

A DISCRETE TIME APPROXIMATIONS FOR CERTAIN CLASS OF ONE-DIMENSIONAL BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS VIA GIRSANOV'S THEOREM

AISSA SGHIR*, DRISS SEGHIR, AND SOUKAINA HADIRI

ABSTRACT. In this paper, we give a discrete time approximations and Monte Carlo simulations for certain classes of one-dimensional backward stochastic differential equations driven by one-dimensional Brownian motion, (BSDEs for short). The key ingredients to prove our results are the well known Girsanov's theorem concerning the martingale property under a change of the probability measure on the underlying filtered space, and the result on explicit solution in the case of linear BSDE.

1. Introduction

The theory of non-linear backward stochastic differential equations, (BSDEs for short), was pioneered by Pardoux and Peng [8],[9]. It became now very popular, and is an important field of research due to its connections with stochastic control, mathematical finance, and partial differential equations, (PDEs for short). BSDEs provide a probabilistic representation of nonlinear PDEs, which extends the famous Feynman-Kac's formula for linear PDEs. As a consequence, BSDEs can be used for designing numerical algorithms to nonlinear PDEs.

The case of linear BSDEs was introduced by Bismut [2] as the adjoint equation associated with stochastic Pontryagin maximum principle in stochastic control theory. In the paper by El Karoui et al. [4], some additional properties are given and several applications to option pricing and recursive utilities are studied. Since then, BSDEs have been studied with great interest. In particular, many efforts have been made to relax the Lipschitz hypothesis on the generator in BSDEs, for instance, Kobylanski [6] have proved the existence of a solution for one-dimensional BSDEs, when the generator is only continuous with linear growth.

In this paper, we consider two types of one-dimensional backward stochastic differential equations driven by one-dimensional Brownian motion:

$$Y_t = \xi + \int_t^T g(s, Z_s) ds - \int_t^T Z_s dW_s, \quad t \in [0, T],$$

Received 2017-6-7; Communicated by the editors.

2010 *Mathematics Subject Classification.* Primary 60H10; Secondary 60H35.

Key words and phrases. Backward stochastic differential equation, discrete time approximation, Monte Carlo simulation, Girsanov's theorem.

* Corresponding author.

$$Y_t = \xi + \int_t^T h(s, Y_s) ds - \int_t^T Z_s dW_s, \quad t \in [0, T],$$

where $(W_t)_{0 \leq t \leq T}$ is a one-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, P)$, T is a fixed finite horizon, $(\mathcal{F}_t)_{0 \leq t \leq T}$ is the natural Brownian filtration, the random functions $g, h : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ are respectively the generators of the BSDEs, and the \mathbb{R} -valued \mathcal{F}_T adapted random variable ξ is the terminal condition. We give a discrete time approximations and Monte Carlo simulations for the $(\mathcal{F}_t)_{0 \leq t \leq T}$ adapted solutions Y of the BSDEs. The key ingredients to prove our results are the well known Girsanov's theorem concerning the martingale property under a change of the probability measure on the underlying filtered space, and the result on explicit solution in the case of linear BSDE.

The reminder of this paper is organized as follows: in section 2, we collect some basic facts about BSDEs and Girsanov's theorem. In section 3, we give the statements and proofs of our main results with examples of simulation of trajectories obtained by our main R-codes.

2. On the Backward Stochastic Differential Equations and the Girsanov's Theorem

2.1. Backward stochastic differential equations. Notice that for an ordinary differential equation (ODE for short), under certain regularity conditions, both the initial value and the terminal value problems are well-posed. In fact, for an ODE, the terminal value problem on $[0, T]$ is equivalent to an initial value problem on $[0, T]$ under the time-reversing transformation: $t \rightarrow T - t$. However, things are fundamentally different for BSDEs when we are looking for a solution that is adapted to the given filtration. One cannot simply reverse the time to get a solution for a terminal value problem of stochastic differential equations (SDEs for short), as it would destroy the adaptiveness. Therefore, the first issue one should address in the stochastic case is how to correctly formulate a terminal value problem for SDEs.

We begin this section by briefly recalling some aspects of the theory of BSDEs, and we refer for example to [4],[6],[8],[9] and [10].

