THE qq–BIT (II):
FUNCTIONAL CENTRAL LIMITS AND MONOTONE
REPRESENTATION OF THE AZEMA MARTINGALE

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ABSTRACT. We prove the functional central limit theorem for the Bernoulli process with left Jordan–Wigner–type q–embeddings. In this limit the associated classical random variables are re–scalings of the classical Azema random variable for each $t \in \mathbb{R}_+$. The corresponding process is found to be realized on a new q–deformation of the monotone Fock space which, in the case $q = 0$, is reduced to the usual monotone Fock space.

1. Introduction

Parametric deformations of $SU(2)$ were introduced by Kulish [26] in the context of quantum groups (see also [15]). This motivated a multiplicity of investigations in physics and in quantum probability looking at the deformation problem from different points of view.

Biedenharn and Mac Farlane [8] [20] introduced a q–deformation of the canonical commutation relations and of the harmonic oscillator as formal interpolation between Bose and Fermi quantization (see [16] for a more detailed description). At the same time a different type of interpolation between Bose and Fermi Fock spaces was introduced by Lindsay and Parthasarathy [17] and the first example of interpolation between the CCR and the CAR not constructed by hands, but naturally emerging for the solution of a concrete problem, was introduced by Parthasarathy in his deep paper on the quantum structure of the Azema martingale [21], [22].

All these lines of research concerning deformations of the standard quantum structures have been pursued in quantum probability. Lenczewsky and Podgorski [16] extended to the framework of quantum groups the proof that the Bose [1] [2] and the Fermi [19] oscillators can be obtained as central limits of quantum Bernoulli processes. They proved such a CLT connecting Sklyanin's $SU_q(2)$ and Biedenharn–Mac Farlane q–deformed harmonic oscillator. Central limit theorems (CLT) based on deformations of the Bose and Fermi commutation relations were proved in [7].

Bozeiko and Speicher [10], [11], constructed the Fock representation for the q–deformed canonical commutation relations (CCR) by proving the positivity (for

some values of q) of the sesqui–linear form uniquely determined by the Fock pre-scription and extended the construction to the second quantized level, which in probabilistic language corresponds to the transition from single random variable to stochastic processes. Speicher [27] realized the same construction for q–commuting creators and annihilators. Schürmann took the move from Parthasarathy's con-struction of the Boson representation of the Azema martingale to construct his theory of q–Levy processes on $*$–bi–algebras [23], [24], [25]. To achieve this goal he introduced the q–deformed version of the left and right Jordan–Wigner (JW) embeddings and proved the corresponding CLT in the context of $*$–bi–algebras.

In the paper [4] we gave a probabilistic and physical interpretation of the q–parameter based on the theory of orthogonal polynomials. Then, starting from Schürmann q–JW–embeddings, we proved a q–CLT of Bernoulli type in the frame-work of usual tensor product algebras rather than $*$–bi–algebras. This approach allowed to obtain an explicit determination of the limit space, not contained in Schürmann's papers, and to identify it, for $q = 0$, to the monotone Fock space introduced by Lu [18], and for $q \neq 0$, to a non–trivial deformation of it.

The description of this deformation was only outlined in [4] and is the object of the present paper, where the results of [4] are extended to the framework of functional CLT.

In particular we show (see Corollary 4.2 below) that Bozeiko's q–symmetrizator [9] emerges naturally from the CLT and that the positivity of our scalar product is guaranteed by the fact that it is obtained as limit of sesqui–linear forms which are evidently positive definite. This is true for **arbitrary** complex values of the deformation parameter.

From [4] we know that the limit process has the property that for any bounded interval $I \subset \mathbb{R}$, the vacuum distribution of the field operator localized in I has the same moments as the Azema martingale (up to a re–scaling depending on the Lebesgue measure of I). These moments were first obtained by Parthasarathy [21] for values of the deformation parameters in $[-1, 1)$ and extended to arbitrary complex values in [4].

However, contrarily to Parthasarathy's boson representation, in the representa-tion obtained in our CLT the limit process is not a classical one because random variables localized in disjoint intervals in general do not commute. It would be interesting to verify if, in analogy with what happens for the Bozeiko–Speicher q–deformed process, there exist classical processes naturally associated to the pro-cess obtained here and to study their properties on the lines of the Bryc and Wang paper [12]. We conjecture that the answer to this question is positive, but for the moment the problem is open.

2. Notations

Recall from [4] the notations:

$$\sigma^{(+1)} := \left(\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right) \quad ; \quad \sigma^{(-1)} := \left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right);$$

and the definition of the left x–JW–embeddings: for any $N \in \mathbb{N}$, $x \in \mathbb{C}$, $k \in \mathbb{N}$,

$$\sigma_k^{(+1)}(x) := \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}_{k)} \otimes \sigma^{(+1)} \otimes \mathbf{1}_{(k} := \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}^{\otimes(k-1)} \otimes \sigma^{(+1)} \otimes \mathbf{1}^{\otimes(N-k)}$$

$$\sigma_k^{(-1)}(x) := \begin{pmatrix} 1 & 0 \\ 0 & \overline{x} \end{pmatrix}_{k)} \otimes \sigma^{(-1)} \otimes \mathbf{1}_{(k} = \left(\sigma_k^{(+1)}(x)\right)^*$$

and the choice of the states:

$$\Phi_{[Nt]} := \bigotimes_{k=1}^{[Nt]} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.1}$$

where $[\,\cdot\,]$ denotes integer part. Our main goal in this paper is to prove the functional version of the Bernoulli central limit theorem for the left q–Jordan–Wigner embedding, that can be stated as follows. Denoting

$$\mathcal{L}\left([0,t]\right) := \{\text{Riemannn–integrable, bounded, } \mathbb{C}\text{–valued functions on } [0,t]\}$$

and defining, for any $t > 0$, $N \in \mathbb{N}$ and $f \in \mathcal{L}\left([0,t]\right)$

$$S_N^{(+1)}(t,f) := \frac{1}{\sqrt{N}} \sum_{k=1}^{[Nt]} f\left(\frac{k}{N}\right) \sigma_k^{(+1)}(x) \tag{2.2}$$

$$S_N^{(-1)}(t,f) := \frac{1}{\sqrt{N}} \sum_{k=1}^{[Nt]} \overline{f}\left(\frac{k}{N}\right) \sigma_k^{(-1)}(x)$$

we study the limits, as $N \to \infty$, of the moments

$$\left\langle \Phi_{[Nt]}, S_N^{(\varepsilon(1))}(t,f_1) \cdots S_N^{(\varepsilon(m))}(t,f_m) \Phi_{[Nt]} \right\rangle \tag{2.3}$$

where: $m \in \mathbb{N}$, $\varepsilon \in \{-1,1\}^m$ and $\{f_k\}_{k=1}^m \subset \mathcal{L}\left([0,t]\right)$ and hereinafter we use the notation

$$X^{(\varepsilon)} := \begin{cases} X \,, & \text{if } \varepsilon = -1 \\ X^+ \,, & \text{if } \varepsilon = 1 \end{cases} \tag{2.4}$$

Notice that, due to the identity $\sigma^{(-1)}\binom{1}{0} = \binom{0}{0}$, the state (2.1) is of **Fock–type** with respect to the operator random variables $(S_N^{(+)}(t,f), S_N^{(-)}(t,f))$, with annihilator $S_N^{(-)}(t,f)$.

It is known (see [4] for details) that, up to terms of order $o(1)$, one can write (2.3) in the form

$$\frac{1}{N^{m/2}} \sum_{1 \le k_1, \cdots, k_m \le [Nt]} \left\langle \Phi_{[Nt]}, \sigma_{k_1}^{(\varepsilon(1))}(x) \cdots \sigma_{k_m}^{(\varepsilon(m))}(x) \Phi_{[Nt]} \right\rangle \prod_{h=1}^m g_h\left(\frac{k_h}{N}\right) \tag{2.5}$$

with

$$g_h := \chi_{[0,t]} \cdot \begin{cases} f_h, & \text{if } \varepsilon(h) = 1 \\ \overline{f}_h, & \text{if } \varepsilon(h) = -1 \end{cases}$$

where, here and in the following, the characteristic (indicator) function of any set I will be denoted

$$\chi_I(x) := \begin{cases} 1 \text{ , if} \\ 0 \text{ , if } x \notin I \end{cases} \tag{2.6}$$

The existence of the limit of (2.5) (hence of (2.3)) can be deduced from [4] with techniques, now rather standard, that allow to pass from a central limit theorem to its functional version.

Once the existence of the limit of (2.3) has been established, one knows from [4], [6] that there exists an interacting Fock space (IFS) (see [4], [6] for a description of this notion) $(\mathcal{H}, a^+, \Phi)$ over some sub–space of $L^2(\mathbb{R}_+)$ such that this limit has the form (in the notation (2.4)):

$$\left\langle \Phi, a^{(\varepsilon(1))}\left(\chi_{[0,t]}f_1\right)\cdots a^{(\varepsilon(m))}\left(\chi_{[0,t]}f_m\right)\Phi\right\rangle$$

Object of the present paper is to determine the explicit form of this IFS and of the associate creation and annihilation operators.

3. The Limit Moments

The product and Fock–type structure of the state (2.1) imply that expression (2.5) is equal to zero whenever either the following happens:
– m is odd;
– $m = 2n$ and

$$\varepsilon \in \{-1, 1\}_-^{2n} := \left\{\varepsilon \in \{-1, 1\}^{2n} : \exists k, \text{such that} \sum_{h=k}^{2n} \varepsilon(h) < 0\right\}$$

So we can restrict our attention to moments of the form

$$\left\langle \Phi_{[Nt]}, S_N^{(\varepsilon(1))}(t, f_1)\cdots S_N^{(\varepsilon(2n))}(t, f_{2n})\Phi_{[Nt]}\right\rangle \tag{3.1}$$

$$\varepsilon \in \{-1, 1\}_+^{2n} = \{-1, 1\}^{2n} \setminus \{-1, 1\}_-^{2n} \tag{3.2}$$

We will fix the following notations: for any $n, N \in \mathbb{N}$:

$$F(N, n) := \{\text{functions from } \{1, \cdots, 2n\} \to \{1, \cdots, N\}\}$$

$$PPF(N, n) := \left\{\mathbf{k} \in F(N, n) : \forall r \in \{1, \cdots, 2n\}, \left|\left\{\mathbf{k}^{-1}(\mathbf{k}(r))\right\}\right| = 2\right\}$$

Moreover, for any $\pi = \{(l_h, r_h)\}_{h=1}^n \in PP(2n)$, one denotes

$$PPF(N, n, \pi) := \{\mathbf{k} \in PPF(N, n) : \forall h \in \{1, \cdots, 2n\} \text{ , } \mathbf{k}(l_h) = \mathbf{k}(r_h)\}$$

Lemma 3.1. *For any $\varepsilon \in \{-1, 1\}_+^{2n}$, the expression (3.1) has the form*

$$o(1) + \frac{1}{N^n}\sum_{\pi=\{(l_h, r_h)\}_{h=1}^n \in PP(2n, \varepsilon)}\sum_{\mathbf{k}\in PPF([Nt], n, \pi)} \tag{3.3}$$

$$\left\langle \Phi_{[Nt]}, \sigma_{\mathbf{k}(1)}^{(\varepsilon(1))}\cdots\sigma_{\mathbf{k}(2n)}^{(\varepsilon(2n))}\Phi_{[Nt]}\right\rangle\prod_{h=1}^n\left(\overline{f_{l_h}}f_{r_h}\right)\left(\frac{\mathbf{k}(l_h)}{N}\right)$$

Proof. This is a consequence of von Waldenfelds Lemma. $\square$

Lemma 3.2. *For any $t > 0$, $n \in \mathbb{N}$, $\varepsilon \in \{-1,1\}_+^{2n}$, $\pi = \{(l_h, r_h)\}_{h=1}^n \in PP(2n, \varepsilon)$ and $\{f_h\}_{h=1}^{2n} \subset \mathcal{L}([0,t])$ one has, introducing the notation (see (2.6))*

$$\chi_{(s}(t) := \begin{cases} 1 \ , \ \ if \ s < t \\ 0 \ , \ \ if \ s \geq t \end{cases} \quad , \ \forall s, t \in \mathbb{R} \tag{3.4}$$

in the notation (2.6)

$$\lim_{N \to \infty} \frac{1}{N^n} \sum_{\mathbf{k} \in PPF([Nt], n, \pi)} \left\langle \Phi_{[Nt]}, \sigma_{\mathbf{k}(1)}^{(\varepsilon(1))} \cdots \sigma_{\mathbf{k}(2n)}^{(\varepsilon(2n))} \Phi_{[Nt]} \right\rangle \prod_{h=1}^n \left(\overline{f_{l_h}} f_{r_h} \right) \left(\frac{\mathbf{k}(l_h)}{N} \right) \tag{3.5}$$

$$= \int_{[0,t]^n} \overline{x}^{\sum_{h,m=1}^n \chi_{(l_h, r_h)}(l_m) \cdot \chi_{(t_h}(t_m)} x^{\sum_{h,m=1}^n \chi_{(l_h, r_h)}(r_m) \cdot \chi_{(t_h}(t_m)} \prod_{h=1}^n \left(\overline{f_{l_h}} f_{r_h} \right)(t_h)$$

and consequently

$$\lim_{N \to \infty} \left\langle \Phi_{[Nt]}, S_N^{(\varepsilon(1))}(t, f_1) \cdots S_N^{(\varepsilon(2n))}(t, f_{2n}) \Phi_{[Nt]} \right\rangle = \sum_{\{(l_h, r_h)\}_{h=1}^n \in PP(2n, \varepsilon)} \tag{3.6}$$

$$\int_{[0,t]^n} \overline{x}^{\sum_{h,m=1}^n \chi_{(l_h, r_h)}(l_m) \cdot \chi_{(t_h}(t_m)} x^{\sum_{h,m=1}^n \chi_{(l_h, r_h)}(r_m) \cdot \chi_{(t_h}(t_m)} \prod_{h=1}^n \left(\overline{f_{l_h}} f_{r_h} \right)(t_h)$$

