COMPLEMENTARY PERFECT DOMINATION NUMBER AND CHROMATIC NUMBER OF A GRAPH

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ABSTRACT: *A subset S of V of a non-trivial graph G is said to be complementary perfect dominating set, if S is a dominating set and <V – S> has a perfect matching. The minimum cardinality taken over all complementary perfect dominating sets is called complementary perfect domination number and is denoted by* $\gamma_{cp.}$ *The minimum number of colours required to colour all the vertices of G in such a way that adjacent vertices do not receive the same colour is the chromatic number* χ *of G. In this paper, we find an upper bound for sum of these two parameters and characterize the corresponding extremal graphs of order upto 2n-5.*

Keywords: *Complementary Perfect domination number, Chromatic number.*

AMS Subject classification: *05C.*

1. INTRODUCTION

Let $G = (V, E)$ be a simple undirected graph. The degree of any vertex u in G is the number of edges incident with u and is denoted by $d(u)$. The maximum degree of a vertex is denoted by $\Delta(G)$. The path on n vertices is denoted by P_n. The vertex connectivity $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected graph.

A subset S of V is called a dominating set in G if every vertex in $V – S$ is adjacent to at least one vertex in S. The minimum cardinality taken over all dominating sets in G is called the *domination number* of G and is denoted by γ. The dominating set is called *total* if the induced subgraph $< S >$ has no isolated vertices and *connected* if $< S >$ is connected. The minimum cardinality taken over all total (connected) dominating sets in G is called *total (connected)* domination number of G and is denoted by γ_t (γ_c). The concept of complementary perfect domination number with applications was introduced by Paulraj Joseph J., Mahadevan G. and Selvam A. [5]

A subset S of V of a non-trivial graph G is said to be complementary perfect dominating set, if S is a dominating set and $< V – S >$ has a perfect matching. The minimum cardinality taken over all complementary perfect dominating sets is called complementary perfect domination number and is denoted by $\gamma_{cp.}$

Several authors have studied the problem of obtaining an upper bound for the sum of a domination parameter and a graph theoretic parameter and characterized the corresponding extremal graphs. In [7], Paulraj Joseph and Arumugam proved that $\gamma + \kappa \leq p$. In [8], Paulraj Joseph and Arumugam proved that $\gamma_c + \chi \leq p + 1$. They also characterized the class of graphs for which the upper bound is attained. They also proved similar results for γ and γ_t. In [6], Paulraj Joseph J. and Mahadevan G. proved that $\gamma_{cc} + \chi \leq 2n - 1$ and characterized the corresponding extremal graphs.

Previous Results

Theorem 1.1[5]: For any graph G, $\gamma_{cp}(G) \leq n - 2$.

Theorem 1.2[5]: For any graph G, $\gamma_{cp}(G) = n$ if and only if G is a star.

Theorem 1.3[1]: For any graph G, $\chi(G) = \Delta(G)$ or $\Delta(G) + 1$.

Theorem 1.4[1]: If G is k-critical, then $\delta(G) \geq k - 1$.

2. MAIN RESULTS

Theorem 2.1: For any connected graph G, $\gamma_{cp} + \chi \leq 2n$, and equality holds if and only if G is isomorphic to K_2.

Proof: Clearly for any graph G, $\gamma_{cp} \leq n$. Also for any graph G, $\chi \leq \Delta + 1$. Hence $\gamma_{cp} + \chi \leq n + (\Delta + 1) = n + (n - 1 + 1) = 2n$. Now assume that $\gamma_{cp} + \chi = 2n$. This is possible only if $\gamma_{cp} = n$ and $\chi = n$. Since $\gamma_{cp} = n$, by theorem [1.2] G is a Star. Since $\chi = n$, G is K_2. Converse is obvious.

Theorem 2.2: For any connected graph G, $\gamma_{cp} + \chi = 2n - 1$ if and only if G is isomorphic to P_3.

Proof: Assume that $\gamma_{cp} + \chi = 2n - 1$. This is possible only if $\gamma_{cp} = n$ and $\chi = n - 1$ (or) $\gamma_{cp} = n - 1$ and $\chi = n$.

Case 1: $\gamma_{cp} = n$ and $\chi = n - 1$.

Since $\gamma_{cp} = n$, by theorem [1.2] G is a star. Since $\chi = n - 1$, $2 = n - 1$ so that $n = 3$. Hence $G \cong K_{1,2} = P_3$.

Case 2: $\gamma_{cp} = n - 1$ and $\chi = n$.

Since $\gamma_{cp} = n - 1$, there exists a complementary perfect dominating set S with $n - 1$ elements. Hence $< V - S >$ has isolate, which is a contradiction. Hence no graph exists. Converse is obvious.

Theorem 2.3: For any connected graph G, $\gamma_{cp} + \chi = 2n - 2$ if and only if G is isomorphic to K_3, K_4, $K_{1,3}$.

Proof: Assume that $\gamma_{cp} + \chi = 2n - 2$. This is possible only if $\gamma_{cp} = n$ and $\chi = n - 2$ (or) $\gamma_{cp} = n - 1$ and $\chi = n - 1$ (or) $\gamma_{cp} = n - 2$ and $\chi = n$.

The case that $\gamma_{cp} = n - 1$ and $\chi = n - 1$ is not possible.

Case 1: $\gamma_{cp} = n$ and $\chi = n - 2$.

Since $\gamma_{cp} = n$, by theorem [1.2] G is a star. Since $\chi = n - 2$, $2 = n - 2$ so that $n = 4$. Hence $G \cong K_{1,3}$.

Case 2: $\gamma_{cp} = n - 2$ and $\chi = n$.

Since $\chi = n$, $G \cong K_n$. If G has even number of vertices, then $\gamma_{cp} = 2$ so that $n = 4$. Hence $G \cong K_4$. If G has odd number of vertices then $\gamma_{cp} = 1$ so that $n = 3$. Hence $G \cong K_3$.

