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ASYMPTOTIC SPECTRAL DISTRIBUTIONS OF DISTANCE-k GRAPHS OF HAMMING GRAPHS

YUJI HIBINO

ABSTRACT. We derive the asymptotic spectral distributions of the distance-k graphs of Hamming graphs $H(N,q)$ as $N \to \infty$. Moreover, we also derive the ones of the distance-k graphs of Hamming graphs $H(N,q)$ as $N, q \to \infty$ in such a way that $N/q \to \lambda > 0$.

1. Introduction

The study of the spectrum of a graph has a long history. Recently, the asymptotic spectral distribution of a growing graph has been discussed in the framework of quantum probability theory (see [5] and references cited therein). In this content, asymptotic spectral analysis for the adjacency matrix of a growing graph is connected with the central limit theorem. We especially have an interest in the case where the spectral structure is dominated by the growing sturucture, namely, the limit distribution does not depend on the detailed structure of G, as the limit distribution of the sum of independent and identically distributed randam variables is the Gaussian distribution independently of the distribution of the random variables. For example, it is well-known that, even if it starts from any graph G, the limit distribution of a growing graph by the power of the Cartesian product, the comb product, the star product is the Gaussian distribution, the arcsine law, the Bernoulli distribution, respectively.

For a given graph $G = (V, E)$ having an adjacency matrix A, we consider the graph $G_N^{(k)} = (V^N, E_N^{(k)})$ defined by $E_N^{(k)} = \{(x, y); x = (x_1, x_2, \ldots, x_N), y = (y_1, y_2, \ldots, y_N) \in V^N$, for some $1 \le j_1 < \cdots < j_k \le N, (x_j, y_j) \in E$ if $j \in \{j_1, \ldots, j_k\}, x_j = y_j$ if $j \notin \{j_1, \ldots, j_k\}\}$. Then the adjacency matrix $A_N^{(k)}$ of $G_N^{(k)}$ is

$$A_N^{(k)} = \sum_{1 \le j_1 < \cdots < j_k \le N} I \otimes \cdots \otimes I \otimes A \otimes I \otimes \cdots \otimes I \otimes A \otimes I \otimes \cdots \otimes I,$$

where A appears k times and sits at j_1-th, ..., j_k-th positions, and where I denotes an identity matrix. The study of the asymptotic spectral distribution of $G_N^{(k)}$ first appeared in [6] for $G = K_2$ (the complete graph with two vertices) and $k = 2$. In [7], it was studied in the case for $G = K_2$ and arbitrary $k \in \mathbb{N}$. In the present

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paper, we shall study the spectral distribution of $G_N^{(k)}$ for $G = K_q$ and arbitrary $k \in \mathbb{N}$. It goes without saying that the results are the generalization of [7].

We note that $G_N^{(1)}$ is the Cartesian product of G. It is well-known that the asymptotic spectral distribution of $G_N^{(1)}$ is the standard Gaussian distribution because of the quantum central limit theorem for the commutative independence. It is worth noting that the limit distribution is determined independently of the details of a factor G.

The Cartesian product of N complete graphs K_q is called a Hamming graph $H(N, q)$ (e.g. [2]). Hence $G_N^{(1)}$ for $G = K_q$ is nothing but the Hamming graph $H(N, q)$. In this case, $G_N^{(k)}$ for $G = K_q$ is regarded as the distance-k graph of the Hamming graph $H(N, q)$.

This paper is organized as follows: In Section 2, we determine the spectral distributions of the distance-k graphs of the Hamming graph $H(N, q)$ in terms of the Krawtchouk polynomials.

In Section 3, we determine the asymptotic spectral distributions of the distance-k graphs of the Hamming graph $H(N, q)$ as N tends to infinity. As the result, the limit distributions are independent of q. We can conjecture that the asymptotic spectral distribution of $G_N^{(k)}$ does not depend on the structure of G just like the central limit theorem. It will be solved in the forthcoming paper [3].