Proof. Due to (3.3), the second statement is a consequence of the first one. To prove the first statement recall, from Lemma 2.4 of [4], that using repeatedly the identities

$$\begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad ; \quad \forall c \in \mathbb{C}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix} \sigma^{(-1)} \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix} \sigma^{(+1)} \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = c \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad ; \quad \forall c, y, z \in \mathbb{C}$$

one finds that, for any $\pi = \{(l_h, r_h)\}_{h=1}^n \in PP(2n)$, $\mathbf{k} \in PPF(N, n, \pi)$ and $p \in \{1, \cdots, N\}$, the $p-th$ tensor factor of the vector $\sigma_{\mathbf{k}(1)}^{(\varepsilon(1))} \cdots \sigma_{\mathbf{k}(2n)}^{(\varepsilon(2n))} \Phi_{[Nt]}$ is equal to

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{if} \quad p \in \{1, \cdots, N\} \setminus Range(\mathbf{k})$$

$$\prod_{j=l_h+1}^{r_h-1} y_j \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{if} \quad p = \mathbf{k}(l_h) \text{ with } h \in \{1, \cdots, n\}$$

where the y_j belong to $\{1, x, \overline{x}\}$. More precisely, for any $j \in \{l_h + 1, \cdots, r_h - 1\}$:
$- y_j = 1$ if $\mathbf{k}(j) < \mathbf{k}(l_h)$;
$- y_j = x$ if $\mathbf{k}(j) > \mathbf{k}(l_h)$ and $\varepsilon(j) = +1$ (equivalently, $j = r_m$ for some m);
$- y_j = \overline{x}$ if $\mathbf{k}(j) > \mathbf{k}(l_h)$ and $\varepsilon(j) = -1$ (equivalently, $j = l_m$ for some m).
In view of this, in the notation (3.4), for any $h \in \{1, \cdots, n\}$, the corresponding $\prod_{j=l_h+1}^{r_h-1} y_j$ can be written in the form

$$\overline{x}^{\sum_{m=1}^n \chi_{(l_h, r_h)}(l_m) \chi_{(\mathbf{k}(l_h)}(\mathbf{k}(l_m))} \cdot x^{\sum_{m=1}^n \chi_{(l_h, r_h)}(r_m) \chi_{(\mathbf{k}(l_h)}(\mathbf{k}(l_m))}$$

Using this result, one gets

$$\sigma_{\mathbf{k}(1)}^{(\varepsilon(1))}\cdots\sigma_{\mathbf{k}(2n)}^{(\varepsilon(2n))}\Phi_{[Nt]}$$

$$=\prod_{h=1}^{n}\left(\overline{x}^{\sum_{m=1}^{n}\chi_{(l_h,r_h)}(l_m)\chi_{(\mathbf{k}(l_h))}(\mathbf{k}(l_m))}x^{\sum_{m=1}^{n}\chi_{(l_h,r_h)}(r_m)\chi_{(\mathbf{k}(l_h))}(\mathbf{k}(l_m))}\right)\Phi_{[Nt]}$$

The new element, with respect to the proof Theorem 2.1 of [4] of is the presence of the test functions. To handle this notice that, since $\chi_{(s}(u)=\chi_{(cs}(cu)$ for any $c>0$, one has

$$\frac{1}{N^n}\sum_{\mathbf{k}\in PPF([Nt],n,\pi)}\left\langle\Phi_{[Nt]},\sigma_{\mathbf{k}(1)}^{(\varepsilon(1))}\cdots\sigma_{\mathbf{k}(2n)}^{(\varepsilon(2n))}\Phi_N\right\rangle\prod_{h=1}^{n}(\overline{f_{l_h}}f_{r_h})\left(\frac{\mathbf{k}(l_h)}{N}\right)\quad(3.7)$$

$$=\frac{1}{N^n}\sum_{\mathbf{k}\in PPF([Nt],n,\pi)}\prod_{h=1}^{n}\left(\overline{x}^{\sum_{m=1}^{n}}\right)$$

$$\chi_{(l_h,r_h)}(l_m)\chi_{(\mathbf{k}(l_h)}(\mathbf{k}(l_m))x^{\sum_{m=1}^{n}\chi_{(l_h,r_h)}(r_m)\chi_{(\mathbf{k}(l_h)}(\mathbf{k}(l_m))}(\overline{f_{l_h}}f_{r_h})\left(\frac{\mathbf{k}(l_h)}{N}\right)$$

$$=\frac{1}{N^n}\sum_{\mathbf{k}\in PPF([Nt],n,\pi)}\prod_{h=1}^{n}(\overline{f_{l_h}}f_{r_h})\left(\frac{\mathbf{k}(l_h)}{N}\right)$$

$$\left(\overline{x}^{\sum_{m=1}^{n}\chi_{(l_h,r_h)}(l_m)\chi_{(\mathbf{k}(l_h)/N}(\mathbf{k}(l_m)/N)}x^{\sum_{m=1}^{n}\chi_{(l_h,r_h)}(r_m)\chi_{(\mathbf{k}(l_h)/N}(\mathbf{k}(l_m)/N)}\right)$$

Since the function $F:[0,t]^n\longmapsto\mathbb{C}$ given by

$$F(t_1,\cdots,t_n):=$$

$$=\prod_{h=1}^{n}\overline{x}^{\sum_{m=1}^{n}\chi_{(l_h,r_h)}(l_m)\cdot\chi_{(t_h}(t_m)}x^{\sum_{m=1}^{n}\chi_{(l_h,r_h)}(r_m)\cdot\chi_{(t_h}(t_m)}(\overline{f_{l_h}}f_{r_h})(t_h)$$

$((t_1,\cdots,t_n)\in[0,t]^n)$ is Riemann–integrable and bounded, as $N\to\infty$, the right hand side of (3.7), i.e.

$$\frac{1}{N^n}\sum_{\mathbf{k}\in PPF([Nt],n,\pi)}F\left(\frac{\mathbf{k}(l_1)}{N},\cdots,\frac{\mathbf{k}(l_n)}{N}\right)$$

tends to the right hand side of (3.5).   □

Corollary 3.3. *For any $t>0$, $n\in\mathbb{N}$ and $\{f_h\}_{h=1}^{2n}\subset\mathcal{L}([0,t])$, denoting*

$$\mathcal{S}_n:=\{permutations\ over\ n\ symbols\}\qquad(3.8)$$

one has

$$\lim_{N\to\infty}\left\langle\Phi_{[Nt]},S_N^{(-1)}(t,f_1)\cdots S_N^{(-1)}(t,f_n)\cdot S_N^{(+1)}(t,f_{n+1})\cdots S_N^{(+1)}(t,f_{2n})\Phi_N\right\rangle$$

$$=\sum_{\alpha\in\mathcal{S}_n}\int_{[0,t]^n}\overline{x}^{\sum_{1\le h<m\le n}\chi_{(t_h}(t_m)}x^{\sum_{1\le h<m\le n}\chi_{(t_{\alpha(h)}}(t_{\alpha(m)})}$$

$$\prod_{h=1}^{n}(\overline{f_h}f_{2n+1-\sigma^{-1}(h)})(t_h)\,dt_1\cdots dt_n$$

$$= \sum_{\alpha \in \mathcal{S}_n} \int_{[0,t]^n} \overline{x}^{\sum_{1 \le h < m \le n} \chi_{(t_h}(t_m)} x^{\sum_{1 \le h < m \le n} \chi_{(t_{\sigma(m)}}} \left(t_{t_{\sigma(h)}}\right) \prod_{h=1}^{n} \left(\overline{f_h} f_{n+\sigma(h)}\right)(t_h)$$

(3.9)

Proof. Notice that the scalar product in (3.9) is equal to

$$\left\langle \Phi_{[Nt]}, S_N^{(\varepsilon(1))}(t, f_1) \cdots S_N^{(\varepsilon(2n))}(t, f_{2n}) \Phi_{[Nt]} \right\rangle$$

with $\varepsilon(h) = -1$ and $\varepsilon(n + h) = 1$ for any $h \in \{1, \cdots, n\}$. Using the fact that the l_h of a pair partition are fixed (and correspond to the -1's) one can identify

$$PP(2n, \varepsilon) = \{\{(h, 2n + 1 - \sigma(h))\}_{h=1}^{n} : \sigma \in \mathcal{S}_n\}$$

If, in this correspondence, the pair partition $\{(l_h, r_h)\}_{h=1}^{n} \in PP(2n, \varepsilon)$ corresponds to
$\{(h, 2n + 1 - \sigma(h))\}_{h=1}^{n}$, then, for any $1 \le h, m \le n$ and any $\sigma \in \mathcal{S}_n$ one has   □

- $\chi_{(l_h, r_h)}(l_m) = \chi_{(h, 2n+1-\sigma(h))}(m)$ and it takes the value 1 if and only if $h < m$;
- $\chi_{(l_h, r_h)}(r_m) = \chi_{(h, 2n+1-\sigma(h))}(2n + 1 - \sigma(m))$ and it takes the value 1 if and only if
  $\sigma(h) < \sigma(m)$.

Consequently, in the right hand side of (3.6), $\sum_{h,m=1}^{n} \chi_{(l_h, r_h)}(l_m) \cdot \chi_{(t_h}(t_m)$ becomes equal to $\sum_{1 \le h < m \le n} \chi_{(t_h}(t_m)$ and

$$x^{\sum_{h,m=1}^{n} \chi_{(h, 2n+1-\sigma(h))}(2n+1-\sigma(m)) \cdot \chi_{(t_h}(t_m)} \prod_{h=1}^{n} f_{2n+1-\sigma(h)}(t_h)$$

$$= x^{\sum_{h,m=1}^{n} \chi_{(\sigma(h), \sigma(m))} \cdot \chi_{(t_h}(t_m)} \prod_{h=1}^{n} f_{2n+1-\sigma(h)}(t_h)$$

$$= x^{\sum_{h,m=1}^{n} \chi_{(\sigma(h), \sigma(m))} \cdot \chi_{(t_{\sigma^{-1}(\sigma(h))}, t_{\sigma^{-1}(\sigma(m))})}} \prod_{h=1}^{n} f_{2n+1-\sigma(h)}\left(t_{\sigma^{-1}(\sigma(h))}\right)$$

Since $\sigma(h)$ runs over $\{1, \cdots, n\}$ as h runs over $\{1, \cdots, n\}$, the above expression is equal to

$$x^{\sum_{1 \le h < m \le n} \chi_{(t_{\sigma^{-1}(h)}}\left(t_{\sigma^{-1}(m)}\right)} \prod_{h=1}^{n} f_{2n+1-h}\left(t_{\sigma^{-1}(h)}\right)$$