On a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, P)$, on which is defined a one-dimensional Brownian motion $(W_t)_{0 \leq t \leq T}$ over a finite time interval $[0, T]$, and its natural filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$, we denote by:

- \mathcal{P} : the σ algebra of $(\mathcal{F}_t)_{0 \leq t \leq T}$ predictable subsets of $[0, T] \times \Omega$,
- \mathbb{S}^2 : the set of real-valued cd-lg $(\mathcal{F}_t)_{0 \leq t \leq T}$ -adapted processes Y such that:

$$\mathbb{E} \left(\sup_{0 \leq t \leq T} |Y_t|^2 \right) < +\infty,$$

- \mathbb{L}^2 : the set of real-valued \mathcal{P} -measurable processes Z such that:

$$\mathbb{E} \left(\int_0^T |Z_t|^2 dt \right) < +\infty,$$

- ξ is the terminal condition which is an \mathcal{F}_T measurable real-valued random variable,

- $f = f(t, y, z)_{0 \leq t \leq T}$ is the generator of BSDE which is an $\mathcal{P} \otimes \mathcal{B}(\mathbb{R} \times \mathbb{R})$ -measurable real-valued map, where $\mathcal{B}(\mathbb{R} \times \mathbb{R})$ denotes the Borel σ -field of $\mathbb{R} \times \mathbb{R}$.

A one-dimensional BSDE in differential form is written as:

$$dY_t = -f(t, Y_t, Z_t)dt + Z_t dW_t, \quad 0 \leq t \leq T, \quad Y_T = \xi,$$

and a solution to this BSDE is a pair $(Y, Z) \in \mathbb{S}^2 \times \mathbb{L}^2$ satisfying:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad t \in [0, T]. \quad (2.1)$$

Existence and uniqueness of the adapted solution (Y, Z) of the BSDE (2.1) is proved under the following Lipschitz and square integrability assumptions (\mathcal{H}_1):

- f is uniformly Lipschitz in (y, z) , i.e., there exists a finite positive constant C_f , such that for all (y, z, y', z') :

$$|f(t, y, z) - f(t, y', z')| \leq C_f(|y - y'| + |z - z'|).$$

- ξ and $(f(t, 0, 0))_{t \in [0, T]}$ satisfies:

$$\mathbb{E} \left(|\xi|^2 + \int_0^T |f(t, 0, 0)|^2 dt \right) < +\infty.$$

Now we deal with the case of linear BSDEs of the form:

$$Y_t = \xi + \int_t^T (a_s Y_s + b_s Z_s + c_s)ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T, \quad (2.2)$$

with the assumptions (\mathcal{H}_2):

- a, b are bounded progressively measurable processes valued in \mathbb{R} ,
- ξ and c satisfies:

$$\mathbb{E} \left(|\xi|^2 + \int_0^T |c_t|^2 dt \right) < +\infty$$

Proposition 2.1. *Under the assumptions (\mathcal{H}_2), the unique adapted solution Y of the linear BSDE (2.2) is given by:*

$$Y_t = \Gamma_t^{-1} \mathbb{E} \left(\xi \Gamma_T + \int_t^T c_s \Gamma_s ds \middle| \mathcal{F}_t \right),$$

where Γ is the solution to the linear SDE:

$$d\Gamma_t = \Gamma_t (a_t dt + b_t dW_t), \quad \Gamma_0 = 1.$$

Remark 2.2.

- The process Γ is given explicitly by:

$$\Gamma_t = \exp \left(\int_0^t b_s dW_s - \frac{1}{2} \int_0^t b_s^2 ds + \int_0^t a_s ds \right).$$

- The linear BSDE (2.2) was the key ingredient to prove a comparison theorem for the BSDE (2.1). (See for example [4] and [6] for some applications of this result).

To end this fact about BSDEs, we give an adapted numerical approximation of the BSDE (1), and we refer for example to [1],[3] and [5]. Let π be a partition of time points $0 = t_0 < t_1 < \dots < t_n = T$ of $[0, T]$, with a fixed time step $\Delta t_i := t_{i+1} - t_i$. We start with the following discrete version of the BSDE (2.1) on the interval $[t_i, t_{i+1}]$, where $\Delta W_{t_i} := W_{t_{i+1}} - W_{t_i}$:

$$\begin{aligned} Y_{t_i} &= Y_{t_{i+1}} + \int_{t_i}^{t_{i+1}} f(s, Y_s, Z_s) ds - \int_{t_i}^{t_{i+1}} Z_s dW_s, \\ Y_{t_i} &\approx Y_{t_{i+1}} + f(t_i, Y_{t_i}, Z_{t_i}) \Delta t_i - Z_{t_i} \Delta W_{t_i}. \end{aligned} \quad (2.3)$$