In Section 4, when both N and q tend to infinity in such a way that N/q tends to a positive constant, we determine the asymptotic spectral distributions of the distance-k graphs of the Hamming graph $H(N, q)$ by the use of the law of small numbers of the Poisson distribution. This result can be regarded as a generalization of [4].

2. Spectral Distributions of Distance-k Graphs of Hamming Graphs

Let $G = (V, E)$ be a simple graph, where V is a set of vertices and E is the set of edges. Here the simple graph means an unweighted and undirected graph containing no loops nor multiple edges.

For an adjacency matrix A of a graph G, let $\mathcal{A}(G)$ be the adjacency algebra, which is the $*$-algebra generated by A. Define the normalized trace by

$$\varphi(a) = \frac{1}{|V|}\mathrm{Tr}(a), \quad a \in \mathcal{A}(G).$$

Then φ becomes a state on $\mathcal{A}(G)$. The adjacency matrix A is regarded as a real random variable of the algebraic probability space $(\mathcal{A}(G), \varphi)$.

We have an interest in the spectral distribution $\mu(dx)$ such that

$$\varphi(A^m) = \int_{\mathbb{R}} x^m \mu(dx), \text{ for } m = 0, 1, 2, \ldots. \tag{2.1}$$

For the normalized trace state, μ coincides with the eigenvalue distribution of the graph G.

Remark 2.1. The vacuum state φ on $\mathcal{A}(G)$ is defined by

$$\varphi(a) = \langle \delta_o, a\delta_o \rangle, \quad a \in \mathcal{A}(G),$$

where o is an origin of G. In the case where G is the complete graph K_q, because of high symmetry, the vacuum state is determined independently of the choice of the origin. Thus the spectral distribution of K_q coincides with the eigenvalue distribution of it even if we employ the vacuum state in place of the trace state.

According to [6], for $N, k \in \mathbb{N}$, $N \geq k$, we define $G_N^{(k)} = (V^N, E_N^{(k)})$ by $E_N^{(k)} = \{(x, y); x = (x_1, x_2, \ldots, x_N), y = (y_1, y_2, \ldots, y_N) \in V^N,$ for some $1 \leq j_1 < \cdots < j_k \leq N, (x_j, y_j) \in E$ if $j \in \{j_1, \ldots, j_k\}, x_j = y_j$ if $j \notin \{j_1, \ldots, j_k\}\}$. Let $A_N^{(k)}$ be the adjacency matrix of $G_N^{(k)}$. Then

$$A_N^{(k)} = \sum_{1 \leq j_1 < \cdots < j_k \leq N} \underbrace{I \otimes \cdots \otimes I \otimes A \otimes I \otimes \cdots \otimes I \otimes A \otimes I \otimes \cdots \otimes I}_{N \text{ times}}, \tag{2.2}$$

where A appears k times and sits at j_1-th, ..., j_k-th positions, and where I denotes an identity matrix of size $|V|$.

If we put $\varphi_N = \varphi \otimes \cdots \otimes \varphi$ (N times), $A_N^{(k)}$ is considered as a real random variable of the algebraic probability space $(\mathcal{A}(G_N^{(k)}), \varphi_N)$. We also have an interest in the asymptotic spectral distribution $\mu_k(dx)$ such that

$$\lim_{N \to \infty} \varphi_N \left((\widetilde{A}_N^{(k)})^m \right) = \int_{\mathbb{R}} x^m \mu_k(dx), \text{ for } m = 0, 1, 2, \ldots,$$

where $\widetilde{A}_N^{(k)}$ is the normalization of the adjacency matrix $A_N^{(k)}$ to have mean 0 and variance 1.