Theorem 2.4: For any connected graph G, $\gamma_{cp} + \chi = 2n - 3$ if and only if G is isomorphic to $K_{1,4}$, G_1 or G_2 given in Figure 2.1.

Proof: Assume that $\gamma_{cp} + \chi = 2n - 3$. This is possible only if $\gamma_{cp} = n$ and $\chi = n - 3$ (or) $\gamma_{cp} = n - 1$ and $\chi = n - 2$ (or) $\gamma_{cp} = n - 2$ and $\chi = n - 1$ (or) $\gamma_{cp} = n - 3$ and $\chi = n$. The cases $\gamma_{cp} = n - 1$ and $\chi = n - 2$ (or) $\gamma_{cp} = n - 3$ and $\chi = n$ are not possible.

Case 1: $\gamma_{cp} = n$ and $\chi = n - 3$.

Since $\gamma_{cp} = n$, by theorem [1.2], G is a star. Since $\chi = n - 3$, $2 = n - 3$ so that $n = 5$. Hence $G \cong K_{1,4}$.

Case 2: $\gamma_{cp} = n - 2$ and $\chi = n - 1$.

Since $\chi = n - 1$, G contains a clique K on $n - 1$ vertices. Let x be a vertex other than the vertices of K_{n-1}. Since G is connected, x is adjacent to at least one vertex say u_i of K_{n-1}.

If the clique K_{n-1} has even number of vertices, then $\{x, u_i, u_j\}$ for some u_j in K_{n-1} forms a γ_{cp}-set of G. Since $\gamma_{cp} = n - 2$, we have $n = 5$. Hence $K = K_4$. Let u_1, u_2, u_3, u_4 be the vertices of K_4. Let x be adjacent to u_1. If $d(x) = 1$, then $G \cong G_1$.

If x is adjacent to one more vertex say u_j of K_{n-1}, then $\{u_j\}$ is a γ_{cp}-set, which is a contradiction.

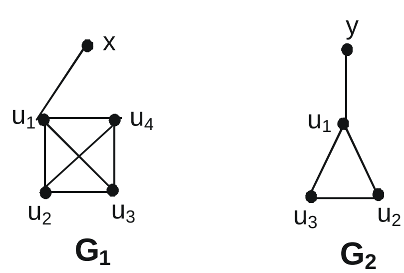

Figure 2.1

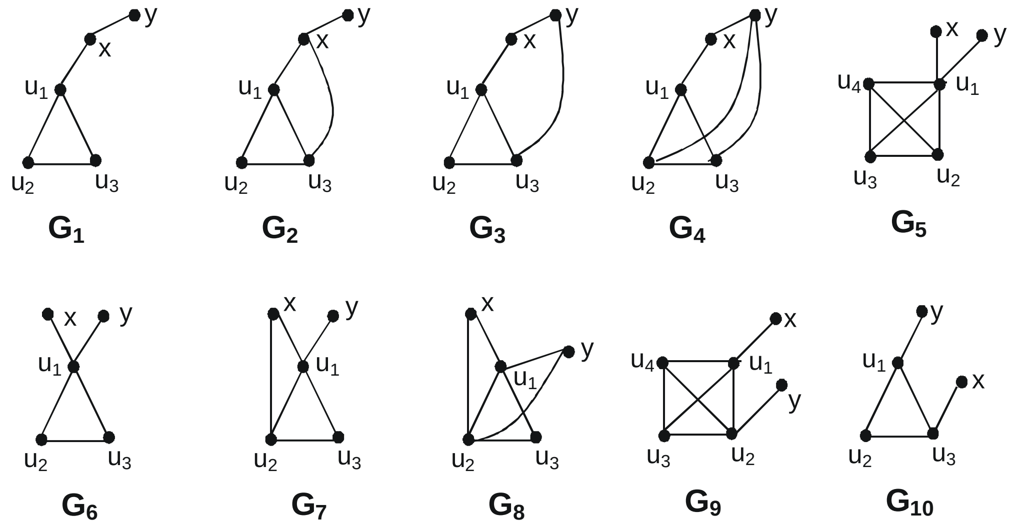

Figure 2.2

If the clique K has an odd number of vertices, then $\{x, u_i\}$ is a γ_{cp}-set of G. Since $\gamma_{cp} = n - 2$, we have $n = 4$. Hence $K = K_{3.}$ Let u_1, u_2, u_3 be the vertices of K_3. Let x be adjacent to u_1. If $d(x) = 1$, then $G \cong G_2$. If x is adjacent to one more vertex say u_j in K_{n-1}, then $\{u_i\}$ is a a γ_{cp}-set which is a contradiction.

Theorem 2.5: For any connected graph G, $\gamma_{cp} + \chi = 2n - 4$ if and only if G is isomorphic to K_5, K_6, $K_{1,5}$, P_4, C_4, or any one of the graphs G_1 to G_{10} given in Figure 2.2.

Proof: Assume that $\gamma_{cp} + \chi = 2n - 4$. This is possible only if $\gamma_{cp} = n$ and $\chi = n - 4$ (or) $\gamma_{cp} = n - 1$ and $\chi = n - 3$ (or) $\gamma_{cp} = n - 2$ and $\chi = n - 2$ (or) $\gamma_{cp} = n - 3$ and $\chi = n - 1$ (or) $\gamma_{cp} = n - 4$ and $\chi = n$. The cases for which $\gamma_{cp} = n - 1$ and $\chi = n - 3$ (or) $\gamma_{cp} = n - 3$ and $\chi = n - 1$ are not possible.

Case 1: $\gamma_{cp} = n$ and $\chi = n - 4$.

Since $\gamma_{cp} = n$, by theorem [1.2] G is a star. Since $\chi = n - 4$, $2 = n - 4$ so that $n = 6$. Hence $G \cong K_{1,5}$.

Case 2: $\gamma_{cp} = n - 2$ and $\chi = n - 2$.

Since $\chi = n - 2$, G contains a clique K on $n - 2$ vertices. Let $S = \{x, y\} = V(G) - V(K)$. Then $\langle S \rangle = K_2$ or K_2.