Moreover,

$$\sum_{\alpha \in \mathcal{S}_n} x^{\sum_{1 \le h < m \le n} \chi\left(t_{\sigma^{-1}(h)}, t_{\sigma^{-1}(m)}\right)} \prod_{h=1}^{n} f_{2n+1-h}\left(t_{\sigma^{-1}(h)}\right)$$

$$= \sum_{\alpha \in \mathcal{S}_n} x^{\sum_{1 \le h < m \le n} \chi_{t_{\sigma(m)}}(t_h)} \prod_{h=1}^{n} f_{2n+1-h}\left(t_{\sigma(h)}\right)$$

$$= \sum_{\alpha \in \mathcal{S}_n} x^{\sum_{1 \le h < m \le n} \chi_{t_{\sigma(m)}}(t_h)} \prod_{h=1}^{n} f_{2n+1-\sigma^{-1}(h)}(t_h)$$

so Lemma 3.2 tells us that

$$\lim_{N \to \infty} \left\langle \Phi_{[Nt]}, S_N^{(-1)}(t, f_1) \cdots S_N^{(-1)}(t, f_n) \cdot S_N^{(+1)}(t, f_{n+1}) \cdots S_N^{(+1)}(t, f_{2n}) \Phi_N \right\rangle$$

$$= \sum_{\alpha \in \mathcal{S}_n} \int_{[0,t]^n} \overline{x}^{\sum_{1 \le h < m \le n} \chi_{(t_h)}(t_m)} x^{\sum_{1 \le h < m \le n} \chi_{t_{\sigma(m)}}(t_h)} \prod_{h=1}^{n} \left(\overline{f_h} f_{2n+1-\sigma^{-1}(h)} \right)(t_h)$$

and this is the first identity in (3.9). On the other hand, using the alternative identification

$$PP(2n, \varepsilon) = \{\{(h, n + \sigma(h))\}_{h=1}^{n} \ : \ \sigma \in \mathcal{S}_n\}$$

we see that, if $\{(l_h, r_h)\}_{h=1}^{n} \in PP(2n, \varepsilon)$ corresponds to $\{(h, 2n+1-\sigma(h))\}_{h=1}^{n}$, then for any $1 \le h, m \le n$ and $\sigma \in \mathcal{S}_n$, one has

$$\chi_{(l_h, r_h)}(l_m) = \chi_{(h, n+\sigma(h))}(m)$$

and it takes the value 1 if and only if $h < m$;

$$\chi_{(l_h, r_h)}(r_m) = \chi_{(h, n+\sigma(h))}(n + \sigma(m))$$

and it takes the value 1 if and only if $\sigma(m) < \sigma(h)$. So

$$x^{\sum_{h,m=1}^{n} \chi_{(h,n+\sigma(h))}(n+\sigma(m)) \cdot \chi_{(t_h)}(t_m)} \prod_{h=1}^{n} f_{n+\sigma(h)}(t_h) \tag{3.10}$$

$$= x^{\sum_{h,m=1}^{n} \chi_{(\sigma(m))}(\sigma(h)) \cdot \chi_{(t_h)}(t_m)} \prod_{h=1}^{n} f_{n+\sigma(h)}(t_h)$$

$$= x^{\sum_{h,m=1}^{n} \chi_{(\sigma(m))}(\sigma(h)) \cdot \chi_{(t_{\sigma^{-1}(\sigma(h))})}(t_{\sigma^{-1}(\sigma(m))})} \prod_{h=1}^{n} f_{n+\sigma(h)}(t_{\sigma^{-1}(\sigma(h))})$$

$$= x^{\sum_{p,q=1}^{n} \chi_{(p,q)} \cdot \chi_{(t_{\sigma^{-1}(q)})}(t_{\sigma^{-1}(p)})} \prod_{q=1}^{n} f_{n+q}(t_{\sigma^{-1}(q)})$$

$$= x^{\sum_{1 \le h < m \le n} \chi_{(t_{\sigma^{-1}(m)})}(t_{\sigma^{-1}(h)})} \prod_{h=1}^{n} f_{n+h}(t_{\sigma^{-1}(h)})$$

$$= x^{\sum_{1 \le h < m \le n} \chi_{(t_{\sigma^{-1}(m)})}(t_{\sigma^{-1}(h)})} \prod_{h=1}^{n} f_{n+\sigma(h)}(t_h)$$

and Lemma 3.2 tells us that

$$\lim_{N \to \infty} \left\langle \Phi_{[Nt]}, S_N^{(-1)}(t, f_1) \cdots S_N^{(-1)}(t, f_n) \cdot S_N^{(+1)}(t, f_{n+1}) \cdots S_N^{(+1)}(t, f_{2n}) \Phi_{[Nt]} \right\rangle \tag{3.11}$$

$$= \sum_{\alpha \in \mathcal{S}_n} \int_{[0,t]^n} \overline{x}^{\sum_{1 \le h < m \le n} \chi_{(t_h)}(t_m)} x^{\sum_{1 \le h < m \le n} \chi_{(t_{\sigma(m)})}(t_{t_{\sigma(h)}})} \prod_{h=1}^{n} \left(\overline{f_h} f_{n+\sigma(h)} \right)(t_h)$$

This is the second identity in (3.9).

Corollary 3.4. *For any $t > 0$, $n \in \mathbb{N}$ and $\{f_h, g_h\}_{h=1}^n \subset \mathcal{L}([0, t])$, one has*

$$\lim_{N \to \infty} \left\langle \Phi_{[Nt]}, S_N^{(-1)}(t, f_1) \cdots S_N^{(-1)}(t, f_n) \cdot S_N^{(+1)}(t, g_n) \cdots S_N^{(+1)}(t, g_1) \Phi_{[Nt]} \right\rangle$$

$$= \sum_{\alpha \in \mathcal{S}_n} \int_{[0,t]^n} \overline{x}^{\sum_{1 \le h < m \le n} \chi_{(t_h}(t_m)} x^{\sum_{1 \le h < m \le n} \chi_{(t_{\alpha(h)}}} \left(t_{\alpha(m)} \right)$$

$$\prod_{h=1}^n \left(\overline{f_h} g_{\alpha^{-1}(h)} \right)(t_h) \, dt_1 \cdots dt_n$$

Proof. The result follows from the first equality in (3.9) taking $g_h := f_{2n+1-h}$ for any $h \in \{1, \cdots, n\}$. $\qquad\square$

3.1. The scalar product in the limit space. In order to write the identity (3.9) in a more transparent way we introduce, for each $n \in \mathbb{N}$ and $(t_1, \cdots, t_n) \in \mathbb{R}_+^n$, the multiplication operator by $x^{\sum_{1 \le h < m \le n} \chi_{(t_h}(t_m)}$

$$(X_n F)(t_1, \cdots, t_n) := \begin{cases} x^{\sum_{1 \le h < k \le n} \chi_{(t_h}(t_k)} F(t_1, \cdots, t_n) \,, & \text{if } x \in \mathbb{C} \setminus \{0\} \\ \chi_{(\mathbb{R}_+^n)_<}(t_1, \cdots, t_n) F(t_1, \cdots, t_n) \,, & \text{if } x = 0 \end{cases} \quad (3.12)$$

where F is any complex valued Borel function on $\mathbb{R}_+^n$ and the symmetrization operator, denoted P_n is, acting on as follows

$$(P_n F)(t_1, \cdots, t_n) := \sum_{\sigma \in \mathcal{S}_n} F\left(t_{\sigma(1)}, \cdots, t_{\sigma(n)} \right) \quad (3.13)$$

where $\mathcal{S}_n$ denotes the permutation group on $\{1, \ldots, n\}$. The standard (i.e. the tensor) scalar product on $L^2(\mathbb{R}_+^n)$ is denoted

$$(\,\cdot\,,\,\cdot\,)_n$$

With respect to this scalar product, $P_n/n!$ is the orthogonal projection onto the sub–space of symmetric functions. From (3.12) and (3.13) it follows that

$$(P_n X_n F)(t_1, \cdots, t_n) = \sum_{\alpha \in \mathcal{S}_n} x^{\sum_{1 \le h < m \le n} \chi_{(t_{\alpha(h)}}\left(t_{\alpha(m)} \right)} F(t_{t_{\alpha(1)}}, \cdots, t_{t_{\alpha(n)}})$$

is adjointable on this domain for the scalar product $(\,\cdot\,,\,\cdot\,)_n$ and $(P_n X_n)^*$, $P_n X_n$, $\lambda_n(x)$ are linear operators leaving invariant the dense sub–space $L^2_{comp-supp}(\mathbb{R}_+^n) \subset L^2(\mathbb{R}_+^n)$, consisting of square–integrable compact support functions. Therefore the linear operator

$$\lambda_n(x) := \frac{1}{n!}(P_n X_n)^* P_n X_n \quad (3.14)$$

leaves $L^2_{comp-supp}(\mathbb{R}_+^n)$ invariant and is positive on this domain. Hence

$$\langle F, G \rangle_n := (F, \lambda_n(x) G)_n = \frac{1}{n!}(P_n X_n F, P_n X_n G)_n \quad ; \quad \forall F, G \in L^2_{comp-supp}(\mathbb{R}_+^n)$$

is a pre–scalar product on $L^2_{comp-supp}(\mathbb{R}_+^n)$. Denote $\mathcal{H}_n$ the completion of $L^2_{comp-supp}(\mathbb{R}_+^n)$ with respect to this pre–scalar product and define the Hilbert space

$$\Gamma_I(L^2_{comp-supp}(\mathbb{R}_+^n), \{\lambda_n(x)\}_n) := \bigoplus_{k=0}^{\infty} \mathcal{H}_k \quad ; \quad \mathcal{H}_0 := \mathbb{C} \cdot \Phi \,, \; \|\Phi\| = 1$$

as the completion of the orthogonal sum of the $\mathcal{H}_n$. On the pre–Hilbert space

$$\Gamma_I^0(L^2_{comp-supp}(\mathbb{R}_+^n), \{\lambda_n(x)\}_n) := \bigoplus_{k=0}^{\infty \ w} \mathcal{H}_k \quad ; \quad \mathcal{H}_0 := \mathbb{C} \cdot \Phi \qquad (3.15)$$

where $\bigoplus_{k=0}^{\infty \ w}$ means weak orthogonal sum (finite linear combinations), the creation operator a_f^+ is well defined for any $f \in L^2_{comp-supp}(\mathbb{R}_+)$ by

$$a^+(f) F := \langle f \otimes F \rangle \quad ; \quad F \in \Gamma_I^0(L^2_{comp-supp}(\mathbb{R}_+^n), \{\lambda_n(x)\}_n) \qquad (3.16)$$

We will prove, see Lemma 4.3 below, that the existence condition for the adjoint of a_f^+:

$$\|F_n\|_n = 0 \ \Rightarrow \ \|f \otimes F_n\|_{n+1} = 0$$

is satisfied. Therefore the triple $(\Gamma_I(L^2_{comp-supp}(\mathbb{R}_+^n), \{\lambda_n(x)\}_n), a^+, \Phi)$ defines an IFS (for Definition, see e.g. [3] or [4], [13]). In the above notations one can rewrite the limit scalar product, i.e. the right hand side of (3.9), in the form

$$\sum_{\alpha \in \mathcal{S}_n} \int_{[0,t]^n} \overline{x^{\sum_{1 \le h < m \le n} \chi_{(t_h)}(t_m)} \prod_{h=1}^n f_h(t_h)} x^{\sum_{1 \le h < m \le n} \chi_{(t_{\alpha(h)})}(t_{\alpha(m)})} \qquad (3.17)$$

$$\prod_{h=1}^n g_h(t_{\alpha(h)}) \, dt_1 \cdots dt_n$$

$$= \int_{[0,t]^n} \left(\overline{[X_n(f_1 \otimes \cdots \otimes f_n)]} [P_n X_n(g_1 \otimes \cdots \otimes g_n)] \right)(t_1, \cdots, t_n) \, dt_1 \cdots dt_n$$

$$= ((f_1 \otimes \cdots \otimes f_n), \lambda_n(x)(g_1 \otimes \cdots \otimes g_n))_n$$

4. The limit Space as a Deformation of the Monotone Fock Space

4.1. Emergence of the q–symmetrizator. In the notation (3.8), for $\sigma \in \mathcal{S}_n$, denote $|\sigma|$ the **index** (or degree) of σ, i.e.

$$|\sigma| := |\{(h, m) \ : \ 1 \le h < m \le n \text{ and } \sigma(h) > \sigma(m)\}| \qquad (4.1)$$

and, for $n \ge 2$, introduce the notations:

$$(\mathbb{R}_+^n)_< := \{(t_1, \cdots, t_n) \in \mathbb{R}_+^n \ : \ 0 \le t_1 < \cdots < t_n\} \qquad (4.2)$$

$$(\mathbb{R}_+^n)_\ge := \{(t_1, \cdots, t_n) \in \mathbb{R}_+^n \ : \ 0 \le t_1 \ge \cdots \ge t_n\} \qquad (4.3)$$

$$= \text{(Lebesgue a.e.) } (\mathbb{R}_+^n)_> := \{(t_1, \cdots, t_n) \in \mathbb{R}_+^n \ : \ 0 \le t_1 > \cdots > t_n\}$$

Lemma 4.1. *For any $n \ge 2$, $\sigma \in \mathcal{S}_n$ and $(t_1, \cdots, t_n) \in \mathbb{R}^n$*

$$\chi_{(\mathbb{R}_+^n)_<}(t_{\sigma(1)}, \cdots, t_{\sigma(n)}) X_n(t_1, \cdots, t_n) = \chi_{(\mathbb{R}_+^n)_<}(t_{\sigma(1)}, \cdots, t_{\sigma(n)}) x^{\frac{1}{2}n(n-1)-|\sigma|}$$
$$(4.4)$$

$$\chi_{(\mathbb{R}_+^n)_>}(t_{\sigma(1)}, \cdots, t_{\sigma(n)}) X_n(t_1, \cdots, t_n) = \chi_{(\mathbb{R}_+^n)_>}(t_{\sigma(1)}, \cdots, t_{\sigma(n)}) x^{|\sigma|} \qquad (4.5)$$

Proof. For any $\sigma \in \mathcal{S}_n$ one has (see (4.1))

$$\left|\sigma^{-1}\right| = \left|\left\{(h,m) \; : \; 1 \leq h < m \leq n, \; h' := \sigma^{-1}(h) > \sigma^{-1}(m) =: m'\right\}\right|$$
$$= \left|\left\{(h,m) \; : \; 1 \leq \sigma(h') < \sigma(m') \leq n, \; h' > m'\right\}\right| = |\sigma|$$

Consequently

$$\frac{1}{2}n\,(n-1) = |\{(h,m) : 1 \leq h < m \leq n\}|$$
$$= |\{(h,m) : 1 \leq h < m \leq n, \; \sigma(h) < \sigma(m)\}|$$
$$+ |\{(h,m) : 1 \leq h < m \leq n, \; \sigma(h) > \sigma(m)\}|$$
$$= |\{(h,m) : 1 \leq h < m \leq n, \; \sigma(h) < \sigma(m)\}| + |\sigma|$$
$$= |\{(h,m) : 1 \leq h < m \leq n, \; \sigma(h) < \sigma(m)\}| + \left|\sigma^{-1}\right|$$

Therefore, for $(t_1, \cdots, t_n) \in \mathbb{R}^n$ such that $t_{\sigma(1)} < \cdots < t_{\sigma(n)}$, one has

$$\sum_{1 \leq h < m \leq n} \chi_{(t_h}(t_m) = |\{(h,m) \; : \; 1 \leq h < m \leq n, \; t_h < t_m\}|$$

$$= \left|\left\{(h,m) \; : \; 1 \leq h < m \leq n, \; t_{\sigma(\sigma^{-1}(h))} < t_{\sigma(\sigma^{-1}(m))}\right\}\right|$$
$$= \left|\left\{(h,m) : 1 \leq h < m \leq n, \; \sigma^{-1}(h) < \sigma^{-1}(m)\right\}\right|$$
$$= \frac{1}{2}n\,(n-1) - |\sigma| = \frac{1}{2}n\,(n-1) - \left|\sigma^{-1}\right|$$

and this gives (4.4). Similarly, for $(t_1, \cdots, t_n) \in \mathbb{R}^n$ such that $t_{\sigma(1)} > \cdots > t_{\sigma(n)}$, one has

$$\sum_{1 \leq h < m \leq n} \chi_{(t_h}(t_m) = |\{(h,m) \; : \; 1 \leq h < m \leq n, \; t_h < t_m\}|$$

$$= \left|\left\{(h,m) \; : \; 1 \leq h < m \leq n, \; t_{\sigma(\sigma^{-1}(h))} < t_{\sigma(\sigma^{-1}(m))}\right\}\right|$$
$$= \left|\left\{(h,m) \; : \; 1 \leq h < m \leq n, \; \sigma^{-1}(h) > \sigma^{-1}(m)\right\}\right| = \left|\sigma^{-1}\right| = |\sigma|$$

and this gives (4.5). $\square$

Lemma 4.1 suggests the following expression of X_n in terms of x– or x^{-1}–symmetrizators.