Remark 2.2. For $k \in \mathbb{N}$, a distance-k graph $G^{[k]} = (V, E^{[k]})$ is defined by

$$E^{[k]} = \{(x, y); x, y \in V, \partial_G(x, y) = k\},$$

where $\partial_G(x, y)$ is the graph distance on G. The distance-k graph $G^{[k]}$ is not necessarily connected even if G is connected. In the case where $G = K_q$, $G_N^{(1)}$ is a Hamming graph $H(N, q)$ and $G_N^{(k)}$ is the distance-k graph of the Hamming graph. However, in general $G_N^{(k)}$ is not necessarily the distance-k graph of $G_N^{(1)}$.

We hereafter suppose that G is the complete graph K_q. Then its adjacency matrix $A = (1 - \delta_{ij})_{1 \leq i,j \leq q}$. It is easy to see that the mean $\varphi_N(A_N^{(k)}) = 0$ and the variance $\varphi_N((A_N^{(k)})^2) = \binom{N}{k}(q-1)^k$.

Lemma 2.3. *For $k \geq 2$,*

$$A_N^{(1)} A_N^{(k)} = (k+1) A_N^{(k+1)} + (q-2)k A_N^{(k)} + (q-1)(N-k+1) A_N^{(k-1)}. \tag{2.3}$$

Proof. For the sake of convenience, we introduce $F_N^{(k)}(X)$ by

$$\sum_{\substack{1 \leq j_1 < \cdots < j_{k-1} \leq N \\ j \notin \{j_1, \ldots, j_{k-1}\}}} \underbrace{I \otimes \cdots \otimes I \otimes A \otimes I \otimes \cdots \otimes I \otimes X \otimes I \otimes \cdots \otimes I \otimes A \otimes I \otimes \cdots \otimes I}_{N \text{ times}},$$

where A appears $(k-1)$ times and sits at j_1-th, ..., j_{k-1}-th positions, and where X appears once and sits at the j-th position.

Using this notation, we have

$$\begin{aligned} A_N^{(1)}A_N^{(k)} &= (k+1)A_N^{(k+1)} + F_N^{(k)}(A^2) \\ &= (k+1)A_N^{(k+1)} + (q-2)F_N^{(k)}(A) + (q-1)F_N^{(k)}(I) \\ &= (k+1)A_N^{(k+1)} + (q-2)kA_N^{(k)} + (q-1)(N-k+1)A_N^{(k-1)} \end{aligned}$$

where we used that $A^2 = (q-2)A + (q-1)I$ since A is the adjacency matrix of K_q. □

Observe that if we use the convention $A_N^{(0)} = I \otimes \cdots \otimes I$, then equation (2.3) holds for $k \in \mathbb{N}$.

Here we introduce Krawtchouk polynomials $\{k_n^{(N,p)}(x)\}$ with the parameters $N \in \mathbb{N}$ and $0 < p < 1$. They are the orthogonal polynomials associated with the binomial distribution $\mathrm{Bin}(N,p)$ and satisfy the three-term recurrence relation

$$\{x-(pN+n-2pn)\}k_n^{(N,p)}(x) = (n+1)k_{n+1}^{(N,p)}(x)+p(1-p)(N-n+1)k_{n-1}^{(N,p)}(x), \quad (2.4)$$

$k_0^{(N,p)}(x) = 1$, $k_1^{(N,p)}(x) = x - Np$. For an additional information, refer to e.g. [1].

Put $K_n^{(N,q)}(x) = q^n k_n^{(N,1/q)}((x+N)/q)$. Then $\{K_n^{(N,q)}(x)\}$ are the orthogonal polynomials associated with the probability measure

$$\beta(dx) = \sum_{j=0}^{N} \binom{N}{j} \left(\frac{1}{q}\right)^j \left(1-\frac{1}{q}\right)^{N-j} \delta_{jq-N}(dx).$$

Lemma 2.4. $A_N^{(k)} = K_k^{(N,q)}(A_N^{(1)})$.