Subcase 1: $\langle S \rangle = K_2$

Since G is connected, there exists a vertex say u_i in K_{n-2} which is adjacent to x (or equivalently y).

Now, Assume that the clique K_{n-2} has even number of vertices.

Then $\{y, u_j\}$ for $i \neq j$ in K_{n-2} forms a γ_{cp}-set of G. Since $\gamma_{cp} = n - 2$, we have $n = 4$. Hence $K = K_2 = uv$. Let x be adjacent to u. If $d(x) = 2$ and $d(y) = 1$, then $G \cong P_4$. If $d(x) = 3$, then $\chi = 3$, which is a contradiction.

Now let $d(x) = d(y) = 2$.

Without loss of generality let x be adjacent to u. Then y is adjacent to u or v. If y is adjacent to u, then $\chi = 3$, which is a contradiction. If y is adjacent to v, then $G \cong C_4$.

Now assume that the clique K has odd number of vertices.

Then $\{y, x, u_i\}$ forms a γ_{cp}-set of G. Since $\gamma_{cp} = n - 2$, we have $n = 5$. Hence $K = K_3$. Let u_1, u_2, u_3 be the vertices of K_3. Without loss of generality let x be adjacent to u_1. If $d(x) = 2$ and $d(y) = 1$, then $G \cong G_1$. Let $d(x) = 3$ and $d(y) = 1$. Without loss of generality, let x be adjacent to both u_1 and u_3. Then $G \cong G_2$. If $d(x) = 4$ and $d(y) = 1$, then $\chi = 4$, which is a contradiction. Let $d(x) = d(y) = 2$. Let x be adjacent to u_1. Then y is adjacent to u_1 or u_3 (or equivalently u_2). If y is adjacent to u_1, then $\{u_1\}$ is a γ_{cp} set which is a contradiction. If y is adjacent to u_3, then $G \cong G_3$. Let $d(x) = 2$ and $d(y) = 3$. Let x be adjacent to u_1. Then y is adjacent to u_1 and one of $\{u_2, u_3\}$ (or) y is adjacent to both u_2 and u_3. If y is adjacent to u_1 and u_2, then $\{u_1\}$ is a γ_{cp} set which is a contradiction. If y is adjacent to u_2 and u_3, then $G \cong G_4$. Let $d(x) = 2$ and $d(y) = 4$, then $c = 4$, which is a contradiction. Let $d(x) = d(y) = 3$. Without loss of generality, let x be adjacent to u_1 and u_2. Then y is adjacent to u_1 and u_2 (or) y is adjacent to u_3 and u_1 (or equivalently u_2). If y is adjacent to u_1 and u_2, then $\chi = 4$, which is a contradiction. If y is adjacent to u_3 and u_1, then $\{u_1\}$ is a γ_{cp}-set, which is a contradiction.

Subcase 2: $< S > = \overline{K}_2$.

Since G is connected, x and y are adjacent to a common vertex or distinct vertices of K_{n-2}.

Subcase 2(a): Let x and y be adjacent to a common vertex say u_i of K_{n-2}.

Now, Assume that the clique K_{n-2} has even number of vertices.

Then $\{x, y, u_i, u_j\}$ for $i \neq j$ forms a γ_{cp}-set of G. Since $\gamma_{cp} = n - 2$, we have $n = 6$. Hence $K = K_4$. Let u_1, u_2, u_3, u_4 be the vertices of K_4. Let u_1 be adjacent to both x and y. If $d(x) = d(y) = 1$, then $G \cong G_5$. Let $d(x) = 2$ and $d(y) = 1$. then $\{y, u_i\}$ forms a γ_{cp}-set of G, which is a contradiction.

Now assume that the clique K has odd number of vertices. Then $\{x, y, u_i\}$ forms a γ_{cp}-set of G. Since a $\gamma_{cp} = n - 2$, we have $n = 5$. Hence $K = K_3$. Let u_1, u_2, u_3 be the vertices of K_3. Let u_1 be adjacent to both x and y. If $d(x) = d(y) = 1$, then $G \cong G_6$. Let $d(x) = 2$ and

$d(y) = 1$. Without loss of generality, let x be adjacent to u_1 and u_2, then $G \cong G_7$. If $d(x) = 3$ and $d(y) = 1$, then $\chi = 4$, which is a contradiction. Let $d(x) = d(y) = 2$. Without loss of generality, let x be adjacent to u_1 and u_2. Then y is adjacent to u_2 or u_3. If y is adjacent to u_2, then $G \cong G_8$. If y is adjacent to u_3, then $\{u1\}$ is a γ_{cp}-set which is a contradiction.

Subcase 2(b): Let x and y are adjacent to distinct vertices of K_{n-2}.

Let x be adjacent to u_i and y is adjacent to u_j for $i \neq j$.

Assume that clique K_{n-2} has even number of vertices.

Then $\{x, y, u_i, u_j\}$ for $i \neq j$ forms a γ_{cp}-set of G. Since $\gamma_{cp} = n - 2$, we have $n = 6$. Hence $K = K_4$. Let u_1, u_2, u_3, u_4 be the vertices of K_4. Without loss of generality, let x_1 be adjacent to u_1 and x_2 be adjacent to u_2. If $d(x) = d(y) = 1$, then $G \cong G_9$. Let $d(x) = 2$ and $d(y) = 1$.Then x is adjacent to u_2 or u_3 (or equivalently to u_4). If x is adjacent to u_2, then $\{y, u_2\}$ forms a γ_{cp}-set of G, which is a contradiction. If x is adjacent to u_3 (or equivalently u_4), then $\{y, u_4\}$ forms a γ_{cp}-set of G. If $d(x) = d(y) = 2$, then no graph exists.

