $X = \{X_t, t \in \{0, 1, \ldots, T\}\}$ is a finite state process. Without loss of generality we suppose for each $t \in \{0, 1, \ldots, T\}$,

$$X_t \in \{e_1, e_2, \ldots, e_N\},$$

where N is the number of the states and e_i is the ith standard unit vector in $\mathbb{R}^N$. Consider a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \le t \le T}, \mathbb{P})$, where

$$\mathcal{F}_t = \sigma(\{X_s, s \le t\}, A \in \mathcal{F} : \mathbb{P}(A) = 0).$$

Then, we can define the martingale process M by $M_t := X_t - \mathbb{E}(X_t | \mathcal{F}_{t-1})$. We shall discuss vector BSDEs of the form:

$$Y_t(\omega) - \sum_{t \le s < T} g(\omega, s, Y_s(\omega), Z_s(\omega)) + \sum_{t \le s < T} Z_s(\omega) M_{s+1}(\omega) = Q(\omega). \tag{2.1}$$

Here g is an adapted functional $g : \Omega \times \{0, \cdots, T\} \times \mathbb{R}^K \times \mathbb{R}^{K \times N} \to \mathbb{R}^K$ and Q an $\mathbb{R}^K$-valued $\mathcal{F}_T$-measurable terminal condition.

Remark 2.1. (see Cohen and Elliott [6]) Since X can take only finitely many possible paths, and all the quantities are finite-dimensional, it is clear that

$$L^1(\mathcal{F}_t) = L^2(\mathcal{F}_t) = L^\infty(\mathcal{F}_t).$$

Remark 2.2. (see Cohen and Elliott [6]) BSDE (2.1) is equivalent to the following one-step equation:

$$Y_t - g(\omega, t, Y_t, Z_t) + Z_t M_{t+1} = Y_{t+1}, \quad Y_T = Q. \tag{2.2}$$

Definition 2.3. (see Cohen and Elliott [6]) For any integer K and adapted process Z in $\mathbb{R}^{K \times N}$, we define the seminorm of Z, $\|Z\|_M$, by:

$$\|Z\|_M^2 := \mathbb{E}[\mathrm{Tr}(\sum_{0 \le s < T} (Z_s M_{s+1})(Z_s M_{s+1})^*)].$$

If $\|Z^1 - Z^2\|_M^2 = 0$, we shall write $Z^1 \sim_M Z^2$. And if $Z_t^1 M_{t+1} = Z_t^2 M_{t+1}$ $\mathbb{P}$-a.s., we shall write $Z_t^1 \sim_{M_{t+1}} Z_t^2$.

Theorem 2.4. *(Existence and Uniqueness) (see Cohen and Elliott* [6]*) Suppose g satisfies the assumptions:*

(A) If $Z_t^1 \sim_{M_{t+1}} Z_t^2$, then for any Y, $g(\omega, t, Y_t, Z_t^1) = g(\omega, t, Y_t, Z_t^2)$ $\mathbb{P}$-a.s. for all t,

(B) For any $z \in \mathbb{R}^{K \times N}$, for all t, for $\mathbb{P}$-almost all ω, the map

$$y \mapsto y - g(\omega, t, y, z)$$

is a bijection $\mathbb{R}^K \to \mathbb{R}^K$.

Then for any terminal condition Q which is essentially bounded, $\mathcal{F}_T$-measurable, and with values in $\mathbb{R}^K$, BSDE (2.1) *has an adapted solution (Y, Z). Moreover, this solution is unique up to indistinguishability for Y and equivalence $\sim_M$ for Z.*

Thus we can define the conditional g-expectation of Q under $\mathcal{F}_t$ as

$$\mathcal{E}^g(Q | \mathcal{F}_t) = Y_t. \tag{2.3}$$

The g-expectation of Q is defined as

$$\mathcal{E}^g(Q) := Y_0 = \mathcal{E}^g(Q|\mathcal{F}_0). \tag{2.4}$$

Theorem 2.5. *(Comparison Theorem) (see Cohen and Elliott [6]) Consider two BSDEs as in (2.1) with drivers g^i, $i = 1, 2$ and essentially bounded terminal values Q^i, $i = 1, 2$. Suppose that g^i satisfies both Assumptions (A) and (B). Let (Y^i, Z^i) be the associated solution. Suppose the following conditions hold:*

(i) $Q^1 \geq Q^2$, $\mathbb{P}$*-a.s.*
(ii) $\mathbb{P}$*-a.s., for all times* t, $g^1(\omega, t, Y_t^2, Z_t^2) \geq g^2(\omega, t, Y_t^2, Z_t^2)$.
(iii) $\mathbb{P}$*-a.s., for all times* t*, the ith component of* g^1*, given by* $e_i^* g^1$*, satisfies*

$$e_i^* g^1(\omega, t, Y_t^2, Z_t^1) - e_i^* g^1(\omega, t, Y_t^2, Z_t^2) \geq \min_{j \in \mathbb{J}_t}\{e_i^*(Z_t^1 - Z_t^2)(e_j - \mathbb{E}(X_{t+1}|\mathcal{F}))\},$$

where $\mathbb{J}_t$ *is the* $\mathcal{F}_t$*-measurable set of indices of possible values of* X_{t+1}*, given* $\mathcal{F}_t$*, i.e.* $\mathbb{J}_t := \{i : \mathbb{P}(X_{t+1} = e_i|\mathcal{F}_t) > 0\}$.

(iv) $\mathbb{P}$*-a.s., for all* t*, if*

$$Y_t^1 - g^1(\omega, t, Y_t^1, Z_t^1) \geq Y_t^2 - g^1(\omega, t, Y_t^2, Z_t^1),$$

then $Y_t^1 \geq Y_t^2$.

Then $Y^1 \geq Y^2$ $\mathbb{P}$*-a.s.*

Properties of solutions of BSDEs, and hence the related g-expectations, are determined by properties of the driver g. Therefore, before we start the next section, we shall state some assumptions for g we may use in the sequel:

(C) For any t, $g(\omega, 0, 0) = 0$ $\mathbb{P}$-s.s.
(D) For any y and t, $g(\omega, t, y, 0) = 0$ $\mathbb{P}$-a.s.
(E) g is independent of y, i.e. for any z fixed, for any t, $g(t, y, z) = g(t, y', z)$ $\mathbb{P}$-a.s. for any y, y'.
(F) g is positive homogeneous in (y, z), i.e. for all t, all $\lambda \geq 0$ and all (y, z),

$$\lambda g(t, y, z) = g(t, \lambda y, \lambda z), \quad \mathbb{P} - a.s.$$

(G) g is convex in (y, z), that is, for all t and all $\alpha \in (0, 1)$, and all (y^1, z^1), $(y^2, ^2)$, $\mathbb{P}$-a.s.

$$g(t, \alpha y^1 + (1-\alpha)y^2, \alpha z^1 + (1-\alpha)z^2) \leq \alpha g(t, y^1, z^1) + (1-\alpha)g(t, y^2, z^2).$$

(H) For any fixed z, for all t, for $\mathbb{P}$-almost all ω, the map $y \mapsto y - g(\omega, t, y, z)$ is increasing, i.e. if $y^1 \geq y^2$ componentwise, then $y^1 - g(t, y^1, z) \geq y^2 - g(t, y^2, z)$ $\mathbb{P}$-a.s. componentwise.
(I) (see Cohen and Elliott [6]) Consider some driver g that satisfies Assumptions (A) and (B). Suppose for all t, and for all essentially bounded Q^1, Q^2 , the corresponding BSDE solutions (Y^1, Z^1), (Y^2, Z^2) satisfy
(iii') $\mathbb{P}$-a.s., for all times t, the ith component of g^1, given by $e_i^* g^1$, satisfies

$$\begin{aligned} &e_i^* g^1(\omega, t, Y_t^2, Z_t^1) - e_i^* g^1(\omega, t, Y_t^2, Z_t^2) \\ &\geq \min_{j \in \mathbb{J}_t}\{e_i^*(Z_t^1 - Z_t^2)(e_j - \mathbb{E}(X_{t+1}|\mathcal{F}))\}, \end{aligned}$$

with equality only if $e_i^* Z_t^1 \sim_{M_{t+1}} e_i^* Z_t^2$.