Proof. By changing variables from (2.4), $\{K_n^{(N,q)}(x)\}$ satisfy the three-term recurrence relation

$$\{x-(q-2)n\}K_n^{(N,q)}(x) = (n+1)K_{n+1}^{(N,q)}(x) + (q-1)(N-n+1)K_{n-1}^{(N,q)}(x), \quad (2.5)$$

$K_0^{(N,q)}(x) = 1$, $K_1^{(N,q)}(x) = x$. The result follows immediately by comparing (2.5) with (2.3). □

Lemma 2.5. *The spectral distribution of* $A_N^{(1)}$ *is* $\beta(dx)$, *namely,*

$$\varphi_N\left((A_N^{(1)})^m\right) = \int_{\mathbb{R}} x^m \beta(dx), \quad (2.6)$$

for $m = 0, 1, 2, \ldots$.

Proof. It is known that the spectrum of the Hamming graph $H(N,q)$, namely the set of eigenvalues of $A_N^{(1)}$, is $jq - N$ with the multiplicity $\binom{N}{j}(q-1)^{N-j}$, $j = 0, 1, 2, \ldots, N$. Thus the spectral distribution of $A_N^{(1)}$ is $\beta(dx)$. □

Theorem 2.6.

$$\varphi_N\left((A_N^{(k)})^m\right) = \int_{\mathbb{R}} \left(K_k^{(N,q)}(x)\right)^m \beta(dx), \quad (2.7)$$

for $m = 0, 1, 2, \ldots$.

Proof. The proof is obvious from Lemma 2.4 and Lemma 2.5. □

3. Asymptotic Spectral Distributions of $H(N,q)$ as $N\to\infty$

Next we would like to determine the asymptotic spectral distribution $\mu_k(dx)$ such that

$$\lim_{N\to\infty}\varphi_N\left(\left(\frac{A_N^{(k)}}{\sqrt{\binom{N}{k}(q-1)^k}}\right)^m\right)=\int_{\mathbb{R}}x^m\mu_k(dx),$$

for $m=0,1,2,\ldots$.

For the sake of simplicity, we put

$$\widetilde{A}_N^{(k)}=\frac{A_N^{(k)}}{\sqrt{\binom{N}{k}(q-1)^k}}.$$

Since $A_N^{(1)}$ is mean 0 and variance $(q-1)N$, for $m=0,1,2,\ldots,$

$$\varphi_N\left((\widetilde{A}_N^{(1)})^m\right)=\int_{\mathbb{R}}\left(\frac{1}{\sqrt{(q-1)N}}\right)^m x^m\beta(dx)=\int_{\mathbb{R}}x^m\widetilde{\beta}(dx), \tag{3.1}$$

where

$$\widetilde{\beta}(dx)=\sum_{j=0}^{N}\binom{N}{j}\left(\frac{1}{q}\right)^j\left(1-\frac{1}{q}\right)^{N-j}\delta_{(jq-N)/\sqrt{(q-1)N}}(dx).$$

Put

$$\widetilde{K}_n^{(N,q)}(x)=\left\{\binom{N}{n}(q-1)^n\right\}^{-1/2}K_n^{(N,q)}(\sqrt{(q-1)N}x).$$

Then $\{\widetilde{K}_n^{(N,q)}(x)\}$ are the orthonormal polynomials associated with $\widetilde{\beta}(dx)$ and satisfy the three-term recurrence relation

$$\begin{aligned}&\left(x-\frac{(q-2)n}{\sqrt{(q-1)N}}\right)\widetilde{K}_n^{(N,q)}(x) \\ &=\sqrt{\frac{(N-n)(n+1)}{N}}\widetilde{K}_{n+1}^{(N,q)}(x)+\sqrt{\frac{n(N-n+1)}{N}}\widetilde{K}_{n-1}^{(N,q)}(x),\end{aligned} \tag{3.2}$$

$\widetilde{K}_0^{(N,q)}(x)=1,\ \widetilde{K}_1^{(N,q)}(x)=x.$

Proposition 3.1.

$$\varphi_N\left((\widetilde{A}_N^{(k)})^m\right)=\int_{\mathbb{R}}\left(\widetilde{K}_k^{(N,q)}(x)\right)^m\widetilde{\beta}(dx), \tag{3.3}$$

for $m=0,1,2,\ldots$.