Corollary 4.2. *Following* [9], *for any* $c \in \mathbb{C}$, *we define the* c–**symmetrizator** Q_c *as follows. For any* $n \in \mathbb{N}$ *and complex valued Borel function* F *on* $\mathbb{R}^n$, *with the usual convention* $0^0 := 1$:

$$(Q_c F)\,(t_1, \cdots, t_n) := \sum_{\sigma \in \mathcal{S}_n} c^{|\sigma|} F\left(t_{\sigma(1)}, \cdots, t_{\sigma(n)}\right) \tag{4.6}$$

where $|\sigma|$ *denotes the degree of the permutation* σ. *For* $\sigma \in \mathcal{S}_n$ *denote* $\hat{\sigma}$ *the operator acting on a function* F *as above by*

$$(\hat{\sigma} F)\,(t_1, \cdots, t_n) := F\left(t_{\sigma(1)}, \cdots, t_{\sigma(n)}\right) \tag{4.7}$$

With these notations, identifying a function with the associated multiplication operator, for any $x \neq 0$ *and* $n \geq 2$ *one has:*

$$X_n = x^{\frac{1}{2}n(n-1)} Q_{x^{-1}}(\chi_{(\mathbb{R}_+^n)_<}) \tag{4.8}$$

$$X_n = Q_x(\chi_{(\mathbb{R}_+^n)_>}) \tag{4.9}$$

Proof. Since the functions

$$(t_1, \cdots, t_n) \longmapsto \sum_{\sigma \in \mathcal{S}_n} \chi_{(\mathbb{R}_+^n)_>} \left(t_{\sigma(1)}, \cdots, t_{\sigma(n)}\right) \ , \ \sum_{\sigma \in \mathcal{S}_n} \chi_{(\mathbb{R}_+^n)_<} \left(t_{\sigma(1)}, \cdots, t_{\sigma(n)}\right)$$

are equal to the constant 1 Lebesgue a.e., (4.4) implies that, for F as in the statement

$$(X_n F)(t_1, \cdots, t_n) = \sum_{\sigma \in \mathcal{S}_n} \chi_{(\mathbb{R}_+^n)_<} \left(t_{\sigma(1)}, \cdots, t_{\sigma(n)}\right) X_n (t_1, \cdots, t_n) F(t_1, \cdots, t_n)$$

$$= \sum_{\sigma \in \mathcal{S}_n} \chi_{(\mathbb{R}_+^n)_<} \left(t_{\sigma(1)}, \cdots, t_{\sigma(n)}\right) x^{\frac{1}{2}n(n-1)-|\sigma|} F(t_1, \cdots, t_n)$$

$$= x^{\frac{1}{2}n(n-1)} \sum_{\sigma \in \mathcal{S}_n} x^{-|\sigma|} \chi_{(\mathbb{R}_+^n)_<} \left(t_{\sigma(1)}, \cdots, t_{\sigma(n)}\right) F(t_1, \cdots, t_n)$$

and this is equivalent to (4.8). Similarly, using (4.5)

$$(X_n F)(t_1, \cdots, t_n) = \sum_{\sigma \in \mathcal{S}_n} \chi_{(\mathbb{R}_+^n)_>} \left(t_{\sigma(1)}, \cdots, t_{\sigma(n)}\right) X_n (t_1, \cdots, t_n) F(t_1, \cdots, t_n)$$

$$= \sum_{\sigma \in \mathcal{S}_n} \chi_{(\mathbb{R}_+^n)_>} \left(t_{\sigma(1)}, \cdots, t_{\sigma(n)}\right) x^{|\sigma|} F(t_1, \cdots, t_n) = (Q_x(\chi_{(\mathbb{R}_+^n)_>})F)(t_1, \cdots, t_n)$$

and this is equivalent to (4.9). $\qquad\square$

4.2. Some examples of $\lambda_n(x)$. In this section we discuss some examples of $\lambda_n(x)$, i.e. $X_n^* P_n X_n$, or equivalently $\frac{1}{n!}(P_n X_n)^* P_n X_n$, and the corresponding IFS $\Gamma_I \left(L^2(\mathbb{R}_+), \{\lambda_n(x)\}_n\right)$ introduced in (3.16). This will help in understanding in what sense the limit Hilbert space generalizes some well known Fock spaces.

4.2.1. *The case $x = 1$.* If $x = 1$, X_n is nothing else than the identity and $P_n X_n / n!$ is the usual symmetrization operator on $L^2(\mathbb{R}_+^n)$. So for any $F \in L^2(\mathbb{R}_+^n)$ and $(t_1, \cdots, t_n) \in \mathbb{R}_+^n$

$$(\lambda_n(1)F)(t_1, \cdots, t_n) = \sum_{\sigma \in \mathcal{S}_n} F\left(t_{\sigma(1)}, \cdots, t_{\sigma(n)}\right)$$

Consequently the IFS $\Gamma_I \left(L^2(\mathbb{R}_+), \{\lambda_n(1)\}_n\right)$ is the **symmetric** Fock space over $L^2(\mathbb{R}_+^n)$.

4.2.2. *The case $x = -1$.* Notice that, for any Borel function $F : \mathbb{R}^n \to \mathbb{C}$,

$$(P_n X_n F)(t_1, \cdots, t_n) = (P_n Q_x(\chi_{(\mathbb{R}_+^n)_>})F)(t_1, \cdots, t_n)$$

$$= \sum_{\alpha \in \mathcal{S}_n} \sum_{\sigma \in \mathcal{S}_n} \chi_{(\mathbb{R}_+^n)_>} \left(t_{\sigma\alpha(1)}, \cdots, t_{\sigma\alpha(n)}\right) x^{|\sigma|} F\left(t_{\alpha(1)}, \cdots, t_{\alpha(n)}\right)$$

$$= \sum_{\alpha \in \mathcal{S}_n} \sum_{\sigma\alpha \in \mathcal{S}_n} \chi_{(\mathbb{R}_+^n)_>} \left(t_{\sigma\alpha(1)}, \cdots, t_{\sigma\alpha(n)}\right) x^{|\sigma|} F\left(t_{\alpha(1)}, \cdots, t_{\alpha(n)}\right)$$

Putting $\sigma\alpha =: \tau \iff \sigma = \tau\alpha^{-1}$, one finds

$$= \sum_{\alpha \in \mathcal{S}_n} \sum_{\tau \in \mathcal{S}_n} \chi_{(\mathbb{R}_+^n)_>} \left(t_{\tau(1)}, \cdots, t_{\tau(n)}\right) x^{|\tau\alpha^{-1}|} F\left(t_{\alpha(1)}, \cdots, t_{\alpha(n)}\right)$$

If $x = -1$

$$(-1)^{|\tau\alpha^{-1}|} = (-1)^{|\tau|}(-1)^{|\alpha^{-1}|} = (-1)^{|\tau|}(-1)^{|\alpha|}$$

and we find

$$(P_n Q_{-1}(\chi_{(\mathbb{R}_+^n)_>})F)(t_1, \cdots, t_n) =$$

$$= \sum_{\alpha \in \mathcal{S}_n} \sum_{\tau \in \mathcal{S}_n} \chi_{(\mathbb{R}_+^n)_>}(t_{\tau(1)}, \cdots, t_{\tau(n)})(-1)^{|\tau|}(-1)^{|\alpha|}F(t_{\alpha(1)}, \cdots, t_{\alpha(n)})$$

$$= \sum_{\tau \in \mathcal{S}_n} (-1)^{|\tau|}\chi_{(\mathbb{R}_+^n)_>}(t_{\tau(1)}, \cdots, t_{\tau(n)})\sum_{\alpha \in \mathcal{S}_n}(-1)^{|\alpha|}F(t_{\alpha(1)}, \cdots, t_{\alpha(n)})$$

$$= Q_{-1}(\chi_{(\mathbb{R}_+^n)_>}))(t_1, \cdots, t_n)Q_{-1}(F)(t_1, \cdots, t_n)$$

Therefore, for any pair of square–integrable Borel functions $F, G : \mathbb{R}^n \to \mathbb{C}$, one has

$$(F, \lambda_n(-1)G)_n = \frac{1}{n!}\left(P_n Q_{-1}\left(\chi_{(\mathbb{R}_+^n)_>}\right)F, P_n Q_{-1}\left(\chi_{(\mathbb{R}_+^n)_>}\right)G\right)_n$$

$$= \frac{1}{n!}\left(Q_{-1}\left(\chi_{(\mathbb{R}_+^n)_>}\right)Q_{-1}(F), Q_{-1}\left(\chi_{(\mathbb{R}_+^n)_>}\right)Q_{-1}(G)\right)_n$$

$$= \frac{1}{n!}\left(\sum_{\sigma \in \mathcal{S}_n}(-1)^{|\sigma|}\hat{\sigma}\chi_{(\mathbb{R}_+^n)_>}Q_{-1}(F), \sum_{\tau \in \mathcal{S}_n}(-1)^{|\tau|}\hat{\tau}\chi_{(\mathbb{R}_+^n)_>}Q_{-1}(G)\right)_n$$

$$= \frac{1}{n!}\sum_{\sigma \in \mathcal{S}_n}\sum_{\tau \in \mathcal{S}_n}(-1)^{|\sigma|}(-1)^{|\tau|}\left(\chi_{(\sigma^{-1}\mathbb{R}_+^n)_>}Q_{-1}(F), \chi_{\tau^{-1}(\mathbb{R}_+^n)_>}Q_{-1}(G)\right)_n$$

$$= \frac{1}{n!}\sum_{\sigma \in \mathcal{S}_n}\sum_{\tau \in \mathcal{S}_n}(-1)^{|\sigma|}(-1)^{|\tau|}\delta_{\sigma,\tau}\left(\chi_{(\sigma^{-1}\mathbb{R}_+^n)_>}Q_{-1}(F), Q_{-1}(G)\right)_n$$

$$= \frac{1}{n!}\sum_{\sigma \in \mathcal{S}_n}\left(\chi_{(\sigma^{-1}\mathbb{R}_+^n)_>}Q_{-1}(F), Q_{-1}(G)\right)_n = \frac{1}{n!}(Q_{-1}(F), Q_{-1}(G))_n$$

Consequently, the IFS $\Gamma_I\left(L^2(\mathbb{R}_+), \{\lambda_n(-1)\}_n\right)$ is the **anti–symmetric or Fermi Fock space** over $L^2(\mathbb{R}_+^n)$.

4.2.3. *The case $x = 0$.* If $x = 0$,

$$(\lambda_n(0)F)(t_1, \cdots, t_n) =$$

$$= 0^{\sum_{1 \le h < m \le n}\chi_{(t_h)}(t_m)}\sum_{\sigma \in \mathcal{S}_n}0^{\sum_{1 \le h < m \le n}\chi_{t_{\sigma(m))}(t_h)}F(t_{\sigma(1)}, \cdots, t_{\sigma(n)})$$

Thus, in the notations (4.2), (4.2) and noticing that $\sum_{1 \le h < m \le n}\chi_{t_m)}(t_h) = 0$ if and only if

$t_1 > \cdots > t_n$, it follows that for any $F \in L^2(\mathbb{R}_+^n)$ and $(t_1, \cdots, t_n) \in \mathbb{R}_+^n$:

$$(\lambda_n(0)F)(t_1, \cdots, t_n) = \quad \text{Lebesgue–a.e.}$$

$$= \chi_{(\mathbb{R}_+^n)_\ge}(t_1, \cdots, t_n)\sum_{\sigma \in \mathcal{S}_n}\chi_{(\mathbb{R}_+^n)_\ge}(t_{\sigma(1)}, \cdots, t_{\sigma(n)})$$

$$F(t_{\sigma(1)}, \cdots, t_{\sigma(n)}) = \left(\chi_{(\mathbb{R}_+^n)_\ge}F\right)(t_1, \cdots, t_n)$$

i.e. $\lambda_n(0)$ can be identified to the multiplication operator by $\chi_{(\mathbb{R}_+^n)_\ge}F$.

