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EXIT-TIME OF GRANULAR MEDIA EQUATION STARTING IN A LOCAL MINIMUM

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Abstract. We are interested in a nonlinear partial differential equation: the granular media one. Thanks to some of our previous results [10, 11], we know that under easily checked assumptions, there is a unique steady state. We point out that we consider a case in which the confining potential is not globally convex. According to recent articles [8, 9], we know that there is weak convergence towards this steady state. However, we do not know anything about the rate of convergence. In this paper, we make a first step to this direction by proving a deterministic Kramers'type law concerning the first time that the solution of the granular media equation leaves a local well. In other words, we show that the solution of the granular media equation is trapped around a local minimum during a time exponentially equivalent to $\exp\left\{\frac{2}{\sigma^2}H\right\}$, H being the so-called exit-cost.

1. Introduction

In this paper, we are interested in the following so-called granular media equation:

$$\frac{\partial}{\partial t}\mu_t^\sigma(x) = \frac{\partial}{\partial x}\left\{\frac{\sigma^2}{2}\frac{\partial}{\partial x}\mu_t^\sigma(x) + \mu_t^\sigma(x)\Big(V'(x) + F' * \mu_t^\sigma(x)\Big)\right\}, \qquad (1.1)$$

where the confining potential V is nonconvex (double-wells) and the interacting potential F is convex. The exact assumptions will be given subsequently.

This partial differential equation has a natural interpretation in terms of stochastic processes. Indeed, let us consider the following so-called McKean-Vlasov diffusion:

$$\begin{cases} X_t^\sigma = X_0 + \sigma B_t - \int_0^t \left(W_s^\sigma\right)'\left(X_s^\sigma\right)ds \\ W_s^\sigma = V + F * \mathcal{L}\left(X_s^\sigma\right) \end{cases}. \qquad (1.2)$$

Here, $*$ denotes the convolution. Since the law of the process intervenes in the drift, this equation is nonlinear - in the sense of McKean. By μ_t^σ, we denote the law at time t of the process X^σ. It is well-known that the family of probability measures $\{\mu_t^\sigma\,;\,t\geq 0\}$ satisfies the granular media equation starting from $\mathcal{L}(X_0)$.

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We will use the recent results about the exit-problem of the McKean-Vlasov diffusion (see [12, 13]) in order to prove a deterministic Kramers'type law for any σ sufficiently small:

$$\exp\left[\frac{2}{\sigma^2}\left(H_0-\delta\right)\right]<T_\kappa(\sigma)<\exp\left[\frac{2}{\sigma^2}\left(H_0+\delta\right)\right]\,,$$

H_0 being the associated exit-cost (which will be described later), δ being an arbitrarily small constant and

$$\begin{aligned}T_\kappa(\sigma):=&\inf\left\{t\geq0\ :\ \mathbb{E}\left[\left(X_t^\sigma-b\right)^2\right]>\kappa^2\right\}\\=&\inf\left\{t\geq0\ :\ \int_{\mathbb{R}}\left(x-b\right)^2\mu_t^\sigma(dx)>\kappa^2\right\}\,,\end{aligned}$$

where b is a minimizer of V corresponding to a local and non global minimum.

We now give the assumptions on V and F.

Assumption 1.1. *The potentials V and F satisfy the following hypotheses:*

- *The coefficients V' and F' are locally Lipschitz, that is, for each $R>0$ there exists $K_R>0$ such that*
$$|V'(x)-V'(y)|+|F'(x)-F'(y)|\leq K_R|x-y|\,,$$
for $x,y\in\{z\in\mathbb{R}\ :\ |z|<R\}$.
- *The function V is continuously differentiable.*
- *The potential V is convex at infinity: $\lim\limits_{|x|\to+\infty}V''(x)=+\infty$.*
- *The potential V has two wells ($a<0$ and $b>0$) and a local maximum located in 0.*
- *The function V'' is convex.*
- *$F(x):=\frac{\alpha}{2}x^2$ with $\alpha>0$.*

[Exit-time of granular media equation]Exit-time of granular media equation starting in a local minimum An example of such potential is $V(x):=\frac{x^4}{4}+\frac{x^3}{3}-\frac{x^2}{2}$. In this case $a=-\frac{1+\sqrt{5}}{2}<0<\frac{-1+\sqrt{5}}{2}=b$.

If the initial law is a Dirac measure, we know that there exists a unique strong solution X^σ to Equation (1.2), see [5, Theorem 2.13]. Moreover, we have: $\sup_{t\in\mathbb{R}_+}\mathbb{E}\left\{|X_t^\sigma|^{2p}\right\}<\infty$ for any $p\in\mathbb{N}^*$.