(iv') $\mathbb{P}$-a.s., for all t, if $Y_t^1 - g^1(\omega, t, Y_t^1, Z_t^1) \geq Y_t^2 - g^1(\omega, t, Y_t^2, Z_t^1)$, then $Y_t^1 \geq Y_t^2$, the inequalities being taken componentwise.

Then we shall call g a *balanced driver*.

Lemma 2.6. *Assumptions (D), (E), (F), (G) and (H) are respectively equivalent to:*

(D) For any $\mathcal{F}_t$-measurable Y and any t, $g(\omega, t, Y, 0) = 0$, $\mathbb{P}$-a.s..
(E) For any $\mathcal{F}_t$-measurable Y, Y', any t and Z, $g(t, Y, Z) = g(t, Y', Z)$.
(F) For all t, $\lambda \geq 0$ and all $\mathcal{F}_t$-measurable Y, Z, $\mathbb{P}$-a.s.,

$$\lambda g(t, Y, Z) = g(t, \lambda Y, \lambda Z)$$

(G) For all t, and all $\alpha \in (0,1)$, and all $\mathcal{F}_t$-measurable Y_t^1, Y_t^2, Z_t^1, Z_t^2, $\mathbb{P}$-a.s.

$$g(t, \alpha Y_t^1 + (1-\alpha) Y_t^2, \alpha Z_t^1 + (1-\alpha) Z_t^2) \leq \alpha g(t, Y_t^1, Z_t^1) + (1-\alpha) g(t, Y_t^2, Z_t^2).$$

(H) For any fixed $\mathcal{F}_t$-measurable Z_t, for all t, and all $\mathcal{F}_t$-measurable Y_t^1, Y_t^2, if $Y_t^1 \geq Y_t^2$ componentwise, then $Y_t^1 - g(t, Y_t^1, Z_t) \geq Y_t^2 - g(t, Y_t^2, Z_t)$ $\mathbb{P}$-a.s. componentwise.

Proof. We need only to prove the equivalence between (D) and (D'), since the others are analogous. That (D') implies (D) is trivial. For the converse implication, suppose there exist some t and $\mathcal{F}_t$-measurable Y, such that $g(\omega, t, Y, 0) \neq 0$ with positive probability. Since Y and $g(t, Y, 0)$ are $\mathcal{F}_t$-measurable and X can take only finitely many paths, $g(t, Y, 0) = \sum_i a_i I_{A_i}$ and $Y = \sum_i y_i I_{A_i}$ for some partition A_i of $\mathcal{F}_t$ and some constants a_i, y_i. It follows that there exist some k s.t. $a_k \neq 0$. Then $\mathbb{P}$-a.s. in A_k, $g(\omega, t, Y, 0) = g(\omega, t, y_k, 0) \neq 0$, which contradicts assumption (D). □

Remark 2.7. In continuous-state setting, assumption (D) (respectively (E), (F), (G) and (H)) is weaker than (D') (resp. (E'), (F'), (G') and (H')). Suppose $\Omega = (0,1)$, $\mathcal{F}_t = \mathfrak{B}(0,1)$ and $\mathbb{P}$ is Lebesgue measure. Let $g(\omega, t, y, z) = \mathbb{I}_{\{y=\omega\}}$. It is obvious that for any y and t, $g(\omega, t, y, 0) = 0$ $\mathbb{P}$-a.s. But for the $\mathcal{F}_t$-measurable random variable $Y(\omega) = \omega$, $g(\omega, t, Y, 0) \equiv 1$.

3. Properties of Conditional g-expectations

In this section, we study properties of conditional g-expectations, upon which Sections 5 and 6 are based. Though some of the properties in the following lemma were proved by Cohen and Elliott [6], for completeness it is worth recalling them.

Proposition 3.1. *Suppose g satisfies Assumptions (A) and (B). Then we have the following properties:*

(i) (Terminal equality) $\mathcal{E}^g(Q|\mathcal{F}_T) = Q$, for all Q.
(ii) ($\mathcal{F}_t$-triviality and recursivity) The following three conditions are equivalent:
   (a) (Recursivity) $\mathcal{E}^g(\mathcal{E}^g(Q|\mathcal{F}_t)|\mathcal{F}_s) = \mathcal{E}^g(Q|\mathcal{F}_s)$ $\mathbb{P}$-a.s. for any $s \leq t$.
   (b) ($\mathcal{F}_t$-triviality) $\mathcal{E}^g(Q|\mathcal{F}_t) = Q$ $\mathbb{P}$-a.s. for any $\mathcal{F}_t$-measurable Q.
   (c) For any y and t, $g(\omega, t, y, 0) = 0$ $\mathbb{P}$-a.s..

(iii) *(Monotonicity) If g is a balanced driver, then for any t and any $Q \geq Q'$ $\mathbb{P}$-a.s. componentwise,*

$$\mathcal{E}^g(Q|\mathcal{F}_t) \geq \mathcal{E}^g(Q'|\mathcal{F}_t)$$

$\mathbb{P}$-a.s. componentwise, with equality only if $Q = Q'$.

(iv) *(Translation invariance) for all t, all $Q \in L^1(\mathcal{F}_T)$ and all $q \in L^1(\mathcal{F}_t)$,*

$$\mathcal{E}^g(Q + q|\mathcal{F}_t) = \mathcal{E}^g(Q|\mathcal{F}_t) + q.$$

if and only if g is independent of y.

(v) *(Regularity) For any t, any $A \in \mathcal{F}_t$, $I_A\mathcal{E}^g(Q|\mathcal{F}_t) = \mathcal{E}^g(I_A Q|\mathcal{F}_t)$ $\mathbb{P}$-a.s. if and only if $g(t, 0, 0) = 0$ $\mathbb{P}$-a.s. for any t.*

(vi) *$\mathcal{E}^g(\cdot|\mathcal{F}_t)$ is positive homogeneous, that is, for all t, all $\lambda \geq 0$, and all $Q \in L(\mathcal{F}_T)$,*

$$\mathcal{E}^g(\lambda Q|\mathcal{F}_t) = \lambda\mathcal{E}^g(Q|\mathcal{F}_t)$$

if and only if g is positive homogeneous in (y, z).

(vii) *If g is balanced and convex in (y, z), then $\mathcal{E}^g(\cdot|\mathcal{F}_t)$ is convex, that is, for all t, all $\alpha \in (0, 1)$, all Q^1, $Q^2 \in L(\mathcal{F}_T)$ and all t,*

$$\mathcal{E}^g(\alpha Q^1 + (1-\alpha)Q^2|\mathcal{F}_t) \leq \alpha\mathcal{E}^g(Q^1|\mathcal{F}_t) + (1-\alpha)\mathcal{E}^g(Q^2|\mathcal{F}_t).$$

Conversely, if $\mathcal{E}^g(\cdot|\mathcal{F}_t)$ is convex and g satisfies (D)and (H), then g is convex.

Proof. (i) is trivial.