Since Y_{t_i} and Z_{t_i} are \mathcal{F}_{t_i} -adapted and taking expectation conditionally on \mathcal{F}_{t_i} on both sides of (2.3) gives us:

$$Y_{t_i} \approx \mathbb{E}(Y_{t_{i+1}} | \mathcal{F}_{t_i}) + f(t_i, Y_{t_i}, Z_{t_i}) \Delta t_i.$$

Now, multiplying (2.3) by ΔW_{t_i} and then taking conditional expectation gives:

$$0 \simeq \mathbb{E}(Y_{t_{i+1}} \Delta W_{t_i} | \mathcal{F}_{t_i}) - Z_{t_i} \Delta t_i.$$

Therefore

$$Z_{t_i} = E\left(Y_{t_{i+1}} \frac{\Delta W_{t_i}}{\Delta t_i} \middle| \mathcal{F}_{t_i}\right).$$

Finally, we get a backward implicit Euler scheme (Y^π, Z^π) for the BSDE (2.1) of the form:

$$\begin{cases} Y_{t_n}^\pi = \xi, \\ Z_{t_i}^\pi = \mathbb{E}\left(Y_{t_{i+1}}^\pi \frac{\Delta W_{t_i}}{\Delta t_i} \middle| \mathcal{F}_{t_i}\right), & i < n, \\ Y_{t_i}^\pi = \mathbb{E}\left(Y_{t_{i+1}}^\pi | \mathcal{F}_{t_i}\right) + f(t_i, Y_{t_i}^\pi, Z_{t_i}^\pi) \Delta t_i, & i < n. \end{cases} \quad (2.4)$$

Remark 2.3. The practical implementation of the numerical scheme (2.4) requires the computation of conditional expectations, (see [1],[3] and [5]), but these results are highly technical in nature which is the reason why BSDEs are not used by practitioners yet.

2.2. Girsanov's theorem. Here, we give some basic facts about the well known Girsanov's theorem, and we refer for example to [7].

Theorem 2.4. *Let $(W_t)_{0 \leq t \leq T}$, be a Brownian motion on a probability space (Ω, \mathcal{F}, P) . Let $(\mathcal{F}_t)_{0 \leq t \leq T}$, be the accompanying filtration, and let $(\theta_t)_{0 \leq t \leq T}$ be a process adapted to this filtration. For $0 \leq t \leq T$, define*

$$Z_t = \exp\left(-\int_0^t \theta_s dW_s - \frac{1}{2} \int_0^t \theta_s^2 ds\right),$$

$$Q(A) = \int_A Z_T dP, \quad \forall A \in \mathcal{F},$$

and

$$W_t^Q = \int_0^t \theta_s ds + W_t.$$

Then, under the Novikov's condition:

$$\mathbb{E}\left(\exp\left(\frac{1}{2}\int_0^T\theta_s^2ds\right)\right) < \infty,$$

the process $(W_t^Q)_{0 \leq t \leq T}$ is a Brownian motion under the new probability measure Q .

Remark 2.5.

- Under P , the process $(Z_t)_{0 \leq t \leq T}$ is a martingale: $dZ_t = -\theta_t Z_t dW_t$.
- Since $Z_0 = 1$ and $(Z_t)_{0 \leq t \leq T}$ is a martingale, we have $\mathbb{E}(Z_T) = 1$.

Therefore

$$Q(\Omega) = \int_{\Omega} Z_T dP = \mathbb{E}(Z_T) = 1,$$

so Q is a probability measure.

- If $(\theta_t)_{0 \leq t \leq T}$ is constant, then

$$W_T^Q = \theta T + W_T \quad \text{and} \quad Z_T = \exp\left(-\theta W_T - \frac{1}{2}\theta^2 T\right).$$

Under P : $W_T \sim \mathcal{N}(0, T)$ and $W_T^Q \sim \mathcal{N}(\theta T, T)$. However, under Q : $W_T^Q \sim \mathcal{N}(0, T)$.

- When we use the Girsanov's theorem to change the probability measure, means change but variances do not. Quadratic variation is unaffected:

$$dW_t^Q dW_t^Q = (\theta_t dt + dW_t)^2 = dW_t dW_t = dt.$$

- Bayes's rule: If X is Q -integrable and \mathcal{F}_T -measurable, then

$$\mathbb{E}^Q(X|\mathcal{F}_t) = \frac{1}{Z_t} \mathbb{E}(X Z_T | \mathcal{F}_t).$$

3. The Main Results

Now, we deal with the first BSDE:

$$Y_t = \xi + \int_t^T g(s, Z_s) ds - \int_t^T Z_s dW_s, \quad t \in [0, T]. \quad (3.1)$$

We start with a nominal reference deterministic trajectories denoted (\bar{y}, \bar{z}) , and obtained as follows:

$$\begin{cases} \bar{z}_t = \mathbb{E}(Z_t), \\ \bar{y}_t = \mathbb{E}(\xi) + \int_t^T \mathbb{E}(g(s, \bar{z}_s)) ds, \quad t \in [0, T]. \end{cases} \quad (3.2)$$

Clearly the pair (\bar{y}, \bar{z}) exists.