From now on, we consider the potential $W_b:=V+F*\delta_b$. Indeed, by classical large deviations result, for any $T>0$, in the small-noise limit, the diffusion $(X_t^\sigma)_{0\leq t\leq T}$ starting at $X_0=b$ is close to the diffusion $(Y_t^\sigma)_{0\leq t\leq T}$ defined like so:

$$Y_t^\sigma=b+\sigma W_t-\int_0^t W_b'\left(Y_s^\sigma\right)ds\,.$$

An important tool to understand the long-time behaviour of μ_t^σ is the set of invariant probabilities. This set has been precisely described in [6, 7, 10, 11]. From these works, we know that there exists an invariant probability near - in the small-noise limit - the distribution δ_b if and only if b is the unique global minimizer of W_b.

Assumption 1.2. *There exists $y \neq b$ suh that $W_b(y) < W_b(b)$.*

Immediately, from Assumption 1.2, we deduce that W_b has another minimizer than b, that is here denoted as a' and a unique local maximizer (since V'' is convex) denoted as c. From now on, we consider the following exit-cost:

$$H_0 := W_b(c) - W_b(b)\,. \tag{1.3}$$

The long-time behaviour of μ_t^σ has been solved in the convex case (see [1, 2, 3, 4]) and in the non-convex case (see [8, 9]).

An important and remaining question is the one of the rate of convergence. In [3], a rate of convergence has been obtained if V is convex but not uniformly strictly convex. Here, with double-wells potential, we can not use this result. It is an easy exercise to show that μ^σ stays a long time (that does depend on σ) close to δ_b in the small-noise limit. The result of the paper is a characterization of this time.

According to [11], with Assumption 1.1 and Assumption 1.2, there exists - if the noise σ is sufficiently small - a unique steady state for Equation (1.1). Consequently, if $\mu_0 = \delta_{x_0}$ where $x_0 \in]0;+\infty[$, we know that μ_t^σ converges weakly towards the unique invariant probability.

The aim of the current work is to study what happens if $x_0 := b$. For doing so, we use the recent results about the exit-time of the associated McKean-Vlasov diffusion in [12, 13].

From now on, we consider the *deterministic* time

$$T_\kappa(\sigma) := \inf\left\{t \geq 0\,:\,\int_{\mathbb{R}} (x-b)^2 \mu_t^\sigma(dx) \geq \kappa^2\right\}$$

for any $\kappa > 0$. In the following, κ is arbitrarily small. In particular, we assume that

$$\kappa^2 \leq \frac{1}{2}(c-b)^2\,.$$

We consider an additional assumption on the interaction:

Assumption 1.3. *We have $\alpha < \frac{V''(b)}{\sqrt{2}}$.*

This last assumption is used in order to be able to apply the results in [13].

We now give the result of the article.

Theorem 1.4. *For any $\kappa \in \left]0;\frac{1}{\sqrt{2}}|c-b|\right[$, for any $\delta > 0$, there exists $\sigma(\kappa,\delta)$ such that for all $0 < \sigma < \sigma(\kappa,\delta)$:*

$$\exp\left[\frac{2}{\sigma^2}\left(H_0 - \delta\right)\right] < T_\kappa(\sigma) < \exp\left[\frac{2}{\sigma^2}\left(H_0 + \delta\right)\right]\,. \tag{1.4}$$

2. Proof of Theorem 1.4

The lower-bound has already been proved in [13, Proposition C]. Indeed, in [13], the constant T_κ does correspond to the first time t such that $\mathbb{E}\left[(X_t-b)^2\right] < \kappa^2$, which here is 0 since $X_0 = b$.

Consequently, we have

$$\sup_{0\leq t\leq\exp\left[\frac{2}{\sigma^2}(H_0-\delta)\right]}\mathbb{E}\left[(X_t-b)^2\right]<\kappa^2\,,$$

if σ is sufficiently small. We deduce $T_\kappa(\sigma)>\exp\left[\frac{2}{\sigma^2}(H_0-\delta)\right]$ if σ is small enough.

We now prove the upper-bound by proceeding by a *reducto ad absurdum*. Set $\delta>0$. We assume that there exists a sequence $(\sigma_n)_n$ which goes to 0 as n goes to infinity such that, for any $n\in\mathbb{N}$, we have:

$$\exp\left[\frac{2}{\sigma_n^2}(H_0+\delta)\right]\leq T_\kappa(\sigma_n)\,, \tag{2.1}$$

We now introduce the two diffusions $X^{+,\kappa}$ and $X^{-,\kappa}$ by

$$X_t^{\pm,\kappa}=b+\sigma_nB_t-\int_0^t\nabla V\left(X_s^{\pm,\kappa}\right)ds-\alpha\int_0^t\left(X_s^{\pm,\kappa}-(b\pm\kappa)\right)ds \tag{2.2}$$