Now assume that the clique K has odd number of vertices. Then $\{x, y, u_i\}$ forms a γ_{cp}-set of G. Since $\gamma_{cp} = n - 2$, we have $n = 5$. Hence $K = K_3$. Let u_1, u_2, u_3 be the vertices of K_3. Without loss of generality, let x be adjacent to u_1 and y be adjacent to u_2. If $d(x) = d(y) = 1$, then $G \cong G_{10}$. Let $d(x) = 2$ and $d(y) = 1$. Then x is adjacent to u_2 or u_3. If x is adjacent to u_3, then $G \cong G_{12}$; If x is adjacent to u_2, then $G \cong G_7$. If $d(x) = 3$ and $d(y) = 1$, then $\chi = 4$, which is a contradiction. If $d(x) = d(y) = 2$, let x be adjacent to u_1 and u_2. Then y is adjacent to u_1 or u_3. If y is adjacent to u_1, then $G \cong G_8$; If y is adjacent to u_3, then $\{u_2\}$ forms a γ_{cp}-set of G, which is a contradiction. Let x be adjacent to u_1 and u_3. Then y is adjacent to u_1 or u_3.

If y is adjacent to u_1, then $\{u_1\}$ forms a γ_{cp}-set of G, which is a contradiction; If y is adjacent to u_3, then $\{u_3\}$ forms a γ_{cp}-set of G, which is a contradiction.

Case 3: $\gamma_{cp} = n - 4$ and $\chi = n$.

Since $\chi = n$, G is K_n. If K_n has even number of vertices, then $\gamma_{cp} = 2$ and hence $n = 6$. Hence $G \cong K_6$. If K_n has odd number of vertices then $\gamma_{cp} = 1$ and hence $n = 5$. Hence $G \cong K_5$.

Theorem 2.4: For any connected graph G, $\gamma_{cp} + \chi = 2n - 5$ if and only if G is isomorphic to K_6, K_7, $K_{1,6}$ or any one of the graphs G_1 to G_{33} given in Figure 2.3.

Proof: If G is any of the graphs given in figure 6.5, then clearly $\gamma_{cp} + \chi = 2n - 5$. Conversely, assume that $\gamma_{cp} + \chi = 2n - 5$. This is possible only if $\gamma_{cp} = n$ and $\chi = n - 5$ (or) $\gamma_{cp} = n - 1$ and $\chi = n - 4$ (or) $\gamma_{cp} = n - 2$ and $\chi = n - 3$ (or) $\gamma_{cp} = n - 3$ and $\chi = n - 2$ (or) $\gamma_{cp} = n - 4$ and $\chi = n - 1$ (or) $\gamma_{cp} = n - 5$ and $\chi = n$. The cases for which $\gamma_{cp} = n - 1$ and $\chi = n - 4$ (or) $\gamma_{cp} = n - 3$ and $\chi = n - 2$ are not possible.

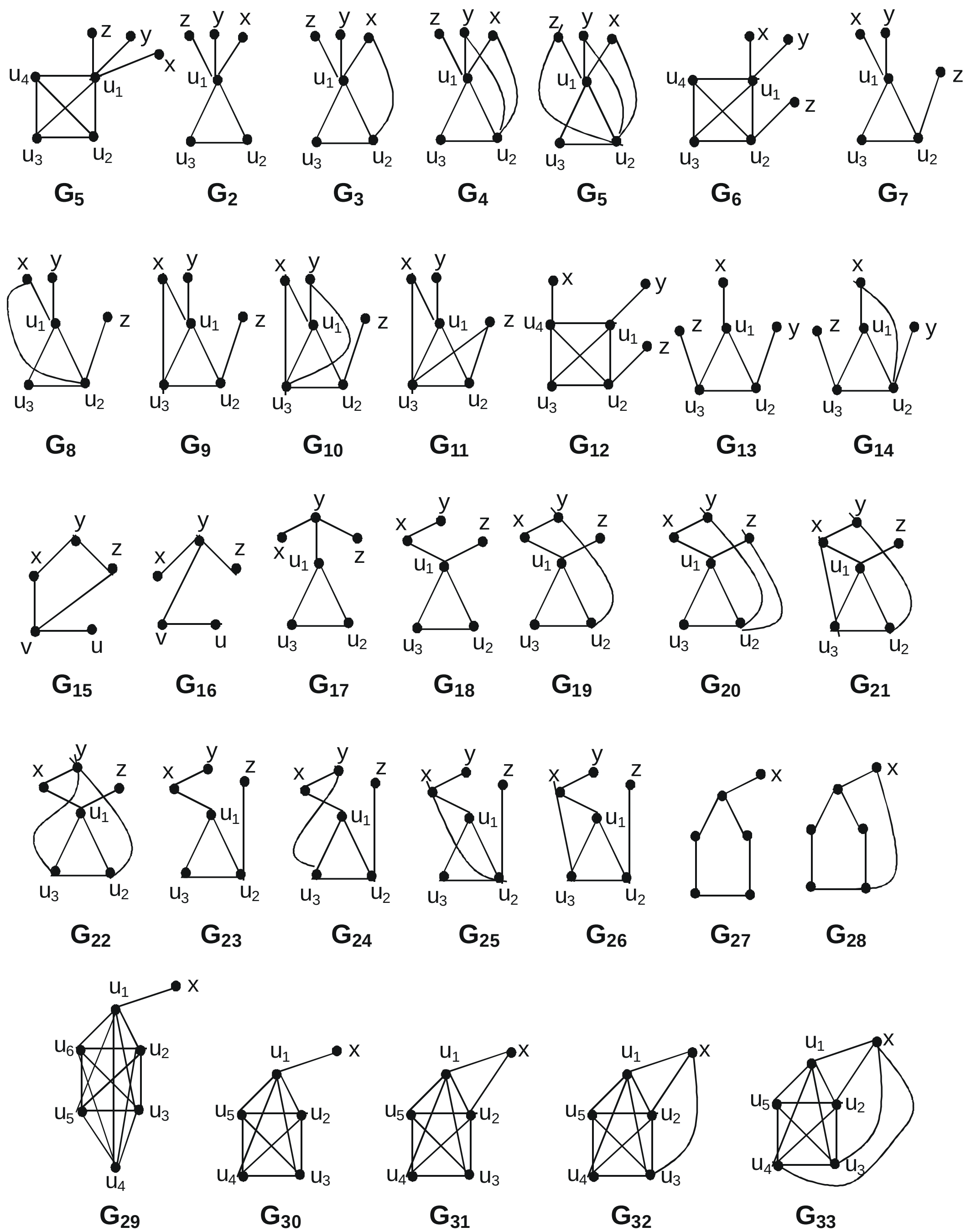
G5 G2 G3 G4 G5 G6 G7
G8 G9 G10 G11 G12 G13 G14
G15 G16 G17 G18 G19 G20 G21
G22 G23 G24 G25 G26 G27 G28
G29 G30 G31 G32 G33