Proof. Normalizing (2.3), we have

$$\widetilde{A}_N^{(1)}\widetilde{A}_N^{(k)}=\sqrt{\frac{(k+1)(N-k)}{N}}\widetilde{A}_N^{(k+1)}+\frac{(q-2)k}{\sqrt{(q-1)N}}A_N^{(k)}+\sqrt{\frac{(N-k+1)k}{N}}\widetilde{A}_N^{(k-1)}. \tag{3.4}$$

Compare this with (3.2), and we obtain $\widetilde{A}_N^{(k)}=\widetilde{K}_k^{(N,q)}(\widetilde{A}_N^{(1)})$. By virtue of (3.1), we complete the proof. □

Now we are in a position to take the limit $N\to\infty$.

Lemma 3.2. *As $N \to \infty$, the asymptotic spectral distribution of $A_N^{(1)}$ is the standard Gaussian distribution, namely,*

$$\lim_{N\to\infty} \varphi_N \left((\widetilde{A}_N^{(1)})^m \right) = \int_{\mathbb{R}} x^m \mathcal{N}(dx), \text{ for } m = 0, 1, 2, \dots,$$

where $\mathcal{N}(dx) = (2\pi)^{-1/2} e^{-x^2/2} dx$.

Proof. Due to the de Moivre-Laplace theorem, the weak limit of the normalized binomial distribution $\widetilde{\beta}(dx)$ is the standard Gaussian distribution $\mathcal{N}(dx)$. Taking the limit $N \to \infty$ of (3.1), we can obtain the result. □

Let $\{H_n(x)\}$ be Hermite polynomials. They are the orthogonal polynomials associated with the distribution $e^{-x^2} dx$ and satisfy the three-term recurrence relation

$$2xH_n(x) = H_{n+1}(x) + 2nH_{n-1}(x), \tag{3.5}$$

$H_0(x) = 1$, $H_1(x) = 2x$.

Put $\widetilde{H}_n(x) = (2^n n!)^{-1/2} H_n(x/\sqrt{2})$. Then $\{\widetilde{H}_n(x)\}$ are the orthonormal polynomials associated with the standard Gaussian distribution $\mathcal{N}(dx)$, and satisfy the three-term recurrence relation

$$x\widetilde{H}_n(x) = \sqrt{n+1}\widetilde{H}_{n+1}(x) + \sqrt{n}\widetilde{H}_{n-1}(x), \tag{3.6}$$

$\widetilde{H}_0(x) = 1$, $\widetilde{H}_1(x) = x$.

Lemma 3.3.

$$\lim_{N\to\infty} \widetilde{K}_n^{(N,q)}(x) = \widetilde{H}_n(x).$$

Proof. By putting $\widetilde{P}_n(x) = \lim_{N\to\infty} \widetilde{K}_n^{(N,q)}(x)$, we have

$$x\widetilde{P}_n(x) = \sqrt{n+1}\widetilde{P}_{n+1}(x) + \sqrt{n}\widetilde{P}_{n-1}(x),$$

because of (3.2). This is identical with the recurrence equation (3.6) of normalized Hermite polynomials $\{\widetilde{H}_n(x)\}$. Therefore, $\widetilde{P}_n(x) = \widetilde{H}_n(x)$ since $\widetilde{P}_0(x) = \widetilde{H}_0(x) = 1$ and $\widetilde{P}_1(x) = \widetilde{H}_1(x) = x$. □

Theorem 3.4. *As $N \to \infty$, the asymptotic spectral distribution of the adjacency matrix $A_N^{(k)}$ is as follows:*

$$\lim_{N\to\infty} \varphi_N \left(\left(\frac{A_N^{(k)}}{\sqrt{\binom{N}{k}(q-1)^k}} \right)^m \right) = \int_{\mathbb{R}} \left(\widetilde{H}_k(x) \right)^m \mathcal{N}(dx), \tag{3.7}$$

for $m = 0, 1, 2, \dots$.