(ii): (ii)(a)⇒(ii)(b): For any $\mathcal{F}_t$-measurable Q, define $Q' = Y_T$ through the recursion

$$Y_{s+1} = Y_s - g(s, Y_s, Z_s) + Z_u M_{s+1},$$

where $Y_t := Q$, and Z_s is arbitrary $\mathcal{F}_s$-measurable random variable. Then

$$\mathcal{E}^g(Q'|\mathcal{F}_t) = Q. \tag{3.1}$$

From (a), let $s = t$, then we obtain

$$\mathcal{E}^g(\mathcal{E}^g(Q'|\mathcal{F}_t)|\mathcal{F}_t) = \mathcal{E}^g(Q'|\mathcal{F}_t). \tag{3.2}$$

(ii)(b) is established by substituting (3.1) into (3.2).
(ii)(b)⇒(ii)(c): For any $y \in \mathbb{R}^K$, by (B) we have $\mathcal{E}^g(y|\mathcal{F}_t) = y$, for all t. By the one-step equation (2.2), $y - g(t, y, Z_t) + Z_t M_{t+1} = y$, for all $t < T$. Taking a conditional expectation gives $Z_t = 0$ and $g(t, y, 0) = 0$.
(ii)(c)⇒(ii)(a): For the proof, see Theorem 7 of Cohen and Elliott [6].

(iii) follows directly from Theorem 2.5.

(iv): The first implication follows using the proof given by Cohen and Elliott [6] for Theorem 7. Conversely, for all t, all $(y, z) \in \mathbb{R}^K \times \mathbb{R}^{K\times N}$ and all $q \in \mathbb{R}^K$, define $Q = Y_T$ through the recursion

$$Y_{s+1} = Y_s - g(s, Y_s, Z_s) + Z_s M_{s+1}$$

where $Y_t := y$, $Z_t = z$. Then $\mathcal{E}^g(Q|\mathcal{F}_t) = y$. Consider the one-step equation:

$$y - g(t, y, z) + zM_{t+1} = Y_{t+1}. \tag{3.3}$$

Since q is a constant, $\mathcal{E}^g(Q+q|\mathcal{F}_t) = y+q$ and $\mathcal{E}^g(Q+q|\mathcal{F}_{t+1}) = Y_{t+1}+q$. Then we have

$$y + q - g(t, y+q, Z) + ZM_{t+1} = Y_{t+1} + q. \tag{3.4}$$

From equation (3.3) and (3.4), it follows that $Z = z$ and $g(t, y+q, z) = g(t, y, z)$.

(v): Cohen and Elliott [6] proved that if $g(\omega, t, y, 0) = 0$, for all y and all t, then $\mathcal{E}^g(\cdot|\mathcal{F}_t)$ satisfies regularity. However, we need only the weaker condition $g(\omega, t, 0, 0) = 0$ and the proof is similar.

Conversely, if $\mathcal{E}^g(\cdot|\mathcal{F}_t)$ satisfies regularity, then for all $A \in \mathcal{F}_t$,

$$I_A \mathcal{E}^g(Q|\mathcal{F}_t) = \mathcal{E}^g(I_A Q|\mathcal{F}_t), \tag{3.5}$$

$$I_A \mathcal{E}^g(Q|\mathcal{F}_{t+1}) = \mathcal{E}^g(I_A Q|\mathcal{F}_{t+1}). \tag{3.6}$$

It follows that

$$I_A Y_t - g(t, Y_t, Z'_t) + Z'_t M_{t+1} = I_A Y_{t+1} \tag{3.7}$$

Combining (2.2) and (3.7), we can obtain

$$Z'_t = I_A Z_t, \quad I_A g(t, Y_t, Z_t) = g(t, I_A Y_t, I_A Z_t).$$

Then $g(t, 0, 0) = 0$, since A is any set in $\mathcal{F}_t$.

(vi): Cohen and Elliott [6] proved that the positive homogeneity of g guarantees the positive homogeneity of $\mathcal{E}^g(\cdot|\mathcal{F}_t)$.

For the converse implication, for any t and (y, z), we can construct a terminal condition Q with the associated solution $(Y_t, Z_t) = (y, z)$ as in the proof of (iv). Then for any $\lambda \geq 0$,

$$\lambda \mathcal{E}^g(Q|\mathcal{F}_t) = \mathcal{E}^g(\lambda Q|\mathcal{F}_t), \tag{3.8}$$

$$\lambda \mathcal{E}^g(Q|\mathcal{F}_{t+1}) = \mathcal{E}^g(\lambda Q|\mathcal{F}_{t+1}). \tag{3.9}$$

i.e.

$$\lambda y - g(t, \lambda y, Z'_t) + Z'_t = \lambda Y_{t+1}. \tag{3.10}$$

Combining (3.3) and (3.10), we obtain $Z'_t = \lambda z$ and $\lambda g(t, y, z) = g(t, \lambda y, \lambda z)$.

(vii): The first implication is analogous to the one given by Cohen and Elliott [7] for Theorem 9.7 for BSDEs on continuous-time finite-state Markov chains. Conversely, suppose $\mathcal{E}^g(\cdot|\mathcal{F}_t)$ is convex and g satisfies (D) and (H). Then taking a convex combination of the BSDEs with terminal condition $Q^1 = y^1 - g(t, y^1, z^1) + z^2 M_{t+1}$ and $Q^2 = y^2 - g(t, y^2, z^2) + z^2 M_{t+1}$ gives the equation

$$\begin{aligned} &\alpha y^1 + (1-\alpha) y^2 - (\alpha g(t, y^1, z^1) + (1-\alpha) g(t, y^2, z^2)) \\ &\quad + (\alpha z^1 + (1-\alpha) z^2) M_{t+1} = \alpha Q^1 + (1-\alpha) Q^2. \end{aligned} \tag{3.11}$$

Consider the BSDE

$$Y_t - g(t, Y_t, Z_t) + Z_t M_{t+1} = Q^\alpha \tag{3.12}$$

with terminal condition $Q^\alpha := \alpha Q^1 + (1-\alpha) Q^2$. Combining (3.11) and (3.12), we obtain $Z_t = \alpha z^1 + (1-\alpha) z^2$. Thus

$$\begin{aligned} \alpha y^1 + &(1-\alpha) y^2 - (\alpha g(t, y^1, z^1) + (1-\alpha) g(t, y^2, z^2)) \\ &= Y_t - g(t, Y_t, \alpha z^1 + (1-\alpha) z^2) \\ &\leq \alpha y^1 + (1-\alpha) y^2 - (\alpha g(t, y^1, z^1) - g(t, \alpha y^1 + (1-\alpha) y^2, \alpha z^1 + (1-\alpha) z^2). \end{aligned}$$

The convexity of g is established. □

4. A Converse Comparison Theorem

Comparison theorem is one of the key results in the theory of BSDEs, as it allows us to compare the solutions of two BSDEs if we can compare the terminal conditions and the drivers. Comparison theorem for BSDEs was first established by Peng [11] in Brownian setting, then generalized by El Karoui et al. [8]. Cohen and Elliott [7] also explored the theory for BSDEs on spaces related to continuous-time finite-state Markov chains. The result for discrete-time finite-state BSDEs was obtained by Cohen and Elliott [6].

Converse comparison theorem is another important result that allows one to compare the drivers whenever we can compare the solutions of two BSDEs with the same terminal condition. In Brownian setting, Peng [12] proved that "$Y_0^1(\xi) = Y_0^2(\xi)$ for each $\xi \in L^2(\mathcal{F}_T)$" implies "$g^1 = g^2$". Results for inequalities were discussed by Briand et al. [4]. We explore the results for discrete-time finite-state models in this section.