Consequently, the IFS $\Gamma_I\left(L^2(\mathbb{R}_+), \{\lambda_n(0)\}_n\right)$ is the **monotone Fock space** over $L^2(\mathbb{R}_+^n)$ introduced in [18] and emerging from the CLT proved in [4].

4.3. Properties of the scalar product for general x.

Lemma 4.3. *For any $f \in C_{\mathbb{C},comp}(\mathbb{R}_+)$, $n \in \mathbb{N}$ and F belonging to the algebraic tensor product $C_{\mathbb{C},comp}(\mathbb{R}_+)^{\odot n}$, $(f \otimes F, \lambda_{n+1}(x)f \otimes F)_{n+1}$ is equal to zero whenever $(F, \lambda_n(x)F)_n = 0$.*

Proof. Writing F as a finite sum $\sum_k c_k f_{n,k} \otimes \cdots \otimes f_{1,k}$, one finds

$$(f \otimes F, \lambda_{n+1}(x)f \otimes F)_{n+1}$$

$$= \lim_{N \to \infty} \left\langle \Phi_{[Nt]}, \left[\sum_k \bar{c}_k S_N^{(-1)}(t, f_{1,k}) \cdots S_N^{(-1)}(t, f_{n,k}) \right] \cdot S_N^{(-1)}(t, f) \cdot S_N^{(+1)}(t, f) \cdot \right.$$

$$\left. \left[\sum_k c_k S_N^{(+1)}(t, f_{n,k}) \cdots S_N^{(+1)}(t, f_{1,k}) \right] \Phi_{[Nt]} \right\rangle \tag{4.10}$$

and the scalar product in the right hand side of (4.10) is less or equal than

$$\left\langle \Phi_{[Nt]}, \right.$$

$$\left. \left| \left[\sum_k \bar{c}_k S_N^{(-1)}(t, f_{1,k}) \cdots S_N^{(-1)}(t, f_{n,k}) \right] \cdot S_N^{(-1)}(t, f) \cdot S_N^{(+1)}(t, f) \right|^2 \Phi_{[Nt]} \right\rangle^{1/2}$$

$$\sum_{k,h} \bar{c}_k c_h$$

$$\left\langle \Phi_{[Nt]}, S_N^{(-1)}(t, f_{1,k}) \cdots S_N^{(-1)}(t, f_{n,k}) S_N^{(+1)}(t, f_{n,h}) \cdots S_N^{(+1)}(t, f_{1,h}) \Phi_{[Nt]} \right\rangle^{1/2}$$

Since, from Lemma 3.2, one knows that both limits exist and are finite and the limit of the second scalar product is

$$\lim_{N \to \infty} \sum_{k,h} \bar{c}_k c_h$$

$$\left\langle \Phi_{[Nt]}, S_N^{(-1)}(t, f_{1,k}) \cdots S_N^{(-1)}(t, f_{n,k}) S_N^{(+1)}(t, f_{n,h}) \cdots S_N^{(+1)}(t, f_{1,h}) \Phi_{[Nt]} \right\rangle$$

$$= (F, \lambda_n(x)F) = 0$$

the thesis follows.   □

Lemma 4.4. *For any $n \in \mathbb{N}$, $x \in \mathbb{C}$ and $F \in L^2\left(\mathbb{R}_+^n\right)$,*

$$(F, \lambda_n(x)F)_n \leq n! \left(|x| \vee 1\right)^{n(n-1)} (F, F)_n \tag{4.11}$$

In particular, $\lambda_n(x)$ is bounded on each n–particle space of the full Fock space over $L^2(\mathbb{R}_+)$.

Proof. From (4.9) one deduces that

$$(F, \lambda_n(x)F)_n$$

$$= \frac{1}{n!} \sum_{\alpha,\tau \in \mathcal{S}_n} \int_{\mathbb{R}^n_+} \overline{\left((Q_n(x)F) \cdot \chi_{(\mathbb{R}^n_+)<}\right)} \left(t_{\sigma(1)}, \cdots, t_{\sigma(n)}\right)$$

$$\left((Q_n(x)F) \cdot \chi_{(\mathbb{R}^n_+)<}\right) \left(t_{\tau(1)}, \cdots, t_{\tau(n)}\right) dt_1 \cdots dt_n$$

$$= \frac{1}{n!} \sum_{\alpha \in \mathcal{S}_n} \int_{\mathbb{R}^n_+} \chi_{(\mathbb{R}^n_+)<} \left(t_{\sigma(1)}, \cdots, t_{\sigma(n)}\right) |(Q_n(x)F)|^2 \left(t_{\sigma(1)}, \cdots, t_{\sigma(n)}\right) dt_1 \cdots dt_n$$

$$= \int_{\mathbb{R}^n_+} \chi_{(\mathbb{R}^n_+)<} (t_1, \cdots, t_n) |(Q_n(x)F)|^2 (t_1, \cdots, t_n) dt_1 \cdots dt_n \qquad (4.12)$$

From (4.6) we see that, for all $(t_1, \cdots, t_n) \in \mathbb{R}^n_+$

$$|(Q_n(x)F)(t_1, \cdots, t_n)| \leq (|x| \vee 1)^{\frac{1}{2}n(n-1)} \sum_{\sigma \in \mathcal{S}_n} |F| \left(t_{\sigma(1)}, \cdots, t_{\sigma(n)}\right)$$

and applying it to (4.12), one gets

$$(F, \lambda_n(x)F)_n \leq (|x| \vee 1)^{n(n-1)}$$

$$\sum_{\sigma,\tau \in \mathcal{S}_n} \int_{\mathbb{R}^n_+} \chi_{(\mathbb{R}^n_+)<} (t_1, \cdots, t_n) |F| \left(t_{\sigma(1)}, \cdots, t_{\sigma(n)}\right) |F| \left(t_{\tau(1)}, \cdots, t_{\tau(n)}\right) dt_1 \cdots dt_n$$

and (4.11) follows from the inequalities

$$\sum_{\sigma,\tau \in \mathcal{S}_n} \int_{\mathbb{R}^n_+} \chi_{(\mathbb{R}^n_+)<} (t_1, \cdots, t_n) |F| \left(t_{\sigma(1)}, \cdots, t_{\sigma(n)}\right) |F| \left(t_{\tau(1)}, \cdots, t_{\tau(n)}\right) dt_1 \cdots dt_n$$

$$\leq \sum_{\sigma \in \mathcal{S}_n} \left[\int_{\mathbb{R}^n_+} \chi_{(\mathbb{R}^n_+)<} (t_1, \cdots, t_n) |F|^2 \left(t_{\sigma(1)}, \cdots, t_{\sigma(n)}\right) dt_1 \cdots dt_n\right]^{\frac{1}{2}} \cdot$$

$$\cdot \sum_{\tau \in \mathcal{S}_n} \left[\int_{\mathbb{R}^n_+} \chi_{(\mathbb{R}^n_+)<} (t_1, \cdots, t_n) |F|^2 \left(t_{\tau(1)}, \cdots, t_{\tau(n)}\right) dt_1 \cdots dt_n\right]^{\frac{1}{2}}$$

$$= \left(\sum_{\sigma \in \mathcal{S}_n} \left[\int_{\mathbb{R}^n_+} \chi_{(\mathbb{R}^n_+)<} (t_1, \cdots, t_n) |F|^2 \left(t_{\sigma(1)}, \cdots, t_{\sigma(n)}\right) dt_1 \cdots dt_n\right]^{\frac{1}{2}}\right)^2$$

$$\leq \left((n!)^{1/2} \left[\sum_{\sigma \in \mathcal{S}_n} \int_{\mathbb{R}^n_+} \chi_{(\mathbb{R}^n_+)<} (t_1, \cdots, t_n) |F|^2 \left(t_{\sigma(1)}, \cdots, t_{\sigma(n)}\right) dt_1 \cdots dt_n\right]^{1/2}\right)^2$$

$$= n! \int_{\mathbb{R}^n_+} |F|^2 (t_1, \cdots, t_n) dt_1 \cdots dt_n = n! (F, F)_n$$

$$\square$$

Lemma 4.3 allows to introduce a structure of interacting Fock space on $(\Gamma_I^0(\{L^2_{comp-supp}(\mathbb{R}^n_+), \lambda_n(x)\}_n)$ (see (3.15)).

Corollary 4.5. *The creation operator defined by (3.16) for any* $f \in L^2_{comp-supp}(\mathbb{R}_+)$, *i.e.*

$$a^+(f) F := f \otimes F \qquad ; \qquad F \in \Gamma_I^0(\{L^2_{comp-supp}(\mathbb{R}^n_+), \lambda_n(x)\}_n)$$

has an adjoint in $\Gamma_I^0(L^2_{comp-supp}(\mathbb{R}^n_+), \{\lambda_n(x)\}_n)$.

Proof. The condition proved in Lemma 4.3 is equivalent to the existence of the desired adjoint. $\qquad\square$

Lemma 4.6. *With the notation*

$$x^{(+)}(s,t) := |x|^{2\chi_{(s}(t)} \quad ; \quad x^{(-)}(s,t) := \overline{x}^{\chi_{(s}(t)} x^{\chi_{(t}(s)} \quad ; \quad \forall s,t \in \mathbb{R} \qquad (4.13)$$

introducing the functions

$$T_{n-1,x,r}(s_1,\ldots,s_{n-1}) := T_{n-1,x,r}(s_r, s_{r+1},\ldots,s_{n-1}) \qquad (4.14)$$

$$:= \left(\prod_{h=r+1}^{n-1} (x^{-1})^{\chi_{(s_r}(s_h)} x^{\chi_{(s_h}(s_r)} \right) \qquad ; \qquad s_1,\ldots,s_{n-1} \in [0,t]$$

(i.e. $T_{n-1,x,r}$ is constant in the variables $(s_1,\ldots,s_{r-1})$) and denoting with the same symbol the corresponding multiplication operators, which act on Borel functions $G : [0,t]^n \to \mathbb{C}$, For any n and $\{f_h, g_h\}_{h=1}^n \subset L^2([0,1])$, one has

$$\left\langle \bigotimes_{h=1}^n f_h, \bigotimes_{h=1}^n g_h \right\rangle_n = \int_0^t dt_n \overline{f}_n(t_n) g_r(t_n) \cdot \left\langle \left(\bigotimes_{h=1}^{n-1} \overline{f}_h \right), \sum_{r=1}^n T_{n-1,x,r} \qquad (4.15)$$

$$\left(\left(\bigotimes_{h=1}^{r-1} \left(x^{(+)}(\cdot, t_n) g_h \right) \right) \otimes \left(x^{(-)}(\cdot, t_n) g_n \right) \otimes \bigotimes_{h=r+1}^{n-1} \left(x^{(-)}(\cdot, t_n) g_h \right) \right) \right\rangle_{n-1}$$

Proof. From (3.17) we know that

$$((f_1 \otimes \cdots \otimes f_n), \lambda_n(x)(g_1 \otimes \cdots \otimes g_n))_n =$$

$$= \int_{[0,t]^n} n! \left(\bigotimes_{h=1}^n \overline{f}_h \right)(t_1,\ldots,t_n)$$

$$\sum_{\alpha \in \mathcal{S}_n} \overline{x}^{\sum_{1 \le h < m \le n} \chi_{(t_h}(t_m)} x^{\sum_{1 \le h < m \le n} \chi_{(t_{\alpha(h)}}(t_{\alpha(m)})} \bigotimes_{h=1}^n g_h(,\ldots,t_{\alpha(n)}) dt_1 \cdots dt_n$$

$$= \sum_{\alpha \in \mathcal{S}_n} \int_{[0,t]^n} \overline{x^{\sum_{1 \le h < m \le n} \chi_{(t_h}(t_m)} \prod_{h=1}^n f_h(t_h)} x^{\sum_{1 \le h < m \le n} \chi_{(t_{\alpha(h)}}(t_{\alpha(m)})} \qquad (4.16)$$

$$\prod_{h=1}^n g_h(t_{\alpha(h)}) dt_1 \cdots dt_n$$

$$= \sum_{\alpha \in \mathcal{S}_n} \int_{[0,t]^n} \overline{x}^{\sum_{1 \le h < m \le n} \chi_{(t_h}(t_m)} x^{\sum_{1 \le h < m \le n} \chi_{(t_{\alpha(h)}}(t_{\alpha(m)})}$$

$$\prod_{h=1}^{n} \overline{f}_h\left(t_h\right) g_h\left(t_{\alpha(h)}\right) dt_1 \cdots dt_n$$