Figure 2.3

Case 1: $\gamma_{cp} = n$ and $\chi = n - 5$.

Since $\gamma_{cp} = n$, by theorem [1.2] G is a star. Since $\chi = n - 5$, $2 = n - 5$ so that $n = 7$. Hence $G \cong K_{1,6}$

Case 2: $\gamma_{cp} = n - 2$ and $\chi = n - 3$.

Then G contains a clique K_{n-3} , or G contains no $K_{n-3.}$

Let G contains a clique $K_{n-3.}$

Let $S = \{x, y, z\} = V(G) - V(K)$. Then $< S > = K_3$ or K_3 or P_3 or $K_2 \cup K_1$.

Subcase: $1 < S > = K_3$.

Since G is connected, x is adjacent to some u_i in K_{n-3}.

If K_{n-3} has even number of vertices, then $\{x, u_i, u_j\}$ for $i \neq j$ in K_{n-3} forms a γ_{cp}-set of G. Since $\gamma_{cp} = n - 2$, we have $n = 5$. But $\chi = n - 3 = 2$, which is a contradiction. Hence no graph exists in this case.

If K_{n-3} has odd number of vertices, then $\{x, u_i\}$ is a γ_{cp}-set of G. Since a $\gamma_{cp} = n - 2$, we have $n = 4$. But $\chi = n - 3 = 1$, which is a contradiction. Hence no graph exists in this case.

Subcase 2: $< S > = \overline{K}_3$.

Since G is connected, one of the vertices of K_{n-3} is adjacent to all the vertices of S or two vertices of S or one vertex of S.

Subcase 2(a): Let u_i for some i in K_{n-3} be adjacent to all the vertices of S.

Now assume that the clique K_{n-3} has even number of vertices.

Then $\{x, y, z, u_i, u_j\}$ is a γ_{cp}-set of G. Since $\gamma_{cp} = n - 2$, we have $n = 7$. Hence $K = K_4$. Let u_1, u_2, u_3, u_4 be the vertices of K_4. Let u_1 be adjacent to all of S. If $d(x) = d(y) = d(z) = 1$, then G is $G \cong G_{1.}$

Let $d(x) = 2$, $d(y) = d(z) = 1$. Let x be adjacent to u_2. Then $\{y, z, u_2\}$ forms a γ_{cp}-set of G, which is a contradiction.

Now assume that K_{n-3} has odd number of vertices.

Then $\{x, y, z, u_i\}$ forms a γ_{cp}-set of G. Since $\gamma_{cp} = n - 2$, we have $n = 6$. Hence $K = K_3$. Let u_1, u_2, u_3 be the vertices of K_3. Let u_1 be adjacent to all the vertices of S.

If $d(x) = d(y) = d(z) = 1$, then $G \cong G_2$.

If $d(x) = 2$, $d(y) = d(z) = 1$, then $G \cong G_3$.

If $d(x) = 3$, $d(y) = d(z) = 1$, then $\chi = 4$, which is a contradiction.

Let $d(x) = d(y) = 2$ and $d(z) = 1$.

Now, let x be adjacent to u_1 and u_2. Then y is adjacent to u_2 or u_3. If y is adjacent to u_2, then $G \cong G_4$. If y is adjacent to u_3, then $\{z, u_1\}$forms a γ_{cp}-set of G, which is a contradiction.

Now, let $d(x) = d(y) = d(z) = 2$.

If $d(x) = d(y) = 2$ and $d(z) = 1$, then the graph is G_4. Now in G_4, z is adjacent to u_3 or u_2. If z is adjacent to u_3, then $\{y, u_1\}$ forms a γ_{cp}-set, which is a contradiction. If z is adjacent to u_2, then $G \cong G_5$.

Subcase 2(b): Let u_i for some i in K_{n-3} is adjacent to x and y, and u_j for some $i \neq j$ in K_{n-3} is adjacent to z.

Now assume that K_{n-3} has even number of vertices.

Then $\{x, y, z, u_i, u_j\}$ forms a γ_{cp}-set of G so that $n = 7$. Hence $K = K_4$. Let u_1, u_2, u_3, u_4 be the vertices of K_4. Let u_1 be adjacent to x and y and let u_2 be adjacent to z. If $d(x) = d(y) = d(z) = 1$, then $G \cong G_6$.

Let $d(x) = 2$, $d(y) = d(z) = 1$.

If $d(x) = d(y) = d(z) = 1$, then the graph is G_6. In G_6, x is adjacent to u_2 or u_3 (or equivalently u_4). If x is adjacent to u_2 then $\{y, z, u_2\}$ forms a γ_{cp}-set of G, which is a contradiction; if x is adjacent to u_3, then $\{y, z, u_3\}$ forms a γ_{cp}-set of G which is a contradiction.

Let $d(x) = d(y) = 1$ and $d(z) = 2$.

If $d(x) = d(y) = d(z) = 1$, then the graph is G_6. In G_6, z is adjacent to u_1 or u_3 (or equivalently u_4). If z is adjacent to u_1 then $\{x, y, u_1\}$ forms a γ_{cp}-set of G, which is a contradiction; if z is adjacent to u_3, then $\{x, y, u_3\}$ forms a γ_{cp}-set of G which is a contradiction.