Theorem 4.1. *(General case) Let Assumptions (A), (B) and (D) hold for g^i, $i = 1, 2$. Assume moreover that g^2 also satisfies (H) and that for all t and all Q,*

$$\mathcal{E}^{g^1}(Q|\mathcal{F}_t) \geq \mathcal{E}^{g^2}(Q|\mathcal{F}_t).$$

Then for all t and all (y, z), we have, $\mathbb{P}$-a.s.,

$$g^1(\omega, t, y, z) \geq g^2(\omega, t, y, z).$$

Proof. For any $t < T$, and any $(y, z) \in \mathbb{R}^K \times \mathbb{R}^{K\times N}$, consider BSDEs with driver g^i respectively and both with terminal condition $Q = y - g^1(t, y, z) + zM_{t+1}$. Since (D) is satisfied, $Y_{t+1}^i = \mathcal{E}^{g^i}(Q|\mathcal{F}_{t+1}) = Q$. The BSDEs will reduce to

$$Y_t^i - g^i(t, Y_t^i, Z_t^i) + Z_t^i M_{t+1} = y - g^1(t, y, z) + zM_{t+1}. \tag{4.1}$$

Since (D) is satisfied, it is obvious that $(Y_t^1, Z_t^1) = (y, z)$ and $Z_t^2 = z$ according to the uniqueness of the solutions. Then we have

$$y - g^1(t, y, z) = Y_t^2 - g^2(t, Y_t^2, z) \leq y - g^2(t, y, z), \tag{4.2}$$

where the last inequality is due to Assumption (H) and $Y_t^2 = \mathcal{E}^{g^2}(Q|\mathcal{F}_t) \leq \mathcal{E}^{g^1}(Q|\mathcal{F}_t) = y$. It follows that $g^1(t, y, z) \geq g^2(t, y, z)$. ☐

Remark 4.2. In the general case, in order to compare the drivers, we need the inequalities of solutions to hold for all times. While in some special cases, for example, when g is independent of y or g is deterministic, just the inequalities of the solutions at time 0 is sufficient to compare the drivers. This will be proved in the following corollaries.

Corollary 4.3. *(Independent case) Let Assumptions (A), (B), (D) and (E) hold for g^i, $i = 1, 2$, g^1 be balanced and g^2 satisfy (H). Assume moreover that for all Q,*

$$\mathcal{E}^{g^1}(Q) \geq \mathcal{E}^{g^2}(Q).$$

Then for all t and all (y, z), we have, $\mathbb{P}$-a.s.,

$$g^1(\omega, t, y, z) \geq g^2(\omega, t, y, z).$$

Proof. First, we need to show that for all t and all Q, with the assumption of the theorem,

$$\mathcal{E}^{g^1}(Q|\mathcal{F}_t) \geq \mathcal{E}^{g^2}(Q|\mathcal{F}_t).$$

Then we can apply Theorem 4.1 to finish the proof. The proof is almost the same as the one given by Briand et al. [4], in Theorem 4.4, for BSDEs driven by Brownian motion. □

Remark 4.4. In the above corollary, if we assume that for all t and all Q,

$$\mathcal{E}^{g^1}(Q|\mathcal{F}_t) \geq \mathcal{E}^{g^2}(Q|\mathcal{F}_t)$$

Then we can omit the assumption that g^2 satisfies (H). The conclusion still holds.

Corollary 4.5. *(Deterministic case) Suppose g^i are defined on: $[0, \cdots, T] \times \mathbb{R}^K \times \mathbb{R}^{K \times N}$ into $\mathbb{R}^K$. Assume that g^i, $i = 1, 2$ satisfies Assumptions (A), (B) and (D). Assume moreover that g^2 satisfies (H) and that for all Q,*

$$\mathcal{E}^{g^1}(Q) \geq \mathcal{E}^{g^2}(Q).$$

Then for all t and all (y, z) we have, $\mathbb{P}$-a.s.,

$$g^1(\omega, t, y, z) \geq g^2(\omega, t, y, z).$$

Proof. The proof is similar with that of Theorem 4.1. Note only that when the g^i are deterministic, Y_t^i in (4.1) are constants so that

$$\mathcal{E}^{g^i}(Q|\mathcal{F}_t) = \mathcal{E}^{g^i}(Q).$$

□

We now consider two counterexamples to Theorem 4.1 when one of the Assumptions (D) and (H) fails.

Example 4.6. Set $T = 2$ and

$$\begin{aligned} g^1(0, y, z) &\equiv 0, \quad g^1(1, y, z) \equiv 4, \\ g^2(0, y, z) &\equiv 1, \quad g^2(1, y, z) \equiv 2. \end{aligned}$$

We can see that both g^1 and g^2 satisfy Assumptions (A), (B) and (H). Moreover, for any Q,

$$\begin{aligned} \mathcal{E}^{g^1}(Q|\mathcal{F}_1) &= \mathcal{E}^{g^2}(Q|\mathcal{F}_1) + 2 \geq \mathcal{E}^{g^2}(Q|\mathcal{F}_1), \\ \mathcal{E}^{g^1}(Q|\mathcal{F}_0) &= \mathcal{E}^{g^2}(Q|\mathcal{F}_0) + 1 \geq \mathcal{E}^{g^2}(Q|\mathcal{F}_0). \end{aligned}$$

However, $g^1(0, y, z) < g^2(0, y, z)$.

Example 4.7. Set $T = 1$ and $K = 1$. Define:

$$g^1(\omega, 0, y, z) = \begin{cases} 0 & \text{if } z = 0, \\ 2y & \text{if } z \neq 0. \end{cases}$$

$$g^2(\omega, 0, y, z) = \begin{cases} 0 & \text{if } z = 0, \\ 2y + 1 & \text{if } z \neq 0. \end{cases}$$

It can be checked that both g^1 and g^2 satisfy (A), (B) and (D), and for any Q,

$$\mathcal{E}^{g^1}(Q|\mathcal{F}_0) \geq \mathcal{E}^{g^2}(Q|\mathcal{F}_0).$$

However, $g^1(0, y, z) < g^2(0, y, z)$.

5. Risk Measures via g-expectations

The theory of dynamic risk measures is one of the important applications of BSDEs. Peng [13] and Rosazza Gianin [14] established results for BSDEs driven by Brownian motion. Cohen and Elliott [7] considered the theory in a continuous-time model driven by a finite-state Markov chain. The discrete-time finite-state case was also studied by Cohen and Elliott [6]. In this section, we adopt the model proposed by Cohen and Elliott [6] and focus on applications in static risk measures.

5.1. Static risk measures via g-expectations. In this subsection, we recall the definition and representation theorems of static risk measures, then concentrate on risk measures defined by g-expectations.

First, we denote the set of financial positions by $\mathcal{X} := L^\infty(\mathcal{F}_T)$. A static risk measure is a map $\rho : \mathcal{X} \to \mathbb{R}$. In this paper, we only consider risk measures ρ such that

$$\rho(Q) = \rho(Q') \quad \text{if } Q = Q' \quad \mathbb{P}\text{-a.s.}. \tag{5.1}$$

We follows Follmer and Schied [9] and give the following definitions.

Definition 5.1. A risk measure ρ is called a *convex risk measure* if it satisfies:

(1) (Monotonicity): If $Q \leq Q'$, then $\rho(Q) \geq \rho(Q')$,
(2) (Cash invariance): If $m \in \mathbb{R}$, then $\rho(Q + m) = \rho(Q) - m$,
(3) (Convexity): $\rho(\alpha Q + (1-\alpha)Q') \leq \alpha\rho(Q) + (1-\alpha)\rho(Q')$, $\forall 0 \leq \alpha \leq 1$.

Definition 5.2. A convex risk measure ρ is called a *coherent risk measure* if it satisfies

(4) (Positive Homogeneity): If $\lambda \geq 0$, then $\rho(\lambda Q) = \lambda\rho(Q)$.