From now on, κ is arbitrarily small. By $b_\kappa^\pm$, we denote the positive critical point (close to b) of the potential $x\mapsto V(x)+\frac{\alpha}{2}\left(x-(b\pm\kappa)\right)^2$. By a simple computation, we get:

$$b_\kappa^\pm=b\pm\frac{\alpha}{V''(b)+\alpha}\kappa+o(\kappa)\,.$$

Now, if κ is small enough, we know that the Freidlin-Wentzell theory may be applied to Diffusion $X^{\pm,\kappa}$ and domain $]c;+\infty[$. So, we deduce that

$$\tau^\pm_{]c;+\infty[}(\sigma_n):=\inf\left\{t\geq0\,:\,X_t^{\pm,\kappa}\leq c\right\}$$

satisfies a Kramers'type law. In particular, we have

$$\lim_{\sigma\to0}\mathbb{P}\left(\exp\left[\frac{2}{\sigma_n^2}\left(H_\kappa^\pm(c)-\delta\right)\right]\leq\tau^\pm_{]c;+\infty[}(\sigma_n)\leq\exp\left[\frac{2}{\sigma_n^2}\left(H_\kappa^\pm(c)+\delta\right)\right]\right)=0\,,$$

for any $\delta>0$. Here, $H_\kappa^\pm(c):=V(c)-V\left(b_\kappa^\pm\right)+\frac{\alpha}{2}\left(c-b\pm\kappa\right)^2$.

The main idea now is to compare the exit-time of X with the ones of $X^{\pm,\kappa}$. We have

$$\sup_{0\leq t\leq\exp\left[\frac{2}{\sigma_n^2}(H_0-\delta)\right]}\mathbb{E}\left[|X_t-b|^2\right]<\kappa^2\,.$$

Consequently, for any $t\in\left[0;\exp\left[\frac{2}{\sigma_n^2}(H_0-\delta)\right]\right]$, we have $X_t^{-,\kappa}\leq X_t\leq X_t^{+,\kappa}$. As a consequence, if we put $\tau(\sigma_n):=\inf\{t\geq0\,:\,X_t\leq c\}$, we have

$$\tau_\kappa^-(\sigma_n)\leq\tau(\sigma_n)\leq\tau_\kappa^+(\sigma_n)\,.$$

However, a Kramers'type law is satisfied by $\tau_\kappa^\pm(\sigma_n)$. So, for any $\xi>0$, we have

$$\lim_{\sigma\to0}\mathbb{P}\left(\exp\left[\frac{2}{\sigma_n^2}\left(H_\kappa^-(c)-\xi\right)\right]\leq\tau(\sigma_n)\leq\exp\left[\frac{2}{\sigma_n^2}\left(H_\kappa^+(c)+\xi\right)\right]\right)=1\,.$$

Consequently, by taking κ sufficiently small, we obtain that for any $\delta>0$, we have

$$\lim_{\sigma\to0}\mathbb{P}\left(\exp\left[\frac{2}{\sigma_n^2}(H_0-\delta)\right]\leq\tau(\sigma_n)\leq\exp\left[\frac{2}{\sigma_n^2}(H_0+\delta)\right]\right)=1\,. \tag{2.3}$$

By $T_c(\sigma_n)$, we denote the first time that X^{σ_n} returns to $]c;+\infty[$. By proceeding similarly, we have the following inequality:

$$\lim_{n\to\infty}\mathbb{P}\left(T_c(\sigma_n)\leq\exp\left[\frac{2}{\sigma_n^2}(H_0+\frac{\delta}{2})\right]\right)=0\,. \tag{2.4}$$

Indeed, the exit-cost for going from the left to the right is $W_b(c)-W_b(a')>W_b(c)-W_b(b)$. We recall that a' is the global minimizer of W_b.

Inequalities (2.3) and (2.4) imply the following limit:

$$\lim_{n\to\infty}\mathbb{P}\left(X^{\sigma_n}_{\exp\left[\frac{2}{\sigma_n^2}\left(H_0+\frac{\delta}{2}\right)\right]}\geq c\right)=0\,.$$

In particular:

$$\lim_{n\to\infty}\mathbb{E}\left[\left|X^{\sigma_n}_{\exp\left[\frac{2}{\sigma_n^2}\left(H_0+\frac{\delta}{2}\right)\right]}-b\right|^2\right]\geq(c-b)^2\geq 2\kappa^2>\kappa^2\,.$$

Last limit means that $T_\kappa(\sigma_n)<\exp\left[\frac{2}{\sigma_n^2}(H_0+\frac{\delta}{2})\right]$ if n is large enough, which is absurd according to (2.1).

We deduce that Hypothesis (2.1) was wrong. Consequently, we obtain the upper-bound:

$$\exp\left[\frac{2}{\sigma^2}\left(H_0+\delta\right)\right]>T_\kappa(\sigma)\,,$$

if σ is small enough. This achieves the proof.

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