Denoting, for $r \in \{1, \ldots, n\}$,

$$\mathcal{S}_n^{(\hat{r})} := \{\alpha \in \mathcal{S}_n \; : \; \alpha(r) = n\}$$

this is equal to

$$= \sum_{r=1}^{n} \sum_{\alpha \in \mathcal{S}_n^{(\hat{r})}} \int_{[0,t]^n} \overline{x}^{\sum_{1 \leq h < m \leq n} \chi_{(t_h}(t_m)} x^{\sum_{1 \leq h < m \leq n} \chi_{(t_{\alpha(h)}}\left(t_{\alpha(m)}\right)}$$

$$\prod_{h=1}^{n} \overline{f}_h\left(t_h\right) g_h\left(t_{\alpha(h)}\right) dt_1 \cdots dt_n$$

Using the identities, valid for any $1 \leq r \leq n$ and any $\alpha \in \mathcal{S}_n^{(\hat{r})}$:

$$\sum_{1 \leq h < m \leq n} \chi_{(t_h}\left(t_m\right) = \sum_{h=1}^{n-1} \chi_{(t_h}\left(t_n\right) + \sum_{1 \leq h < m \leq n-1} \chi_{(t_h}\left(t_m\right)$$

$$= \sum_{1 \leq h < m \leq n} \chi_{(t_{\alpha(h)}}\left(t_{\alpha(m)}\right) =$$

$$= \sum_{h=1}^{r-1} \chi_{(t_{\alpha(h)}}\left(t_n\right) + \sum_{m=r+1}^{n} \chi_{(t_n}\left(t_{\alpha(m)}\right) + \sum_{1 \leq h < m \leq n; h \neq r \neq m} \chi_{(t_{\alpha(h)}}\left(t_{\alpha(m)}\right)$$

one can write this in the form

$$= \sum_{r=1}^{n} \sum_{\alpha \in \mathcal{S}_n^{(\hat{r})}} \int_{[0,t]^n} \overline{x}^{\sum_{h=1}^{n-1} \chi_{(t_h}(t_n) + \sum_{1 \leq h < m \leq n-1} \chi_{(t_h}(t_m)}$$

$$\cdot x^{\sum_{h=1}^{r-1} \chi_{(t_{\alpha(h)}}(t_n) + \sum_{m=r+1}^{n} \chi_{(t_n}\left(t_{\alpha(m)}\right) + \sum_{1 \leq h < m \leq n; h \neq r \neq m} \chi_{(t_{\alpha(h)}}\left(t_{\alpha(m)}\right)}$$

$$\overline{f}_r\left(t_r\right) g_r\left(t_n\right) \prod_{h=1, h \neq r}^{n} \overline{f}_h\left(t_h\right) g_h\left(t_{\alpha(h)}\right) dt_1 \cdots dt_n$$

$$= \sum_{r=1}^{n} \sum_{\alpha \in \mathcal{S}_n^{(\hat{r})}} \int_{[0,t]^n} \overline{x}^{\sum_{h=1}^{n-1} \chi_{(t_h}(t_n)} x^{\sum_{h=1}^{r-1} \chi_{(t_{\alpha(h)}}(t_n) + \sum_{m=r+1}^{n} \chi_{(t_n}\left(t_{\alpha(m)}\right)} \overline{f}_r\left(t_r\right) g_r\left(t_n\right)$$

$$\overline{x}^{\sum_{1 \leq h < m \leq n-1} \chi_{(t_h}(t_m)} \cdot x^{\sum_{1 \leq h < m \leq n; h \neq r \neq m} \chi_{(t_{\alpha(h)}}\left(t_{\alpha(m)}\right)}$$

$$\prod_{h=1, h \neq r}^{n} \overline{f}_h\left(t_h\right) g_h\left(t_{\alpha(h)}\right) dt_1 \cdots dt_n$$

Using the fact that

$$\sum_{h=1}^{n-1} \chi_{(t_h}\left(t_n\right) = \sum_{h=1}^{n} \chi_{(t_h}\left(t_n\right) = \sum_{h=1}^{n} \chi_{(t_{\alpha(h)}}\left(t_n\right) = \sum_{h=1}^{r-1} \chi_{(t_{\alpha(h)}}\left(t_n\right) + \sum_{h=r+1}^{n} \chi_{(t_{\alpha(h)}}\left(t_n\right)$$

this becomes

$$= \sum_{r=1}^{n} \sum_{\alpha \in \mathcal{S}_n^{(\hat{r})}} \int_{[0,t]^n} \overline{x}^{\sum_{h=1}^{r-1} \chi_{(t_{\alpha(h)}}(t_n)} \overline{x}^{\sum_{h=r+1}^{n} \chi_{(t_{\alpha(h)}}(t_n)}$$

$$x^{\sum_{h=1}^{r-1} \chi_{(t_{\alpha(h)})}(t_n) + \sum_{h=r+1}^{n} \chi_{(t_n}(t_{\alpha(h)})} \overline{f}_r(t_r) g_r(t_n)$$

$$\overline{x}^{\sum_{1 \le h < m \le n-1} \chi_{(t_h}(t_m)} \cdot x^{\sum_{1 \le h < m \le n; h \ne r \ne m} \chi_{(t_{\alpha(h)})}(t_{\alpha(m)})}$$

$$\prod_{h=1, h \ne r}^{n} \overline{f}_h(t_h) g_h(t_{\alpha(h)}) \, dt_1 \cdots dt_n$$

$$= \sum_{r=1}^{n} \sum_{\alpha \in \mathcal{S}_n^{(\hat{r})}} \int_{[0,t]^n}$$

$$\left(|x|^2\right)^{\sum_{h=1}^{r-1} \chi_{(t_{\alpha(h)})}(t_n)} \overline{x}^{\sum_{h=r+1}^{n} \chi_{(t_{\alpha(h)})}(t_n)} x^{\sum_{h=r+1}^{n} \chi_{(t_n}(t_{\alpha(h)})} \overline{f}_r(t_r) g_r(t_n)$$

$$\overline{x}^{\sum_{1 \le h < m \le n-1} \chi_{(t_h}(t_m)} \cdot x^{\sum_{1 \le h < m \le n; h \ne r \ne m} \chi_{(t_{\alpha(h)})}(t_{\alpha(m)})}$$

$$\prod_{h=1, h \ne r}^{n} \overline{f}_h(t_h) g_h(t_{\alpha(h)}) \, dt_1 \cdots dt_n$$

$$= \sum_{r=1}^{n} \sum_{\alpha \in \mathcal{S}_n^{(\hat{r})}} \int_{[0,t]^n} dt_1 \cdots dt_n$$

$$\left(\prod_{h=1}^{r-1} |x|^{2\chi_{(t_{\alpha(h)})}(t_n)}\right) \left(\prod_{h=r+1}^{n} \overline{x}^{\chi_{(t_{\alpha(h)})}(t_n)} x^{\chi_{(t_n}(t_{\alpha(h)})}\right) \overline{f}_r(t_r) g_r(t_n)$$

$$\overline{x}^{\sum_{1 \le h < m \le n-1} \chi_{(t_h}(t_m)} \cdot x^{\sum_{1 \le h < m \le n; h \ne r \ne m} \chi_{(t_{\alpha(h)})}(t_{\alpha(h)})} \prod_{h=1, h \ne r}^{n} \overline{f}_h(t_h) g_h(t_{\alpha(h)})$$

$$= \sum_{r=1}^{n} \sum_{\alpha \in \mathcal{S}_n^{(\hat{r})}} \int_{[0,t]^n} dt_1 \cdots dt_n \left(\prod_{h=1}^{r-1} x^{(+)}\left(t_{\alpha(h)}, t_n\right)\right)$$

$$\left(\prod_{h=r+1}^{n} x^{(-)}\left(t_{\alpha(h)}, t_n\right)\right) \overline{f}_r(t_r) g_r(t_n) \overline{x}^{\sum_{1 \le h < m \le n-1} \chi_{(t_h}(t_m)} \cdot$$

$$\cdot x^{\sum_{1 \le h < m \le n; h \ne r \ne m} \chi_{(t_{\alpha(h)})}(t_{\alpha(m)})} \overline{f}_n(t_n) g_n\left(t_{\alpha(n)}\right) \prod_{h=1, h \ne r}^{n-1} \overline{f}_h(t_h) g_h(t_{\alpha(h)})$$

$$= \sum_{r=1}^{n} \sum_{\alpha \in \mathcal{S}_n^{(\hat{r})}} \int_{[0,t]^n} \left(\prod_{h=1}^{r-1} x^{(+)}\left(t_{\alpha(h)}, t_n\right)\right) \left(\prod_{h=r+1}^{n} x^{(-)}\left(t_{\alpha(h)}, t_n\right)\right) \overline{f}_n(t_n) g_r(t_n)$$

$$\overline{x}^{\sum_{1 \le h < m \le n-1} \chi_{(t_h}(t_m)} \cdot x^{\sum_{1 \le h < m \le n; h \ne r \ne m} \chi_{(t_{\alpha(h)})}(t_{\alpha(m)})} \overline{f}_r(t_r) g_n\left(t_{\alpha(n)}\right)$$

$$\prod_{h=1, h \ne r}^{n-1} \overline{f}_h(t_h) g_h(t_{\alpha(h)}) \, dt_1 \cdots dt_n$$

$$= \sum_{r=1}^{n} \sum_{\alpha \in \mathcal{S}_n^{(\hat{r})}} \int_{[0,t]^n} \left(\prod_{h=1}^{r-1} x^{(+)}\left(t_{\alpha(h)}, t_n\right)\right) \left(\prod_{h=r+1}^{n-1} x^{(-)}\left(t_{\alpha(h)}, t_n\right)\right)$$

$$x^{(-)}\left(t_{\alpha(n)}, t_n\right) \overline{f}_n(t_n) g_r(t_n)$$

$$\overline{x}^{\sum_{1 \le h < m \le n-1} \chi_{(t_h}(t_m)} \cdot x^{\sum_{1 \le h < m \le n-1; h \ne r \ne m} \chi_{(t_{\alpha(h)})}(t_{\alpha(m)})} \cdot x^{\sum_{1 \le h \le n-1; h \ne r} \chi_{(t_{\alpha(h)})}(t_{\alpha(n)})}$$

$$\overline{f}_r\left(t_r\right)g_n\left(t_{\alpha(n)}\right)\prod_{h=1,h\neq r}^{n-1}\overline{f}_h\left(t_h\right)g_h\left(t_{\alpha(h)}\right)dt_1\cdots dt_n$$

Defining the permutation τ, on the set $\{1,\ldots,r,\ldots,n-1\}$ by

$$\tau(h):=\begin{cases}\alpha(h)\ ,\ \text{if }h<r\\ \alpha\left(n\right)\ ,\ \text{if }h=r\\ \alpha(h)\ ,\ \text{if }h\in\{r+1,\cdots,n-1\}\end{cases}$$

and noticing that, as α runs over the set $\{\alpha\in\mathcal{S}_n\ :\ \alpha\left(r\right)=n\}$, τ runs over the set of all permutations of $\{1,\ldots,r,\ldots,n-1\}$, this becomes

$$=\sum_{r=1}^n\sum_{\tau\in\mathcal{S}_{n-1}}\int_{[0,t]^n}\left(\prod_{h=1}^{r-1}x^{(+)}\left(t_{\tau(h)},t_n\right)\right)\left(\prod_{h=r+1}^{n-1}x^{(-)}\left(t_{\tau(h)},t_n\right)\right)$$

$$x^{(-)}\left(t_{\tau(r)},t_n\right)\overline{f}_n\left(t_n\right)g_r\left(t_n\right)$$

$$\overline{x}^{\sum_{1\leq h<m\leq n-1}\chi_{(t_h}(t_m)}\cdot x^{\sum_{1\leq h<m\leq n-1;h\neq r\neq m}\chi_{(t_{\tau(h)}}\left(t_{\tau(m)}\right)}\cdot x^{\sum_{1\leq h\leq n-1;h\neq r}\chi_{(t_{\tau(h)}}\left(t_{\tau(r)}\right)}$$

$$\overline{f}_r\left(t_r\right)g_n\left(t_{\tau(r)}\right)\prod_{h=1,h\neq r}^{n-1}\overline{f}_h\left(t_h\right)g_h\left(t_{\tau(h)}\right)dt_1\cdots dt_n$$

$$=\sum_{r=1}^n\sum_{\tau\in\mathcal{S}_{n-1}}\int_{[0,t]^n}\left(\prod_{h=1}^{r-1}x^{(+)}\left(t_{\tau(h)},t_n\right)\right)\left(\prod_{h=r+1}^{n-1}x^{(-)}\left(t_{\tau(h)},t_n\right)\right)$$

$$x^{(-)}\left(t_{\tau(r)},t_n\right)\overline{f}_n\left(t_n\right)g_r\left(t_n\right)\overline{x}^{\sum_{1\leq h<m\leq n-1}\chi_{(t_h}(t_m)}\cdot x^{\sum_{1\leq h<m\leq n-1;h\neq r\neq m}\chi_{(t_{\tau(h)}}\left(t_{\tau(m)}\right)}$$

$$\cdot x^{\sum_{h=1}^{r-1}\chi_{(t_{\tau(h)}}\left(t_{\tau(r)}\right)}\cdot x^{\sum_{h=r+1}^{n-1}\chi_{(t_{\tau(h)}}\left(t_{\tau(r)}\right)}\overline{f}_r\left(t_r\right)g_n\left(t_{\tau(r)}\right)$$

$$\prod_{h=1,h\neq r}^{n-1}\overline{f}_h\left(t_h\right)g_h\left(t_{\tau(h)}\right)dt_1\cdots dt_n$$