Proof. Lemma 3.3 assures that the coefficients of $\widetilde{K}_k^{(N,q)}(x)$ converge to those of $\widetilde{H}_k(x)$. Since the convergence of Lemma 3.2 is weak,

$$\lim_{N\to\infty} \int_{\mathbb{R}} x^n \widetilde{\beta}(dx) = \int_{\mathbb{R}} x^n \mathcal{N}(dx),$$

for $n = 0, 1, 2, \dots$. From Proposition 3.1, we obtain

$$\lim_{N\to\infty} \varphi_N \left((\widetilde{A}_N^{(k)})^m \right) = \lim_{N\to\infty} \int_{\mathbb{R}} \left(\widetilde{K}_k^{(N,q)}(x) \right)^m \widetilde{\beta}(dx) = \int_{\mathbb{R}} \left(\widetilde{H}_k(x) \right)^m \mathcal{N}(dx),$$

for $m = 0, 1, 2, \ldots$, by noting that both $\left(\widetilde{K}_k^{(N,q)}(x)\right)^m$ and $\left(\widetilde{H}_k(x)\right)^m$ are polynomials of degree km. Thus the proof is completed. □

Remark 3.5. It is noteworthy that the right-hand side of (3.7) does not depend on q. Even if G is not K_q, Lemma 3.2 is a straightforward consequence from the quantum central limit theorem for the commutative independence since $A_N^{(1)}$ is the sum of the commutative independent random variables. This fact suggests us that we have the same limit distribution for $G_N^{(k)}$ even if it starts from any graph G. (see [3])

4. Asymptotic Spectral Distributions of $H(N, q)$ as $N, q \to \infty$

In this section, we consider the asymptotic spectral distributions of the distance-k graphs of Hamming graphs as both N and q tend to infinity. We need to assume a good balance between N and q so that N/q tends to $\lambda > 0$ (constant).

Lemma 4.1. *As $N, q \to \infty$ and $N/q \to \lambda > 0$, the asymptotic spectral distribution of $A_N^{(1)}$ is the normalized Poisson distribution, namely,*

$$\lim_{\substack{N,q\to\infty \\ N/q\to\lambda}} \varphi_N\left((\widetilde{A}_N^{(1)})^m\right) = \int_{\mathbb{R}} x^m \mathcal{P}(dx), \ \textit{for } m = 0, 1, 2, \ldots,$$

where

$$\mathcal{P}(dx) = \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} e^{-\lambda} \delta_{(j-\lambda)/\sqrt{\lambda}}(dx).$$

Proof. Due to the law of small numbers of the Poisson distribution, when N and q tend to infinity in such a way that N/q tends to a constant $\lambda > 0$, the weak limit of the binomial distribution $\mathrm{Bin}(N, 1/q)$ is the Poisson distribution $\mathrm{Pois}(\lambda)$. Thus, the weak limit of $\widetilde{\beta}(dx)$ is $\mathcal{P}(dx)$. □

Remark 4.2. This lemma means that the Hamming graph $H(N, q)$ asymptotically corresponds to the Poisson distribution when $N, q \to \infty$ with $N/q \to \lambda > 0$. This fact was already obtained in [4].

Here we introduce Charlier polynomials $\{C_n(x, \lambda)\}$ with the parameter $\lambda > 0$. They are the orthogonal polynomials associated with the Poisson distribution $\mathrm{Pois}(\lambda)$ and satisfy the three-term recurrence relation

$$(x - n - \lambda)C_n(x, \lambda) = -\lambda C_{n+1}(x, \lambda) - nC_{n-1}(x, \lambda), \tag{4.1}$$

$C_0(x, \lambda) = 1$, $C_1(x, \lambda) = 1 - x/\lambda$. For more details, refer to e.g. [1].