Now assume that K_{n-3} has odd number of vertices.

Then $\{x, y, z, u_i\}$ forms a γ_{cp}-set of G and hence $n = 6$. Hence $K = K_3$. Let u_1, u_2, u_3 be the vertices of K_3. Let u_1 be adjacent to both x and y and let u_2 be adjacent to z. If $d(x) = d(y) = d(z)= 1$, then $G \cong G_7$.

Let $d(x) = 2$, and $d(y) = d(z) = 1$.

Now, if $d(x) = d(y) = d(z) = 1$, then the graph is G_7. In G_7, x is adjacent to u_2 or u_3. If x is adjacent to u_2, then $G \cong G_8$; if x is adjacent to u_3, then $G \cong G_9$.

If $d(x) = 3$ and $d(y) = d(z) = 1$. Then $\chi = 4$, which is a contradiction.

Now Let $d(x) = d(y) = 2$ and $d(z) = 1$.

If $d(x) = 2$, and $d(y) = d(z) = 1$, then the graphs are G_8 or G_9. Now in G_8, y is adjacent to u_2 or u_3. If y is adjacent to u_2, then $G \cong G_4$; If y is adjacent to u_3, then $\{z, u_1\}$ forms a γ_{cp}-set of G, which is a contradiction. In G_9, y is adjacent to u_2 or u_3. If y is adjacent to u_2, then $\{z, u_1\}$ forms a γ_{cp}-set of G, which is a contradiction; If y is adjacent to u_3, then $G \cong G_{10}$.

Now Let $d(x) = d(y) = d(z) = 2$.

If $d(x) = d(y) = 2$ and $d(z) = 1$, then the graph is G_4 or G_{10}. Now in G_4, z is adjacent to u_2 or u_3. If z is adjacent to u_2, then $G \cong G_5$; If z is adjacent to u_3, then $\{y, u_1\}$ forms a γ_{cp} set of G, which is a contradiction. In G_{10}, z is adjacent to u_1 or u_3. If z is adjacent to u_1, then $\{u_1, y\}$ forms a γ_{cp} set of G, which is a contradiction. If z is adjacent to u_3, then $\{x, u_3\}$ forms a γ_{cp} set of G, which is a contradiction.

Now let $d(x) = d(y) = 1$ and $d(z) = 2$.

If $d(x) = d(y) = d(z) = 1$, then the graph is G_7. In G_7, z is adjacent to u_1 or u_3. If z is adjacent to u_1, then $G \cong G_3$; If z is adjacent to u_3, then $G \cong G_{11}$.

Now Let $d(x) = 2$, $d(y) = 1$ and $d(z) = 2$.

Now if $d(x) = 2$, $d(y) = d(z) = 1$, then the graphs are G_8 or G_9. In G_8, z is adjacent to u_1 or u_3. If z is adjacent to u_1, then $G \cong G_4$; If z is adjacent to u_3, then $\{y, u_2\}$ forms a γ_{cp} set of G, which is a contradiction. In G_9, z is adjacent to u_1 or u_3. If z is adjacent to u_1, then $\{y, u_1\}$ forms a γ_{cp} set of G, which is a contradiction. If z is adjacent to u_3, then $\{y, u_3\}$ forms a γ_{cp} set of G, which is a contradiction.

If let $d(x) = d(y) = 1$ and $d(z) = 3$. Then $\chi = 4$, which is a contradiction.

Subcase 2(c): Let u_i be adjacent to x and u_j for $i \neq j$ be adjacent to y and u_k for $i \neq j \neq k$ be adjacent to z.

Assume that the Clique K_{n-3} has even number of vertices.

Then $\{x, y, z, u_i, u_j\}$ forms a γ_{cp}-set of G so that $n = 7$. Hence $K = K_4$. Let u_1, u_2, u_3, u_4 be the vertices of K_4. Let u_1 be adjacent to x and u_2 be adjacent to y and u_3 be adjacent to z.

If $d(x) = d(y) = d(z) = 1$, then $G \cong G_{12}$.

Let $d(x) = 2$ and $d(y) = d(z) = 1$. Then clearly no graph exists satisfying the hypothesis.

Assume that the Clique K_{n-3} has odd number of vertices.

Then $\{x, y, z, u_i\}$ forms a γ_{cp}-set of G and hence $n = 6$. Hence $K = K_3$. Let u_1, u_2, u_3 be the vertices of K_3. Let u_1 be adjacent to x and u_2 be adjacent to y and u_3 be adjacent to z. If $d(x) = d(y) = d(z) = 1$, then $G \cong G_{13}$.

If $d(x) = 2$ and $d(y) = d(z) = 1$, then $G \cong G_{14}$

If $d(x) = 3$ and $d(y) = d(z) = 1$, then $\chi = 4$, which is a contradiction.

If $d(x) = d(y) = 2$ and $d(z) = 1$, then $G \cong G_{10}$.

If $d(x) = d(y) = d(z) = 2$, then no graph exists satisfying the hypothesis.

Subcase 3: $< S > = P_3 = (x\ y\ z)$.

Assume that the clique $K = K_{n-3}$ have even number of vertices.

Since G is connected, atleast one of the vertices say u_i of K_{n-3}, is adjacent to x (or equivalently z) or y.

If u_i is adjacent to x, then $\{z, u_i, u_j\}$ for $i \neq j$ forms a γ_{cp} set of G. Since $\gamma_{cp} = n - 2$, we have $n = 5$. Hence $K = K_2 = uv$. Let x be adjacent to u. If $d(x) = d(y) = 2$ and $d(z) = 1$, then $G \cong P_5$. If $d(x) = 3$, then $\chi = 3$, which is a contradiction. If $d(x) = d(y) = d(z) = 2$, then $G \cong G_{15}$.