Lemma 5.3. *(See Follmer and Schied* [9]*) Any risk measure ρ that satisfies monotonicity and cash invariance is Lipschitz continuous w.r.t. the essential supremum norm $\| \cdot \|$:*

$$| \rho(Q) - \rho(Q') | \leq \| Q - Q' \| .$$

Now, let g satisfy Assumptions (A) and (B). Set $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $K = 1$. Consider the risk measure defined by g-expectations, which is given by (2.4), as follows:

$$\rho^g(Q) := \mathcal{E}^g(-Q). \tag{5.2}$$

Proposition 5.4. *(i) If g is balanced and independent of y, then ρ^g is continuous, that is, if $Q_n \to Q$ $\mathbb{P}$-a.s. in $L^\infty(\mathcal{F}_T)$, then $\rho(Q_n) \to \rho(Q)$.*

(ii) If g is balanced, independent of y and convex, then ρ^g is a convex risk measure. Moreover, if g is also positive homogeneous, then ρ^g is a coherent risk measure.

Proof. (i): According to (iii) and (iv) in Proposition 3.1, if g is balanced and independent of y, then ρ^g must satisfy monotonicity and cash invariance. Applying Lemma 5.3,

$$| \rho(Q) - \rho(Q_n) | \leq \| Q - Q_n \| .$$

Thus we need only to show that if $Q_n \to Q$, then $\| Q - Q_n \| \to 0$. In fact, Let $\{A_i\}$ be a partition of $\mathcal{F}_T$. Then Q_n and Q have the representation

$$Q_n = \sum_i q_n^i I_{A_i}, \quad Q = \sum_i q^i I_{A_i}.$$

Thus $Q_n \to Q$ implies $q_n^i \to q^i$ for any i. It follows that $\| Q - Q_n \| \to 0$.

(ii) follows by combining Proposition 3.1, Definitions 5.1 and 5.2. □

We recall the representations of convex and coherent risk measures from Follmer and Schied [9]. We denote by $\mathcal{M}_{1,f} := \mathcal{M}_{1,f}(\Omega, \mathcal{F})$ the set of all finitely additive set functions $\mathbb{Q} : \mathcal{F} \to [0, 1]$ which are normalized with $\mathbb{Q}(\Omega) = 1$, and by $\mathcal{M}_1(\mathbb{P}) := \mathcal{M}_1(\Omega, \mathcal{F}, \mathbb{P})$ the set of all probability measures on $(\Omega, \mathcal{F})$ which are absolutely continuous with respect to $\mathbb{P}$. Then we define the *acceptance set* of ρ by

$$\mathcal{A}_\rho := \{Q \in \mathcal{X} | \rho(Q) \leq 0\},$$

and the *minimal penalty function* of ρ by

$$\alpha_{\min}(\mathbb{Q}) := \sup_{Q \in \mathcal{A}_\rho} \mathbb{E}_{\mathbb{Q}}[-Q] \quad \text{for } \mathbb{Q} \in \mathcal{M}_{1,f}.$$

Theorem 5.5. *(see Follmer and Schied* [9]*) Suppose ρ is a convex risk measure. Then the following conditions are equivalent*

(i) ρ can be represented as:

$$\rho(Q) = \sup_{\mathbb{Q} \in \mathcal{M}_1(\mathbb{P})} (\mathbb{E}_{\mathbb{Q}}[-Q] - \alpha_{\min}(\mathbb{Q})), \ Q \in L^\infty. \tag{5.3}$$

(ii) ρ is continuous from above: If $Q_n \searrow Q$ $\mathbb{P}$-a.s., then $\rho(Q_n) \nearrow \rho(Q)$.

Theorem 5.6. *(see Follmer and Schied* [9]*) For a coherent risk measure ρ the following conditions are equivalent:*

(i) There exists a set $\mathcal{Q} \in \mathcal{M}_1(\mathbb{P})$ such that the supremum is attained:

$$\rho(Q) = \max_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[-Q] \quad \forall Q \in \mathcal{X}. \tag{5.4}$$

In this case $\mathcal{Q}$ can be chosen as $\{\mathbb{Q} \in \mathcal{M}_1(\mathbb{P}) \mid \alpha_{\min}(\mathbb{Q}) = 0\}$.

(ii) ρ is continuous from below: If $Q_n \nearrow Q$ $\mathbb{P}$-a.s. then $\rho(Q_n) \searrow \rho(Q)$.

Then combining Proposition 5.4, Theorem 5.5 and 5.6, we can obtain the sufficient conditions for drivers that guarantee similar representations of the related risk measures.

Corollary 5.7. *(i) If g is balanced, independent of y and convex, then ρ^g has the representation (5.3).*

(ii) Moreover, if g is also positive homogeneous in (y, z), then ρ^g has the representation (5.4).

5.2. Connection between static risk measures and drivers. On the other hand, it is natural to ask under what conditions can a static risk measure be represented in terms of g-expectation. This question will be answered at the end of this section. The connection between $\mathcal{F}_t$-consistent nonlinear expectations and g-expectations together with the connection between static evaluation and $\mathcal{F}_t$-consistent nonlinear expectations were studied by Cohen and Elliott [6]. What we need is to combine these two.

Let g satisfy Assumption (A) and (B) and define the dynamic risk measure for all $Q \in L^\infty(\mathcal{F}_T)$ and all t,

$$\rho_t^g(Q) := \mathcal{E}^g(-Q|\mathcal{F}_t). \tag{5.5}$$

Applying Proposition 3.1, we can obtain properties of dynamic risk measures defined by conditional g-expectations.

Proposition 5.8. *Let $\{\rho_t^g\}_{t\in[0,t]}$ be the dynamic risk measure defined in (5.5). Then we can obtain the following properties:*

(i) (Terminal equality) $\rho_T^g(Q) = -Q$, for all Q.

(ii) ($\mathcal{F}_t$-triviality and Recursivity) If for any y, $g(\omega, t, y, 0) = 0$ $\mathbb{P}$-a.s., then $\rho_s^g(-\rho_t^g(Q)) = \rho_s^g(Q)$, $\mathbb{P}$-a.s. for any $s \leq t$, and $\rho_t^g(Q) = -Q$ $\mathbb{P}$-a.s. for any $\mathcal{F}_t$-measurable Q.

(iii) (Monotonicity) If g is a balanced driver, for any $Q \geq Q'$ $\mathbb{P}$-a.s. componentwise, then

$$\rho_t^g(Q) \leq \rho_t^g(Q'),$$

$\mathbb{P}$-a.s. componentwise, with equality only if $Q = Q'$.

(iv) (Translation invariance) If g is independent of y, then for all t, all $Q \in L^\infty(\mathcal{F}_T)$ and all $q \in L^\infty(\mathcal{F}_t)$,

$$\rho_t^g(Q + q) = \rho_t^g(Q) - q.$$

(v) (Regularity) If $g(t, 0, 0) = 0$ $\mathbb{P}$-a.s., then for any $A \in \mathcal{F}_t$, $I_A\rho_t^g(Q) = \rho_t^g(I_AQ)$ $\mathbb{P}$-a.s..

(vi) If g is positive homogeneous, then $\rho_t^g(\cdot)$ is positive homogeneous, that is, for all t, all $\lambda \geq 0$ and all $Q \in L^\infty(\mathcal{F}_T)$,

$$\rho^g(\lambda Q) = \lambda\rho^g(Q).$$

(vii) If g is balanced and convex, then $\rho_t^g(\cdot)$ is convex, that is, for all t, all $\alpha \in (0, 1)$, and all Q^1, $Q^2 \in L^\infty(\mathcal{F}_T)$,

$$\rho_t^g(\alpha Q^1 + (1-\alpha)Q^2) \leq \alpha\rho_t^g(Q^1) + (1-\alpha)\rho_t^g(Q^2).$$

Now we turn to the question: under what conditions can a dynamic risk measure be represented in term of a conditional g-expectation. It will be answered in Corollary 5.12. As in Cohen and Elliott [6], we follow Peng [13] and give the following definition.