Put $\widetilde{C}_n^\lambda(x) = (-1)^n(\lambda^n/n!)^{1/2} C_n(\sqrt{\lambda}x + \lambda, \lambda)$. Then $\{\widetilde{C}_n^\lambda(x)\}$ are the orthonormal polynomials associated with $\mathcal{P}(dx)$, and satisfy the three-term recurrence relation

$$\left(x - \frac{n}{\sqrt{\lambda}}\right)\widetilde{C}_n^\lambda(x) = \sqrt{n+1}\widetilde{C}_{n+1}^\lambda(x) + \sqrt{n}\widetilde{C}_{n-1}^\lambda(x), \tag{4.2}$$

$\widetilde{C}_0^\lambda(x) = 1$, $\widetilde{C}_1^\lambda(x) = x$.

Lemma 4.3. $\lim_{\substack{N,q\to\infty\\ N/q\to\lambda}} \widetilde{K}_n^{(N,q)}(x) = \widetilde{C}_n^{\lambda}(x)$.

Proof. By putting $\widetilde{P}_n(x) = \lim_{N,q\to\infty; N/q\to\lambda} \widetilde{K}_n^{(N,q)}(x)$, we have

$$\left(x - \frac{n}{\sqrt{\lambda}}\right) \widetilde{P}_n(x) = \sqrt{n+1}\widetilde{P}_{n+1} + \sqrt{n}\widetilde{P}_{n-1},$$

because of (3.2). This is identical with the recurrence equation (4.2) of normalized Charlier polynomials $\{\widetilde{C}_n^{\lambda}(x)\}$. Therefore, $\widetilde{P}_n(x) = \widetilde{C}_n^{\lambda}(x)$ since $\widetilde{P}_0(x) = \widetilde{C}_0^{\lambda}(x) = 1$ and $\widetilde{P}_1(x) = \widetilde{C}_1^{\lambda}(x) = x$. □

Theorem 4.4. *As $N, q \to \infty$ and $N/q \to \lambda > 0$, the asymptotic spectral distribution of the adjacency matrix $A_N^{(k)}$ is as follows:*

$$\lim_{\substack{N,q\to\infty\\ N/q\to\lambda}} \varphi_N\left(\left(\frac{A_N^{(k)}}{\sqrt{\binom{N}{k}(q-1)^k}}\right)^m\right) = \int_{\mathbb{R}} \left(\widetilde{C}_k^{\lambda}(x)\right)^m \mathcal{P}(dx), \tag{4.3}$$

for $m = 0, 1, 2, \ldots$.

Proof. Since the convergence of Lemma 4.1 is weak,

$$\lim_{\substack{N,q\to\infty\\ N/q\to\lambda}} \int_{\mathbb{R}} x^n \widetilde{\beta}(dx) = \int_{\mathbb{R}} x^n \mathcal{P}(dx),$$

for $n = 0, 1, 2, \ldots$. From Proposition 3.1 and Lemma 4.3, we obtain

$$\lim_{\substack{N,q\to\infty\\ N/q\to\lambda}} \varphi_N\left((\widetilde{A}_N^{(k)})^m\right) = \lim_{\substack{N,q\to\infty\\ N/q\to\lambda}} \int_{\mathbb{R}} \left(\widetilde{K}_k^{(N,q)}(x)\right)^m \widetilde{\beta}(dx) = \int_{\mathbb{R}} \left(\widetilde{C}_k^{\lambda}(x)\right)^m \mathcal{P}(dx),$$

for $m = 0, 1, 2, \ldots$, by noting that both $\left(\widetilde{K}_k^{(N,q)}(x)\right)^m$ and $\left(\widetilde{C}_k^{\lambda}(x)\right)^m$ are polynomials of degree km. Thus the proof is completed. □

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Yuji HIBINO: Department of Mathematics, Graduate School of Science and Engineering, Saga University, Honjo 1, Saga, 8408502, JAPAN

E-mail address: hibinoy@cc.saga-u.ac.jp