If u_i is adjacent to y, then $\{x, z, u_j\}$ for $i \neq j$ forms a γ_{cp} set of G. Since $\gamma_{cp} = n - 2$, $n = 5$. Hence $K = K_2 = uv$. Let u be adjacent to y. If $d(x) = d(z) = 1$ and $d(y) = 3$, then $G \cong G_{16}$. In all other cases on the degrees of the vertices of x, y and z, no new graph exists satisfying the hypothesis.

Assume that the clique K_{n-3} have odd number of vertices.

Since G is connected, at least one of the vertices say u_i, of K_{n-3}, is adjacent to x (or equivalently z) or y.

If u_i is adjacent to x, then $\{z, u_i\}$ forms a γ_{cp}-set of G. Since $\gamma_{cp} = n - 2$, we have $n = 4$. Hence $K = K_1$ which is a contradiction.

If u_i is adjacent to y, then $\{x, z\ u_j, u_k\}$ forms a γ_{cp}-set of G and hence $n = 6$. Hence $K = K_3$. Let u_1, u_2, u_3 be the vertices of K_3.

Let u_1 be adjacent to y.

Let $d(y) = 3$.

If $d(x) = d(z) = 1$, then $G \cong G_{17}$. In all other cases on the degrees of x and z, no new graph exists satisfying the hypothesis.

Let $d(y) = 4$.

If $d(x) = d(z) = 1$, then $G \cong G_{11}$. In all other cases on the degrees of x and z, no new graph exists satisfying the hypothesis.

Let $d(y) = 5$. Then $\chi = 5$, which is a contradiction.

Subcase 4: $< S > = K_2 \cup K_1$.

Let xy be the edge in $< S >$.

Assume that the clique K_{n-3} have even number of vertices.

Since G is connected x (or equivalently y) is adjacent to atleast one of the vertices say u_i of K_{n-3}. Without loss of generality let x be adjacent to u_i. Then z is adjacent to the same u_i or u_j for $i \neq j$.

If z is adjacent to u_i, then $\{y, z, u_j\}$ for $i \neq j$ forms a γ_{cp} set of G and hence $n = 5$. Hence $K = K_2 = uv$. Let u be adjacent to both x and z. If $d(x) = 2$ and $d(y) = d(z) = 1$, then $G \cong G_{16}$; If $d(x) = d(y) = 2$ and $d(z) = 1$, then $G \cong G_{15}$. Let $d(x) = d(y) = d(z) = 2$. Then $\chi = 3$, which is a contradiction.

If z is adjacent to u_j, for $i \neq j$, then $\{y, z, u_j\}$ forms a γ_{cp} set of G and hence $n = 5$. Hence $K = K_2 = uv$. Let x be adjacent to u and z be adjacent to v. If $d(x) = 2$ and $d(y) = d(z) = 1$, then $G \cong P_5$. All other cases on the degrees of x, y and z no new graph exists satisfying the hypothesis.

Assume that the clique K_{n-3} has odd number of vertices.

Since G is connected x (or equivalently y) is adjacent to atleast one of the vertices say u_i of K_{n-3}. Without loss of generality let x be adjacent to u_i. Then z is adjacent to the same u_i or u_j for $i \neq j$.

If z is adjacent to u_i, then $\{y, z, u_j, u_k\}$ for $i \neq j \neq k$ forms a γ_{cp}-set of G and hence $n = 6$. Hence $K = K_3$. Let u_1, u_2, u_3 be the vertices of K_3. Let u_1 be adjacent to both x and z. If $d(x) = 2$ and $d(y) = d(z) = 1$, then $G \cong G_{18}$.

Let $d(x) = d(y) = 2$ and $d(z) = 1$.

If $d(x) = 2$ and $d(y) = d(z) = 1$, then the graph is G_{18}. In G_{18}, y is adjacent to u_1 or u_2 (or equivalently u_3). If y is adjacent to u_1, then $\{z, u_1\}$ forms a γ_{cp}-set of G, which is a contradiction; If y is adjacent to u_2, then $G \cong G_{19}$.

Let $d(x) = d(y) = d(z) = 2$.

Now if $d(x) = d(y) = 2$ and $d(z) = 1$, then the graph is G_{19}. In G_{19}, z is adjacent to u_2 or u_3. If z is adjacent to u_2, then $G \cong G_{20}$; if z is adjacent to u_3, then $\{x, u_1\}$ forms a γ_{cp}-set of G, which is a contradiction.

If $d(x) = d(y) = 2$ and $d(z) = 3$, then $\chi = 4$, which is a contradiction.

If $d(x) = 3$ and $d(y) = d(z) = 1$, then $G \cong G_9$.

Now let $d(x) = 3$, $d(y) = 2$ and $d(z) = 1$.

Now, if $d(x) = d(y) = 2$, $d(z) = 1$, then the graph is G_{19}. In G_{19}, x is adjacent to u_2 or u_3. If x is adjacent to u_2, then $\{z, u_2\}$ forms a γ_{cp}-set of G, which is a contradiction; If x is adjacent to $u_{3,}$ then $G \cong G_{21}$.

If $d(x) = 3$, $d(y) = 2$ and $d(z) = 2$, then no new graph exists.

If $d(x) = 3$, $d(y) = 3$ and $d(z) = 1$, then no new graph exits.

If $d(x) = 2$, $d(y) = 3$ and $d(z) = 1$, then $G \cong G_{22}$.

If $d(x) = 2$, $d(y) = 3$ and $d(z) = 2$, then no new graph exits.

If z is adjacent to u_j, then $\{y, z, u_j , u_k\}$ for $i \neq j \neq k$ forms a γ_{cp} set of G and hence $n = 6$. Hence $K = K_3$. Let u_1, u_2, u_3 be the vertices of K_3. Let u_1 be adjacent to x and u_2 be adjacent to z.

If $d(x) = 2$ and $d(y) = d(z) = 1$, then $G \cong G_{23}$.

Now let $d(x) = d(y) = 2$ and $d(z) = 1$.