$$=\sum_{r=1}^n\sum_{\tau\in\mathcal{S}_{n-1}}\int_{[0,t]^n}\left(\prod_{h=1}^{r-1}x^{(+)}\left(t_{\tau(h)},t_n\right)\right)\left(\prod_{h=r+1}^{n-1}x^{(-)}\left(t_{\tau(h)},t_n\right)\right)$$

$$x^{(-)}\left(t_{\tau(r)},t_n\right)\overline{f}_n\left(t_n\right)g_r\left(t_n\right)$$

$$\overline{x}^{\sum_{1\leq h<m\leq n-1}\chi_{(t_h}(t_m)}\cdot x^{\sum_{1\leq h<m\leq n-1;h\neq r}\chi_{(t_{\tau(h)}}\left(t_{\tau(m)}\right)}\cdot x^{\sum_{h=r+1}^{n-1}\chi_{(t_{\tau(h)}}\left(t_{\tau(r)}\right)}$$

$$\overline{f}_r\left(t_r\right)g_n\left(t_{\tau(r)}\right)\prod_{h=1,h\neq r}^{n-1}\overline{f}_h\left(t_h\right)g_h\left(t_{\tau(h)}\right)dt_1\cdots dt_n$$

$$=\sum_{r=1}^n\sum_{\tau\in\mathcal{S}_{n-1}}\int_{[0,t]^n}\left(\prod_{h=1}^{r-1}x^{(+)}\left(t_{\tau(h)},t_n\right)\right)\left(\prod_{h=r+1}^{n-1}x^{(-)}\left(t_{\tau(h)},t_n\right)\right)$$

$$x^{(-)}\left(t_{\tau(r)},t_n\right)\overline{f}_n\left(t_n\right)g_r\left(t_n\right)\overline{x}^{\sum_{1\leq h<m\leq n-1}\chi_{(t_h}(t_m)}\cdot x^{\sum_{1\leq h<m\leq n-1}\chi_{(t_{\tau(h)}}\left(t_{\tau(m)}\right)}$$

$$\cdot(x^{-1})^{\sum_{m=r+1}^{n-1}\chi_{(t_{\tau(r)}}\left(t_{\tau(m)}\right)}\cdot x^{\sum_{h=r+1}^{n-1}\chi_{(t_{\tau(h)}}\left(t_{\tau(r)}\right)}$$

$$\overline{f}_r\left(t_r\right)g_n\left(t_{\tau(r)}\right)\prod_{h=1,h\neq r}^{n-1}\overline{f}_h\left(t_h\right)g_h\left(t_{\tau(h)}\right)dt_1\cdots dt_n$$

$$= \sum_{r=1}^{n} \sum_{\tau \in \mathcal{S}_{n-1}} \int_{[0,t]^n} \left(\prod_{h=1}^{r-1} x^{(+)} \left(t_{\tau(h)}, t_n \right) \right) \left(\prod_{h=r+1}^{n-1} x^{(-)} \left(t_{\tau(h)}, t_n \right) \right)$$

$$x^{(-)} \left(t_{\tau(r)}, t_n \right) \overline{f}_n \left(t_n \right) g_r \left(t_n \right) \overline{x}^{\sum_{1 \le h < m \le n-1} \chi_{(t_h}(t_m)} \cdot x^{\sum_{1 \le h < m \le n-1} \chi_{(t_{\tau(h)}} \left(t_{\tau(m)} \right)}$$

$$\cdot \prod_{h=r+1}^{n-1} (x^{-1})^{\chi_{(t_{\tau(r)}} \left(t_{\tau(h)} \right)} \cdot x^{\chi_{(t_{\tau(h)}} \left(t_{\tau(r)} \right)}$$

$$\prod_{h=1}^{r-1} \overline{f}_h \left(t_h \right) g_h \left(t_{\tau(h)} \right) \overline{f}_r \left(t_r \right) g_n \left(t_{\tau(r)} \right) \prod_{h=r+1}^{n-1} \overline{f}_h \left(t_h \right) g_h \left(t_{\tau(h)} \right) dt_1 \cdots dt_n$$

$$= \sum_{r=1}^{n} \sum_{\tau \in \mathcal{S}_{n-1}} \int_0^t dt_n \int_{[0,t]^{n-1}} dt_1 \cdots dt_{n-1}$$

$$\overline{f}_n \left(t_n \right) g_r \left(t_n \right) \overline{x}^{\sum_{1 \le h < m \le n-1} \chi_{(t_h}(t_m)} \cdot x^{\sum_{1 \le h < m \le n-1} \chi_{(t_{\tau(h)}} \left(t_{\tau(m)} \right)}$$

$$\left(\prod_{h=1}^{r-1} x^{(+)} \left(t_{\tau(h)}, t_n \right) \right) \left(\prod_{h=r+1}^{n-1} x^{(-)} \left(t_{\tau(h)}, t_n \right) \right) x^{(-)} \left(t_{\tau(r)}, t_n \right)$$

$$\prod_{h=r+1}^{n-1} (x^{-1})^{\chi_{(t_{\tau(r)}} \left(t_{\tau(h)} \right)} x^{\chi_{(t_{\tau(h)}} \left(t_{\tau(r)} \right)}$$

$$\left(\prod_{h=1}^{r-1} \overline{f}_h \left(t_h \right) g_h \left(t_{\tau(h)} \right) \right) \overline{f}_r \left(t_r \right) g_n \left(t_{\tau(r)} \right) \left(\prod_{h=r+1}^{n-1} \overline{f}_h \left(t_h \right) g_h \left(t_{\tau(h)} \right) \right)$$

$$= \sum_{r=1}^{n} \sum_{\tau \in \mathcal{S}_{n-1}} \int_0^t dt_n \overline{f}_n \left(t_n \right) g_r \left(t_n \right) \int_{[0,t]^{n-1}} dt_1 \cdots dt_{n-1}$$

$$\overline{x}^{\sum_{1 \le h < m \le n-1} \chi_{(t_h}(t_m)} \cdot x^{\sum_{1 \le h < m \le n-1} \chi_{(t_{\tau(h)}} \left(t_{\tau(m)} \right)}$$

$$\left(\prod_{h=1}^{n-1} \overline{f}_h \left(t_h \right) \right) \left(\prod_{h=1}^{r-1} \left(x^{(+)} \left(t_{\tau(h)}, t_n \right) g_h \left(t_{\tau(h)} \right) \right) \right) \left(x^{(-)} \left(t_{\tau(r)}, t_n \right) g_n \left(t_{\tau(r)} \right) \right)$$

$$\prod_{h=r+1}^{n-1} (x^{-1})^{\chi_{(t_{\tau(r)}} \left(t_{\tau(h)} \right)} x^{\chi_{(t_{\tau(h)}} \left(t_{\tau(r)} \right)} \prod_{h=r+1}^{n-1} \left(x^{(-)} \left(t_{\tau(h)}, t_n \right) g_h \left(t_{\tau(h)} \right) \right)$$

Introducing, for any $n \ge 2$ and $r \in \{1, \ldots, n-1\}$, the functions (4.14) and the corresponding multiplication operators the above expression can be written in the form

$$\sum_{r=1}^{n} \sum_{\tau \in \mathcal{S}_{n-1}} \int_0^t dt_n \overline{f}_n \left(t_n \right) g_r \left(t_n \right) \int_{[0,t]^{n-1}} dt_1 \cdots dt_{n-1}$$

$$\overline{x}^{\sum_{1 \le h < m \le n-1} \chi_{(t_h}(t_m)} \cdot x^{\sum_{1 \le h < m \le n-1} \chi_{(t_{\tau(h)}} \left(t_{\tau(m)} \right)} \left(\prod_{h=1}^{n-1} \overline{f}_h \left(t_h \right) \right)$$

$$\left(\prod_{h=1}^{r-1} \left(x^{(+)} \left(\cdot, t_n \right) g_h \right) \left(t_{\tau(h)} \right) \right) \left(x^{(-)} \left(\cdot, t_n \right) g_n \right) \left(t_{\tau(r)} \right)$$

$$T_{n-1,x,r}\left(t_{\tau(r)},t_{\tau(r+1)},\ldots,t_{\tau(n-1)}\right)\prod_{h=r+1}^{n-1}\left(x^{(-)}\left(\,\cdot\,,t_n\right)g_h\right)\left(t_{\tau(h)}\right)$$

$$=\sum_{r=1}^{n}\sum_{\tau\in\mathcal{S}_{n-1}}\int_0^t dt_n\overline{f}_n\left(t_n\right)g_r\left(t_n\right)$$

$$\int_{[0,t]^{n-1}}dt_1\cdots dt_{n-1}\overline{x}^{\sum_{1\le h<m\le n-1}\chi_{(t_h}(t_m)}\cdot x^{\sum_{1\le h<m\le n-1}\chi_{(t_{\tau(h)}}\left(t_{\tau(m)}\right)}$$

$$\left(\bigotimes_{h=1}^{n-1}\overline{f}_h\right)\left(t_1,\ldots,t_{n-1}\right)T_{n-1,x,r}\left(t_{\tau(1)},\ldots,t_{\tau(n-1)}\right)$$

$$\bigotimes_{h=1}^{r-1}\left(x^{(+)}\left(\cdot,t_n\right)g_h\right)\otimes\left(x^{(-)}\left(\cdot,t_n\right)g_n\right)\otimes\bigotimes_{h=r+1}^{n-1}\left(x^{(-)}\left(\cdot,t_n\right)g_h\right)\left(t_{\tau(1)},\ldots t_{\tau(n-1)}\right)$$

$$=\int_0^t dt_n\overline{f}_n\left(t_n\right)g_r\left(t_n\right)\int_{[0,t]^{n-1}}dt_1\cdots dt_{n-1}$$

$$\left(X_{n-1}\bigotimes_{h=1}^{n-1}\overline{f}_h\right)\left(t_1,\ldots,t_{n-1}\right)\sum_{\tau\in\mathcal{S}_{n-1}}X_{n-1}$$

$$\left(\sum_{r=1}^{n}T_{n-1,x,r}\left(\bigotimes_{h=1}^{r-1}\left(x^{(+)}\left(\cdot,t_n\right)g_h\right)\right)\otimes\left(x^{(-)}\left(\cdot,t_n\right)g_n\right)\otimes\right.$$

$$\left.\otimes\bigotimes_{h=r+1}^{n-1}\left(x^{(-)}\left(\cdot,t_n\right)g_h\right)\left(t_{\tau(1)},\ldots t_{\tau(n-1)}\right)\right)$$

$$=\int_0^t dt_n\overline{f}_n\left(t_n\right)g_r\left(t_n\right)\left\langle X_{n-1}\left(\bigotimes_{h=1}^{n-1}\overline{f}_h\right),n!P_{n-1}X_{n-1}\sum_{r=1}^{n}T_{n-1,x,r}\right.$$

$$\left.\left(\left(\bigotimes_{h=1}^{r-1}\left(x^{(+)}\left(\cdot,t_n\right)g_h\right)\right)\otimes\left(x^{(-)}\left(\cdot,t_n\right)g_n\right)\otimes\bigotimes_{h=r+1}^{n-1}\left(x^{(-)}\left(\cdot,t_n\right)g_h\right)\right)\right\rangle_{n-1}$$

$$=\int_0^t dt_n\overline{f}_n\left(t_n\right)g_r\left(t_n\right)\left\langle\left(\bigotimes_{h=1}^{n-1}\overline{f}_h\right),\lambda_{n-1}(x)\sum_{r=1}^{n}T_{n-1,x,r}\right.$$

$$\left.\left(\left(\bigotimes_{h=1}^{r-1}\left(x^{(+)}\left(\,\cdot\,,t_n\right)g_h\right)\right)\otimes\left(x^{(-)}\left(\,\cdot\,,t_n\right)g_n\right)\otimes\bigotimes_{h=r+1}^{n-1}\left(x^{(-)}\left(\,\cdot\,,t_n\right)g_h\right)\right)\right\rangle_{n-1}$$

$$=\int_0^t dt_n\overline{f}_n\left(t_n\right)g_r\left(t_n\right)\cdot\left\langle\left(\bigotimes_{h=1}^{n-1}\overline{f}_h\right),\sum_{r=1}^{n}T_{n-1,x,r}\right.$$

$$\left.\left(\left(\bigotimes_{h=1}^{r-1}\left(x^{(+)}\left(\,\cdot\,,t_n\right)g_h\right)\right)\otimes\left(x^{(-)}\left(\,\cdot\,,t_n\right)g_n\right)\otimes\bigotimes_{h=r+1}^{n-1}\left(x^{(-)}\left(\,\cdot\,,t_n\right)g_h\right)\right)\right\rangle_{n-1}$$

and this proves the statement. $\square$

5. The Limit of the Quantum Moments

In this section we identify the limit of the quantum moments (2.3) of the partial sums with the corresponding quantum moments in the deformed monotone space. With the scalar products defined in section 4.3,

$$\mathbb{C} \oplus L^2\left([0,1]\right) \oplus \left\{ L^2\left([0,1]^2\right), \langle \cdot, \cdot \rangle_2 \right\} \oplus \left\{ L^2\left([0,1]^3\right), \langle \cdot, \cdot \rangle_3 \right\} \oplus \cdots =: \bigoplus_{n=0}^{\infty} \mathcal{H}_n \quad (5.1)$$