Definition 5.9. A system of operators

$$\mathcal{E}(\cdot|\mathcal{F}_t) : L^1(\mathcal{F}_T) \to L^1(\mathcal{F}_t), \quad 0 \leq t \leq T$$

is called an $\mathcal{F}_t$- *consistent nonlinear expectation* if it satisfies the following properties:

(i) (Monotonicity) For any $Q \geq Q'$ $\mathbb{P}$-a.s.,

$$\mathcal{E}^g(Q|\mathcal{F}_t) \geq \mathcal{E}^g(Q'|\mathcal{F}_t),$$

$\mathbb{P}$-a.s. with equality only if $Q = Q'$ $\mathbb{P}$-a.s..

(ii) ($\mathcal{F}_t$-triviality) $\mathcal{E}^g(Q|\mathcal{F}_t) = Q$ $\mathbb{P}$-a.s. for any $\mathcal{F}_t$-measurable Q.

(iii) (Recursivity) $\mathcal{E}^g(\mathcal{E}^g(Q|\mathcal{F}_t)|\mathcal{F}_s) = \mathcal{E}^g(Q|\mathcal{F}_s)$ $\mathbb{P}$-a.s. for any $s \leq t$.

(iv) (Regularity) For any $A \in \mathcal{F}_t$, $I_A\mathcal{E}^g(Q|\mathcal{F}_t) = \mathcal{E}^g(I_AQ|\mathcal{F}_t)$ $\mathbb{P}$-a.s..

The following theorem relates an $\mathcal{F}_t$-consistent nonlinear expectation to conditional g-expectations.

Theorem 5.10. *(see Cohen and Elliott [6]) For some family of operators $\mathcal{E}(\cdot|\mathcal{F}_t) : L^1(\mathcal{F}_T) \to L^1(\mathcal{F}_t)$, the following conditions are equivalent:*

(i) $\mathcal{E}(\cdot|\mathcal{F}_t)$ is an $\mathcal{F}_t$-consistent nonlinear expectation satisfying translation invariance as (iv) in Proposition 3.1.

(ii) There exists a driver g, which is balanced, independent of y and satisfies $g(t, y, 0) = 0$, such that, for all Q, $Y_t = \mathcal{E}(Q|\mathcal{F}_t)$ is the solution to a BSDE with terminal condition Q and driver g.

Furthermore, these two statements are related by the equation

$$g(\omega, t, y, z) = \mathcal{E}(zM_{t+1}|\mathcal{F}_t).$$

Remark 5.11. Theorem 5.10 is also true in the vector case and for $\mathcal{E}(\cdot|\mathcal{F}_t)$ with restriction on $\mathcal{Q}_t \subset L^1(\mathcal{F}_T)$, as was proved by Cohen and Elliott [6].

Corollary 5.12. *In Theorem 5.10, if we replace $\mathcal{E}(\cdot|\mathcal{F}_t)$ by $\{\rho_t\}_{t\in[0,T]}$, condition (i) by*

(i') For any t, ρ_t satisfies monotonicity, $\mathcal{F}_t$-triviality, recursivity, regularity and translate invariance as in Proposition 5.8.

and "$Y_t = \mathcal{E}(Q|\mathcal{F}_t)$" in (ii) by "$Y_t = \rho_t(-Q)$", the theorem still holds.

The next theorem shows the connection between static evaluation and $\mathcal{F}_t$-consistent nonlinear expectation.

Theorem 5.13. *(see Cohen and Elliott [6]) Consider a measurable, scalar valued map $\mathcal{E} : L^1(\mathcal{F}_T) \to L^1(\mathcal{F}_0)$. Suppose this map satisfies:*

(i) ($\mathcal{F}_t$-consistency) For any $Q \in L^1(\mathcal{F}_T)$ and any $t \leq T$, there exists an $\mathcal{F}_t$-measurable random variable Y_t such that

$$\mathcal{E}(I_AQ) = \mathcal{E}(I_AY_t),$$

for any $A \in \mathcal{F}_t$.

(ii) ($\mathcal{F}_0$-triviality) $\mathcal{E}(Q) = Q$ for all $\mathcal{F}_0$-measurable Q.

(iii) (Monotonicity) For any $Q \geq Q'$ $\mathbb{P}$-a.s.,

$$\mathcal{E}(Q) \geq \mathcal{E}(Q'),$$

$\mathbb{P}$-a.s. with equality only if $Q = Q'$ $\mathbb{P}$-a.s..

Then there exists a unique $\mathcal{F}_t$-consistent nonlinear expectation $\mathcal{E}(\cdot|\mathcal{F}_t)$ such that

$$\mathcal{E}(Q) = \mathcal{E}(Q|\mathcal{F}_0),$$

for all $Q \in L^1(\mathcal{F}_T)$. This nonlinear expectation is given by

$$\mathcal{E}(Q|\mathcal{F}_t) = Y_t,$$

with Y_t as in (i). In fact, the assumptions are necessary.

The following corollary answers the question proposed at the beginning of this subsection.

Corollary 5.14. *For a static risk measure ρ, the following conditions are equivalent:*

(i) ρ satisfies:
- *(a) ($\mathcal{F}_t$-consistency) For any $Q \in L^1(\mathcal{F}_T)$ and any $t \leq T$, there exists an $\mathcal{F}_t$-measurable random variable Y_t such that*

$$\rho(I_A Q) = \rho(I_A Y_t),$$

  for any $A \in \mathcal{F}_t$.
- *(b) (Constancy) $\rho(Q) = -Q$ for all constant Q.*
- *(c) (Monotonicity) For any $Q \geq Q'$ $\mathbb{P}$-a.s.,*

$$\rho(Q) \leq \rho(Q'),$$

  $\mathbb{P}$-a.s. with equality only if $Q = Q'$ $\mathbb{P}$-a.s..
- *(d) (Translation invariance) For any $Q \in L^1(\mathcal{F}_T)$, any $t \leq T$ and the associated Y_t as in (i)(a) fixed (note that monotonicity ensures the uniqueness of Y_t, see the proof given by Cohen and Elliott* [6] *for Theorem 8), then for any $q \in \mathcal{F}_t$,*

$$\rho(I_A(Q + q)) = \rho(I_A(Y_t + q)),$$

  for any $A \in \mathcal{F}_t$.

(ii) There exists a driver g_ρ, which is balanced, independent of y and satisfies $g_\rho(t, y, 0) = 0$, such that, for all Q, $Y_0 = \mathcal{E}^{g_\rho}(-Q) = \rho(Q)$ is the solution to a BSDE with terminal condition Q and driver g_ρ.

Proof. The proof is just a combination of Theorems 5.10 and 5.13. □

6. Application to Optimal Design of Dynamic Risk Measures

Optimal design of dynamic risk measures has been widely studied. Barrieu and El Karoui [1] discussed a related problem in a continuous diffusion setting. Barrieu and El Karoui [2] adopted techniques in BSDEs driven by Brownian motion to characterize the optimal solution of the inf-convolution problem. The problem of default risk which is characterized by a single jump process was explored by Shen and Elliott [15]. In this section, we apply theory of discrete-time finite-state BSDEs to this specific problem.

Throughout this section, we set $K = 1$ and consider the set of financial positions $\mathcal{X}$ defined in Subsection 5.1. Firstly we recall the optimal design problem.

6.1. Static optimal design problem.

Definition 6.1. For any Q in $\mathcal{X}$, the *entropic risk measure* is defined as:

$$e^{\gamma}(Q) = \sup_{\mathbb{Q}\in\mathcal{M}_1(\mathbb{P})} \left(\mathbb{E}_{\mathbb{Q}}(-Q) - \gamma h(\mathbb{Q}|\mathbb{P})\right) = \gamma \ln \mathbb{E}_{\mathbb{P}}\left(\exp\left(-\frac{Q}{\gamma}\right)\right). \tag{6.1}$$

Here γ is the risk tolerance coefficient, $\mathcal{M}_1(\mathbb{P})$ is the set of probability measures on the considered space and $h(\mathbb{Q}|\mathbb{P})$ is the relative entropy of $\mathbb{Q}$ with respect to $\mathbb{P}$.