Now if $d(x) = 2$ and $d(y) = d(z) = 1$, then the graph is G_{23}. In G_{23}, y is adjacent to u_1 or u_2 or u_3. If y is adjacent to u_1, then $\{u_1, z\}$ is a γ_{cp} set of G which is a contradiction; if y is adjacent to u_2, then $G \cong G_{19}$; If y is adjacent to u_3, then $G \cong G_{24}$.

If $d(x) = d(y) = d(z) = 2$, then no new graph exists satisfying the hypothesis.

Let $d(x) = 3$ and $d(y) = d(z) = 1$.

Now, if $d(x) = 2$ and $d(y) = d(z) = 1$, the graph is G_{23}. In G_{23}, x is adjacent to u_2, or u_3. If x is adjacent to u_2, then $G \cong G_{25}$; If x is adjacent to u_3, then $G \cong G_{26}$.

Let $d(x) = 3$, $d(y) = 2$ and $d(z) = 1$.

Now, if $d(x) = 3$ and $d(y) = d(z) = 1$, then the graphs are G_{25} or G_{26}. In G_{25}, y is adjacent to u_1, or u_2, or u_3. If y is adjacent to u_1, then $\{u_1, z\}$ forms a γ_{cp} set of G, which is a contradiction; If y is adjacent to $u_{2,}$ then $\{z, u_2\}$ forms a γ_{cp}-set of G, which is a contradiction; If y is adjacent to u_3, the $\{z, u_3\}$ forms a γ_{cp} set of G, which is a contradiction. In G_{26}, y is adjacent to u_1 or u_2 or u_3. If y is adjacent to u_1, then $\{u_1, z\}$ forms a γ_{cp} set of G, which is a contradiction. If y is adjacent to u_2, then $\{x, z\}$ forms a γ_{cp}-set of G, which is a contradiction; if y is adjacent to u_3, then $\{z, u_3\}$ forms a γ_{cp}-set of G, which is a contradiction.

If $d(x) = 3$, $d(y) = 2$ and $d(z) = 2$, then no new graph exists.

Now let G contains no K_{n-3}.

Then clearly $n \geq 6$.

If $n = 6$, then $\gamma_{cp} = 4$, and $\chi = 3$ and G contains no K_3. Therefore G contains G, since $\chi = 3$. Let v vertex of C_5 which is not in C_5. Since G is connected and since G contains no K_3, v cannot be adjacent to two adjacent vertices of C_5. i.e., $d(v) = 1$ or 2 and hence the only possible graphs are isomorphic to G_{27} or G_{28}.

If $n \geq 8$, then $\gamma_{cp} = n - 2$ and $\chi \geq 5$ and G contains no K_5. In this case, if S is a γ_{cp}-set of G, then $< S >$ cannot contain K_3 or P_4 (otherwise $\gamma_{cp}(G) \leq n - 2$). Therefore $< S >$ is acyclic and hence $\chi(< S >) = 2$. This implies that $\chi(G) \leq 4$, which is a contradiction.

If $n = 7$, then $\gamma_{cp} = 5$ and $\chi = 4$, G contains no K_4.

If S is a γ_{cp}-set of G, then $< S >$ is any one of the following graphs given in Figure 2.4.

If $< S > \cong H_7$, then $\chi(G) \leq 3$, which is a contradiction.

If $< S > = H_1$ to H_6, then $\chi(< S >) = 2$, and since $\chi(G) = 4$, G contains K_4, which is a contradiction.

Case 3: $\gamma_{cp} = n - 4$ and $\chi = n - 1$.

Since $\chi = n - 1$, G contains a clique K on $n - 1$ vertices. Let x be the vertex other than the vertices of K_{n-1}. Since, G is connected, x is adjacent to at least one of the vertices say u_i of K_{n-1}.

Now assume that the clique K_{n-1} has even number of vertices.

Then $\{x, u_i, u_j\}$ for $i \neq j$ forms a γ_{cp} set of G. Since $\gamma_{cp} = n - 4$, we have $n = 7$. Hence $K = K_6$. Let $u_1, u_2, u_3, u_4, u_5, u_6$ be the vertices of K_6. Let x be adjacent to u_1. If $d(x) = 1$, then $G \cong G_{29}$. If $d(x) = 2$, then $\{u_1\}$ forms a γ_{cp} set of G, which is a contradiction.

Now assume that the clique K_{n-1} has odd number of vertices.

Then $\{x, u_i\}$ forms a γ_{cp}-set of G. Since $\gamma_{cp} = n - 4$, we have $n = 6$. Hence $K = K_5$. Let u_1, u_2, u_3, u_4, u_5, be the vertices of K_5 . If $d(x) = 1$, then $G \cong G_{30}$. If $d(x) = 2$, then $G \cong G_{31.}$ If $d(x) = 3$, then $G \cong G_{32}$. If $d(x) = 4$, then $G \cong G_{33}$.

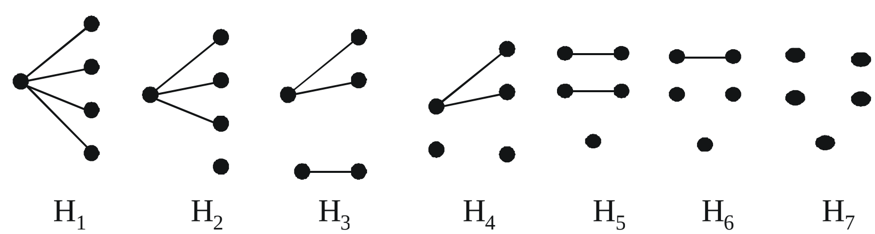

Figure 2.4

Case 4: $\gamma_{cp} = n - 5$ and $\chi = n$.

Since $\chi = n$, G is K_n. If K_n has even number of vertices, then $\gamma_{cp} = 2$ and hence $n = 7$. Hence $G \cong K_7$. If K_n has odd number of vertices then $\gamma_{cp} = 1$ and hence $n = 6$. Hence $G \cong K_6$.

The authors are working similar results for the induced complementary perfect domination number and chromatic number of a graph, which will be reported later.

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