Recall that the creation operator with test function $f \in L^2\left([0,1]\right)$ is defined by linear extension of

$$a^+\left(f\right) F := f \otimes F \quad ; \quad \forall n, \forall F \in L^2\left([0,1]^n\right)$$

and the annihilation operator with the test function $f \in L^2\left([0,1]\right)$ by

$$a\left(f\right) := \left(a^+\left(f\right)\right)^* \quad \text{on} \bigoplus_{n=1}^{\infty} \mathcal{H}_n \quad \text{and } a\left(f\right)\left(\mathcal{H}_0\right) = \{0\}$$

With these definitions, one has that

Lemma 5.1. *For any* $f, f_1, \cdots, f_n \in L^2\left([0,1]\right)$

$$a\left(f\right)\left(f_n \otimes \cdots \otimes f_1\right) = \sum_{r=1}^{n} \int_0^1 dt \left(\overline{f} f_r\right)(t)$$

$$\left(f_n x^{(-)}\left(\cdot, t\right)\right) \otimes \cdots \otimes \left(f_{r+1} x^{(-)}\left(\cdot, t\right)\right) \otimes \left(f_{r-1} x^{(+)}\left(\cdot, t\right)\right) \otimes \cdots \otimes \left(f_1 x^{(+)}\left(\cdot, t\right)\right)$$

Proof. The thesis follows from Lemma 4.6. □

Theorem 5.2 (fclt-jw-th2). *For any* $n \in \mathbb{N}$, $\varepsilon \in \{-1,1\}^n$ *and* $\{f_h\}_{h=1}^n \subset \mathcal{L}\left([0,1]\right)$,

$$\lim_{N \to \infty} \left\langle \Phi_{[Nt]}, S_N^{(\varepsilon(1))}\left(f_1\right) \cdots S_N^{(\varepsilon(n))}\left(f_n\right) \Phi_{[Nt]} \right\rangle = \left\langle \Phi, a^{(\varepsilon(1))}\left(f_1\right) \cdots a^{(\varepsilon(n))}\left(f_n\right) \Phi \right\rangle$$

where
– $a^+\left(f\right), a\left(f\right)$ are the creation–annihilation operators defined on the IFS $\Gamma\left(L^2\left([0,1]\right); \{\langle \cdot, \cdot \rangle_n\}_{n \in \mathbb{N}}\right)$ introduced in (5.1) with test function $f \in L^2\left([0,1]\right)$;
– Φ is the vacuum vector.

Proof. We need to prove that for any $n \in \mathbb{N}$, $\varepsilon \in \{-1,1\}_+^{2n}$ and $\{f_h\}_{h=1}^{2n} \subset \mathcal{L}\left([0,1]\right)$,

$$\lim_{N \to \infty} \left\langle \Phi_{[Nt]}, S_N^{(\varepsilon(1))}\left(f_1\right) \cdots S_N^{(\varepsilon(2n))}\left(f_{2n}\right) \Phi_{[Nt]} \right\rangle$$

$$= \left\langle \Phi, a^{(\varepsilon(1))}\left(f_1\right) \cdots a^{(\varepsilon(2n))}\left(f_{2n}\right) \Phi \right\rangle$$

Because of Lemma 3.2, this is equivalent to prove that

$$\left\langle \Phi, a^{(\varepsilon(1))}\left(f_1\right) \cdots a^{(\varepsilon(2n))}\left(f_{2n}\right) \Phi \right\rangle = \sum_{\{(l_h, r_h)\}_{h=1}^n \in PP(2n,\varepsilon)} \int_{[0,1]^n} dt_1 \cdots dt_n \quad (5.2)$$

$$\overline{x}^{\sum_{h,m=1}^n \chi_{(l_h,r_h)}(l_m) \cdot \chi_{(t_h}(t_m)} x^{\sum_{h,m=1}^n \chi_{(l_h,r_h)}(r_m) \cdot \chi_{(t_h}(t_m)} \prod_{h=1}^n \left(\overline{f_{l_h}} f_{r_h}\right)(t_h)$$

(5.2) holds trivially for $n = 1$. Suppose by induction that it holds for $n - 1$. For any given $\varepsilon \in \{-1, 1\}_+^{2n}$ and $\{(p_h, q_h)\}_{h=1}^n \in PP(2n, \varepsilon)$, one knows that $p_n = \max \{k : \varepsilon(k) = -1\}$ and that q_n runs over $\{p_n + 1, \cdots, 2n\}$. Consequently, in virtue of Lemma 5.1,

$$a^{(\varepsilon(p_n))}(f_{p_n}) a^{(\varepsilon(p_n+1))}(f_{p_n+1}) \cdots a^{(\varepsilon(2n))}(f_{2n}) \Phi$$

$$= a(f_{p_n}) a^+(f_{p_n+1}) \cdots a^+(f_{2n}) \Phi = \sum_{q_n=p_n+1}^{2n} \int_0^1 dt \, (\overline{f_{p_n}} f_{q_n})(t)$$

$$a^+\left(f_{p_n+1} x^{(-)}(\cdot, t)\right) \cdots a^+\left(f_{q_n-1} x^{(-)}(\cdot, t)\right) a^+\left(f_{q_n+1} x^{(+)}(\cdot, t)\right) \cdots$$

$$\cdots a^+\left(f_{2n} x^{(+)}(\cdot, t)\right) \Phi$$

So $\left\langle \Phi, a^{(\varepsilon(1))}(f_1) \cdots a^{(\varepsilon(2n))}(f_{2n}) \Phi \right\rangle$ is equal to

$$\sum_{r_n=l_n+1}^{2n} \int_0^1 dt \, (\overline{f_{p_n}} f_{q_n})(t) \left\langle \Phi, a^{(\varepsilon(1))}(f_1) \cdots a^{(\varepsilon(2n))}(f_{p_n-1}) \right. \tag{5.3}$$

$$a^+\left(f_{p_n+1} x^{(-)}(\cdot, t)\right) \cdots a^+\left(f_{q_n-1} x^{(-)}(\cdot, t)\right) a^+\left(f_{q_n+1} x^{(+)}(\cdot, t)\right) \cdots$$

$$\left. \cdots a^+\left(f_{2n} x^{(+)}(\cdot, t)\right) \Phi \right\rangle$$

By the induction assumption, the scalar product in (5.3) is equal to the sum, over all pair partitions $\{(l_h, r_h)\}_{h=1}^{n-1}$ of $\{1, \cdots, 2n\} \setminus \{(p_n, q_n)\}$ such that $l_h = p_h$ for any $h = 1, \cdots, n-1$, of integrals of the form

$$\int_{[0,1]^{n-1}} dt_1 \cdots dt_{n-1} \overline{x}^{\sum_{h,m=1}^{n-1} \chi_{(l_h, r_h)}(l_m) \cdot \chi_{(t_h}(t_m)} x^{\sum_{h,m=1}^{n-1} \chi_{(l_h, r_h)}(r_m) \cdot \chi_{t_m)}(t_h)}$$

$$\prod_{1 \leq h \leq n-1; r_h < p_n} (\overline{f_{l_h}} f_{r_h})(t_h) \prod_{1 \leq h \leq n-1; r_h \in (p_n, q_n)} x^{(-)}(t_h, t_n) (\overline{f_{l_h}} f_{r_h})(t_h)$$

$$\prod_{1 \leq h \leq n-1; r_h > q_n} x^{(+)}(t_h, t_n) (\overline{f_{l_h}} f_{r_h})(t_h)$$

So, writing $\{(l_h, r_h)\}_{h=1}^{n-1} \cup \{(p_n, q_n)\}$ as $\{(l_h, r_h)\}_{h=1}^n$ (i.e. $l_n := p_n, r_n := q_n$), one has

$$\left\langle \Phi, a^{(\varepsilon(1))}(f_1) \cdots a^{(\varepsilon(2n))}(f_{2n}) \Phi \right\rangle = \sum_{\{(l_h, r_h)\}_{h=1}^n \in PP(2n, \varepsilon)} \int_{[0,1]^n} dt_1 \cdots dt_n$$

$$\overline{x}^{\sum_{h,m=1}^{n-1} \chi_{(l_h, r_h)}(l_m) \cdot \chi_{(t_h}(t_m)} x^{\sum_{h,m=1}^{n-1} \chi_{(l_h, r_h)}(r_m) \cdot \chi_{(t_h}(t_m)}$$

$$\prod_{h=1}^{n-1} (\overline{f_{l_h}} f_{r_h})(t_h) \prod_{1 \leq h \leq n-1; r_h \in (l_n, r_n)} x^{(-)}(t_h, t_n) \prod_{1 \leq h \leq n-1; r_h > r_n} x^{(+)}(t_h, t_n)$$

Since $\chi_{t_n)}(t_n) = 0$ and, for any $h \in \{1, \cdots, n\}$, the following equivalences hold:

$$\chi_{(r_n}(r_h) = 1 \iff r_h > r_n \iff r_h > r_n > l_n \geq l_h \iff \chi_{(l_h, r_h)}(r_n) = 1$$

one has

$$\sum_{h=1}^{n-1} \chi_{(r_n}(r_h)\chi_{t_n)}(t_h) = \sum_{h=1}^{n} \chi_{(r_n}(r_h)\chi_{t_n)}(t_h) = \sum_{h=1}^{n} \chi_{(l_h,r_h)}(r_n) \cdot \chi_{t_n)}(t_h)$$

$$\sum_{h=1}^{n-1} \chi_{(l_n,r_n)}(r_h)\chi_{t_h)}(t_n) = \sum_{h=1}^{n} \chi_{(r_n}(r_h)\chi_{t_h)}(t_n) = \sum_{h=1}^{n} \chi_{(l_n,r_n)}(r_h)\chi_{t_h)}(t_n)$$

Therefore the definition (4.13) of $x^{(\pm)}$ implies that

$$\prod_{1 \le h \le n-1; r_h \in (l_n,r_n)} x^{(-)}(t_h, t_n) \prod_{1 \le h \le n-1; r_h > r_n} x^{(+)}(t_h, t_n) \tag{5.4}$$

$$= \prod_{1 \le h \le n-1; r_h \in (l_n,r_n)} \overline{x}^{\chi_{t_n)}(t_h)} x^{\chi_{t_h)}(t_n)} \prod_{1 \le h \le n-1; r_h > r_n} \overline{x}^{\chi_{t_n)}(t_h)} x^{\chi_{t_n)}(t_h)}$$

$$= \overline{x}^{\sum_{h=1}^{n-1} \chi_{(l_n}(r_h)\chi_{t_n)}(t_h)} x^{\sum_{h=1}^{n-1} \chi_{(l_n,r_n)}(r_h)\chi_{t_h)}(t_n) + \sum_{h=1}^{n-1} \chi_{(r_n}(r_h)\chi_{t_n)}(t_h)}$$

Thanks to these facts, one finds that

$$\overline{x}^{\sum_{h,m=1}^{n-1} \chi_{(l_h,r_h)}(l_m) \cdot \chi_{(t_h}(t_m)} x^{\sum_{h,m=1}^{n-1} \chi_{(l_h,r_h)}(r_m) \cdot \chi(t_h,t_m)}$$

$$\prod_{1 \le h \le n-1; r_h \in (l_n,r_n)} x^{(-)}(t_h, t_n) \prod_{1 \le h \le n-1; r_h > r_n} x^{(+)}(t_h, t_n)$$

$$= \overline{x}^{\sum_{h,m=1}^{n-1} \chi_{(l_h,r_h)}(l_m) \cdot \chi_{(t_h}(t_m)} x^{\sum_{h,m=1}^{n-1} \chi_{(l_h,r_h)}(r_m) \cdot \chi_{(t_h}(t_m)}$$

$$\overline{x}^{\sum_{h=1}^{n} \chi_{(l_n}(r_h) \cdot \chi_{t_n)}(t_h)} x^{\sum_{h=1}^{n} \chi_{(l_n,r_n)}(r_h) \cdot \chi_{t_h)}(t_n) + \sum_{h=1}^{n} \chi_{(l_h,r_h)}(r_n) \cdot \chi_{t_n)}(t_h)}$$

$$= \overline{x}^{\sum_{h,m=1}^{n-1} \chi_{(l_h,r_h)}(l_m) \cdot \chi_{(t_h}(t_m)} x^{\sum_{h,m=1}^{n-1} \chi_{(l_h,r_h)}(r_m) \cdot \chi_{(t_h}(t_m)}$$

$$\overline{x}^{\sum_{m=1}^{n} \chi_{(l_n,r_n)}(l_m) \cdot \chi(t_n,t_m) + \sum_{h=1}^{n} \chi_{(l_h,r_h)}(l_n) \cdot \chi_{t_n)}(t_h)}$$

$$x^{\sum_{m=1}^{n} \chi_{(l_n,r_n)}(r_m) \cdot \chi(t_n,t_m) + \sum_{h=1}^{n} \chi_{(l_h,r_h)}(r_n) \cdot \chi_{t_n)}(t_h)}$$

$$= \overline{x}^{\sum_{h,m=1}^{n} \chi_{(l_h,r_h)}(l_m) \cdot \chi_{(t_h}(t_m)} x^{\sum_{h,m=1}^{n} \chi_{(l_h,r_h)}(r_m) \cdot \chi_{(t_h}(t_m)}$$

and this is (5.2). □

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