We consider a problem about an optimal transaction between two economic agents, denoted by A and B respectively. Agent A is exposed towards a non-hedgeable risk of a financial position Q. Thus agent A wants to issue a financial product S and sell it to agent B for a forward price at time T denoted by π to reduce his exposure.

Suppose both agent use entropic risk measures, with tolerance coefficients γ and γ', to assess the risk of their financial positions. Agent A wants to determine the structure (S, π) as to minimize his global risk measure

$$\inf_{S,\pi} e^{\gamma}(Q - S + \pi)$$

with the constraint

$$e^{\gamma'}(S - \pi) \leq e^{\gamma'}(0) = 0.$$

Using the cash translation invariance property and binding the constraint at the optimum, the pricing rule of the S-structure is fully determined by the buyer as

$$\pi^*(S) = -e^{\gamma'}(S).$$

Using the cash translation invariance property again, the optimization problem simply becomes

$$\inf_{S}\left(e^{\gamma}(Q - S) + e^{\gamma'}(S)\right).$$

6.2. Dynamic optimal design problem. We extend the notion of static entropic risk measures to a dynamic one defined on space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathbb{P})$

$$e_t^{\gamma}(Q) = \gamma \ln \mathbb{E}\left(\exp\left(-\frac{Q}{\gamma}\right)|\mathcal{F}_t\right).$$

Then we can obtain the following proposition:

Proposition 6.2. *$-e_t^{\gamma}(Q)$ is an $\mathcal{F}_t$-consistent, dynamically translation invariant nonlinear expectation.*

Proof. Monotonicity, $\mathcal{F}_t$-triviality, recursivity and dynamically translation invariance are obvious. We only show the proof for regularity. For any $A \in \mathcal{F}_t$,

$$\begin{aligned}
&-e_t^{\gamma}(I_A Q) = I_A(-e_t^{\gamma}(Q)) \\
&\Leftrightarrow \ln \mathbb{E}\left(\exp\left(-\frac{QI_A}{\gamma}\right)|\mathcal{F}_t\right) = I_A \ln \mathbb{E}\left(\exp\left(-\frac{Q}{\gamma}\right)|\mathcal{F}_t\right) \\
&\Leftrightarrow \mathbb{E}\left(\exp\left(-\frac{QI_A}{\gamma}\right)|\mathcal{F}_t\right) = \left[\mathbb{E}\left(\exp\left(-\frac{Q}{\gamma}\right)|\mathcal{F}_t\right)\right]^{I_A}
\end{aligned}$$

It is obvious that

$$\mathbb{E}\left(\exp\left(-\frac{QI_A}{\gamma}\right)|\mathcal{F}_t\right)I_A = \left[\mathbb{E}\left(\exp\left(-\frac{Q}{\gamma}\right)|\mathcal{F}_t\right)\right]^{I_A} I_A,$$

and that

$$\mathbb{E}\left(\exp\left(-\frac{QI_A}{\gamma}\right)|\mathcal{F}_t\right)I_{A^c} = \left[\mathbb{E}\left(\exp\left(-\frac{Q}{\gamma}\right)|\mathcal{F}_t\right)\right]^{I_A} I_{A^c}.$$

□

Combine the above proposition and Theorem 5.10, we can prove the following result:

Theorem 6.3. *$(-e_t^\gamma(Q-S), Z_t^*)$ is the solution of the following BSDE*

$$-e_t^\gamma(Q-S) + \sum_{t\le s<T} g(\omega, s, Z_s^*(\omega)) + \sum_{t\le s<T} Z_s^* M_{t+1}(\omega) = Q - S, \tag{6.2}$$

where

$$g(\omega,t,z) = e_t^\gamma(zM_{t+1}) = \gamma \ln \mathbb{E}\left(\exp\left(-\frac{zM_{t+1}}{\gamma}\right)|\mathcal{F}_t\right).$$

Remark 6.4. we can rewrite (6.2) as

$$e_t^\gamma(Q-S) - \sum_{t\le s<T} g^\gamma(\omega, s, Z_s^\gamma(\omega)) + \sum_{t\le s<T} Z_s^\gamma M_{s+1}(\omega) = -(Q-S), \tag{6.3}$$

where $Z_s^\gamma = -Z_s^*$ and

$$g^\gamma(\omega,t,z) = e_t^\gamma(-zM_{t+1}) = \gamma \ln \mathbb{E}\left(\exp\left(\frac{zM_{t+1}}{\gamma}\right)|\mathcal{F}_t\right).$$

Thus the dynamic risk measure $e_t^\gamma(Q)$ can be represented as the solution of the BSDE with driver g^γ and terminal condition $-Q$.

We now study for any $t \in [0,T]$ the inf-convolution of the dynamic entropic risk measures e_t^γ and $e_t^{\gamma'}$ and the inf-convolution of the corresponding g_t^γ and g_t^γ. We define

$$(e^\gamma \Box e^{\gamma'})_t(Q) := \inf_S (e_t^\gamma(Q-S) + e_t^{\gamma'}(S)), \tag{6.4}$$

$$g^\gamma \Box g^{\gamma'}(\omega,t,z) := \inf_{z'} (g^\gamma(\omega,t,z-z') + g^{\gamma'}(\omega,t,z')). \tag{6.5}$$

We now prove the following theorem:

Theorem 6.5. *Inf-convolutions* (6.4) *and* (6.5) *have properties as follows:*

$$g^\gamma \Box g^{\gamma'}(\omega,s,z) = g^{\gamma+\gamma'}(\omega,s,z), \tag{6.6}$$

$$\begin{aligned}(e^\gamma \Box e^{\gamma'})_t(Q) &= \sum_{t\le s<T} g^{\gamma+\gamma'}(\omega,s,Z_s(\omega)) - \sum_{t\le s<T} Z_u M_{s+1}(\omega) - Q \\ &= e_t^{\gamma+\gamma'}(Q).\end{aligned} \tag{6.7}$$

Proof. Consider the function

$$\begin{aligned} f(z') &:= g^{\gamma}(\omega,t,z-z') + g^{\gamma'}(\omega,t,z') \\ &= \gamma \ln \mathbb{E}\left(\exp\left(\frac{(z-z')M_{t+1}}{\gamma}\right)|\mathcal{F}_t\right) + \gamma' \ln \mathbb{E}\left(\exp\left(\frac{z'M_{t+1}}{\gamma'}\right)|\mathcal{F}_t\right). \end{aligned}$$

In order to show $f(z')$ is a convex function, we need only to show that

$$f^*(z') := \ln \mathbb{E}\left(\exp\left(\frac{z'M_{t+1}}{\gamma'}\right)|\mathcal{F}_t\right)$$

is convex. In fact, denote $L = \exp\left(\frac{z'M_{t+1}}{\gamma'}\right)$, for any $x = (x^1, x^2, \cdots, x^N)^*$,

$$\begin{aligned} & x^* H(f^*) x \\ &= \sum_{i,j} \frac{1}{\gamma'^2 \mathbb{E}^2\left(L|\mathcal{F}_t\right)} x^i [\mathbb{E}\left(L|\mathcal{F}_t\right) \mathbb{E}\left(M^i_{t+1} M^j_{t+1} L|\mathcal{F}_t\right) \\ &\quad - \mathbb{E}\left(M^i_{t+1} L|\mathcal{F}_t\right) \mathbb{E}\left(M^j_{t+1} L|\mathcal{F}_t\right)] x^j \\ &= \frac{1}{\gamma'^2 \mathbb{E}^2(L|\mathcal{F}_t)} \Big\{ \sum_i [\mathbb{E}(L|\mathcal{F}_t)\mathbb{E}((M^i_{t+1})^2 L|\mathcal{F}_t) - \mathbb{E}^2(M^i_{t+1} L|\mathcal{F}_t)](x^i)^2 \\ &\quad + 2\sum_{i<j} [\mathbb{E}(L|\mathcal{F}_t)\mathbb{E}(M^i_{t+1} M^j_{t+1} L|\mathcal{F}_t) - \mathbb{E}(M^i_{t+1} L|\mathcal{F}_t)\mathbb{E}(M^j_{t+1} L|\mathcal{F}_t)] x^i x^j \Big\} \\ &= \frac{1}{\gamma'^2 \mathbb{E}^2(L|\mathcal{F}_t)} [\mathbb{E}(L|\mathcal{F}_t)\mathbb{E}((x^* M_{t+1})^2 L|\mathcal{F}_t) - \mathbb{E}^2((x^* M_{t+1})L|\mathcal{F}_t))]. \end{aligned}$$

Applying Hölder's inequality gives $x^* H(f^*)x \geq 0$, which means $f(z')$ is a convex function. Therefore for each ω the minimum of (6.4) with respect to z' occurs when

$$\nabla f(z') = \frac{\mathbb{E}\left(M_{t+1}\exp\left(\frac{z'M_{t+1}}{\gamma'}\right)|\mathcal{F}_t\right)}{\mathbb{E}\left(\exp\left(\frac{z'M_{t+1}}{\gamma'}\right)|\mathcal{F}_t\right)} - \frac{\mathbb{E}\left(M_{t+1}\exp\left(\frac{(z-z')M_{t+1}}{\gamma}\right)|\mathcal{F}_t\right)}{\mathbb{E}\left(\exp\left(\frac{(z-z')M_{t+1}}{\gamma}\right)|\mathcal{F}_t\right)} = 0.$$

We denote by $z^{*\gamma'}$ the value at which the minimum is attained. It is obvious that $z^{*\gamma'}$ is unique up to equivalence $\sim_{M_{t+1}}$, and $z^{*\gamma'} = \frac{\gamma'}{\gamma+\gamma'}z$. Therefore

$$\begin{aligned} g^{\gamma}\Box g^{\gamma'}(\omega,s,z) &= \inf_{z'}(g^{\gamma}(\omega,t,z-z') + g^{\gamma'}(\omega,t,z')) \\ &= g^{\gamma}(\omega,t,z-z^{*\gamma'}) + g^{\gamma'}(\omega,t,z^{*\gamma'}) \\ &= (\gamma+\gamma')\ln \mathbb{E}\left(\exp\left(\frac{zM_{t+1}}{\gamma+\gamma'}\right)|\mathcal{F}_t\right) \\ &= g^{\gamma+\gamma'}(\omega,s,z). \end{aligned}$$

Similar with (6.3), we have

$$e_t^{\gamma'}(S) - \sum_{t\leq s<T} g^{\gamma'}(\omega,s,Z_s^{\gamma'}(\omega)) + \sum_{t\leq s<T} Z_s^{\gamma'} M_{s+1}(\omega) = -S. \tag{6.8}$$

Adding (6.3) and (6.8), we have

$$\begin{aligned}
&e_t^{\gamma}(Q-S)+e_t^{\gamma'}(S)\\
&=\sum_{t\le s<T}[g^{\gamma}(\omega,s,Z_s^{\gamma}(\omega))+g^{\gamma'}(\omega,s,Z_s^{\gamma'}(\omega))]-\sum_{t\le s<T}Z_uM_{s+1}(\omega)-Q\\
&=\sum_{t\le s<T}[g^{\gamma}(\omega,s,Z_s-Z_s^{\gamma'}(\omega))+g^{\gamma'}(\omega,s,Z_s^{\gamma'}(\omega))]-\sum_{t\le s<T}Z_uM_{s+1}(\omega)-Q\\
&\ge\sum_{t\le s<T}g^{\gamma+\gamma'}(\omega,s,Z_s)-\sum_{t\le s<T}Z_uM_{s+1}(\omega)-Q,
\end{aligned}$$

where $Z_s=Z_s^{\gamma}+Z_s^{\gamma'}$. Thus $e_t^{\gamma}(Q-S)+e_t^{\gamma'}(S)$ can be regarded as the solution to the BSDE with terminal condition $-Q$ and the driver

$$g(\omega,s,z)=g^{\gamma}(\omega,s,z-Z_s^{\gamma'}(\omega))+g^{\gamma'}(\omega,s,Z_s^{\gamma'}(\omega)).$$

Similar with (6.3), $e_t^{\gamma+\gamma'}(Q)$ is the solution to the BSDE with terminal condition $-Q$ and the driver $g^{\gamma+\gamma'}$. $g\ge g^{\gamma+\gamma'}$ implies that for any S,

$$e_t^{\gamma}(Q-S)+e_t^{\gamma'}(S)\ge e_t^{\gamma+\gamma'}(Q). \tag{6.9}$$

Taking $S^*=\frac{\gamma'}{\gamma+\gamma'}Q$, we can show that $\gamma Z_s^{*\gamma'}M_{s+1}=\gamma' Z_s^{*\gamma}M_{s+1}$. Then

$$Z_s^{*\gamma'}M_{s+1}=\frac{\gamma'}{\gamma+\gamma'}(Z_s^{*\gamma'}+Z_s^{*\gamma})M_{s+1}=\frac{\gamma'}{\gamma+\gamma'}Z_sM_{s+1}.$$

Consequently,

$$e_t^{\gamma}(Q-S^*)+e_t^{\gamma'}(S^*)=\sum_{t\le s<T}g^{\gamma+\gamma'}(\omega,s,Z_s(\omega))-\sum_{t\le s<T}Z_uM_{s+1}(\omega)-Q.$$

This means $e_t^{\gamma}(Q-S^*)+e_t^{\gamma'}(S^*)$ is the solution to the BSDE with the driver function $g^{\gamma+\gamma'}$ and the terminal condition $-Q$. Then it also equals to $e_t^{\gamma+\gamma'}(Q)$ due to the uniqueness of the solution. According to (6.9), we have

$$e_t^{\gamma+\gamma'}(Q)=\inf_S(e_t^{\gamma}(Q-S)+e_t^{\gamma'}(S))=(e^{\gamma}\Box e^{\gamma'})_t(Q).$$

□

Remark 6.6. Note that S^* defined in the above theorem is optimal. And for any constant c, S^*+c is also optimal.

7. Conclusion

In this paper, we have systematically studied properties of nonlinear expectations defined using the discrete-time finite-state BSDEs. A converse comparison theorem has also been established. Properties, especially the representation theorems, for risk measure defined on discrete-time finite-state space have been explored. We also have obtained the solution for optimal design of dynamic risk measures by BSDE approach. The optimal solution for the value of insurance is proportional to the non-hedgeable contingent claim. The ratio is the Agent B's risk tolerance to the total risk tolerance.

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Robert Elliott: School of Mathematical Sciences, North Terrace Campus, University of Adelaide, SA, Australia; Haskayne School of Business, University of Calgary, Calgary, Alberta, Canada
E-mail address: relliott@ucalgary.ca

Yin Lin: Department of Statistics and Actuarial Science, The University of Hong Kong, Pokfulam Road, Hong Kong
E-mail address: linyin@connect.hku.hk

Hailiang Yang: Department of Statistics and Actuarial Science, The University of Hong Kong, Pokfulam Road, Hong Kong
E-mail address: hlyang@hku.hk