We denote the error between Y_t and \bar{y}_t by:

$$\tilde{Y}_t = Y_t - \bar{y}_t.$$

Now, combining (3.1) and (3.2), we obtain the following dynamics of \tilde{Y}_t :

$$\tilde{Y}_t = \tilde{\xi} + \int_t^T \tilde{g}(s, Z_s) ds - \int_t^T Z_s dW_s, \quad t \in [0, T], \quad (3.3)$$

where $\tilde{\xi} = \xi - \mathbb{E}(\xi)$ and $\tilde{g}(s, Z_s) = g(s, Z_s) - \mathbb{E}(g(s, \bar{z}_s))$.

To obtain a linear approximation of the BSDE (3.3), we make a Taylor series expansion of $\tilde{g}(s, Z_s)$ around \bar{z}_s :

$$\tilde{g}(s, Z_s) \approx \tilde{g}(s, \bar{z}_s) + (Z_s - \bar{z}_s) \frac{\partial g(s, \bar{z}_s)}{\partial z},$$

and we obtain the approximate linear model:

$$\tilde{Y}_t = \tilde{\xi} + \int_t^T (\tilde{b}_s Z_s + \tilde{c}_s) ds - \int_t^T Z_s dW_s, \quad t \in [0, T], \quad (3.4)$$

where

$$\tilde{b}_s = \frac{\partial g(s, \bar{z}_s)}{\partial z} \quad \text{and} \quad \tilde{c}_s = \tilde{g}(s, \bar{z}_s) - \bar{z}_s \frac{\partial g(s, \bar{z}_s)}{\partial z}.$$

We suppose that the process \tilde{b} satisfy the Novikov's condition. Then by the Girsanov's theorem, the process:

$$W_t^Q = W_t - \int_0^t \tilde{b}_s ds$$

is a Brownian motion.

We put:

$$\tilde{\tilde{Y}}_t = \tilde{Y}_t - \int_0^t \tilde{c}_s ds \quad \text{and} \quad \tilde{\tilde{\xi}} = \tilde{\xi} + \int_0^T \tilde{c}_s ds.$$

Therefore, (3.4) becomes

$$\tilde{\tilde{Y}}_t = \tilde{\tilde{\xi}} - \int_t^T Z_s dW_s^Q,$$

and

$$\tilde{\tilde{Y}}_t = \mathbb{E}^Q(\tilde{\tilde{\xi}} | \mathcal{F}_t).$$

Finally, we are ready to state the first approximation result of this section.

Theorem 3.1. *If the process \tilde{b} satisfy the Novikov's condition, then we get the following approximation of the solution Y of BSDE (3.1):*

$$Y_{estimated}^t = \mathbb{E}^Q \left(\xi + \int_0^T \tilde{c}_s ds \middle| \mathcal{F}_t \right) - \int_0^t \tilde{c}_s ds + \int_t^T \mathbb{E} \left(g(s, \bar{z}_s) \right) ds.$$

Remark 3.2.

- If the generator g is deterministic (non random), then $\tilde{g}(s, \bar{z}_s) = 0$ and we have:

$$Y_{estimated}^t = \mathbb{E}^Q(\xi | \mathcal{F}_t) + \int_t^T \left(g(s, \bar{z}_s) - \bar{z}_s \frac{\partial g(s, \bar{z}_s)}{\partial z} \right) ds.$$

- The conditional expectation $\mathbb{E}^Q(\xi | \mathcal{F}_t)$ can be obtained easily under P by using the Bayes's rule.
- If ξ has the form: $\xi = l_1(W_T^Q)$, then by the Markov property, we have:

$$Y_{estimated}^t = l_2(W_t) + \int_t^T \left(g(s, \bar{z}_s) - \bar{z}_s \frac{\partial g(s, \bar{z}_s)}{\partial z} \right) ds := J_t^1 + J_t^2,$$

and we obtain the discretization scheme for the process $Y_{estimated}^t$ as follows:

- (1) $Y_{\text{estimated}}^T = Y_T = \xi$.
- (2) The deterministic nominal reference trajectory \bar{z} is given by (2.4) as:

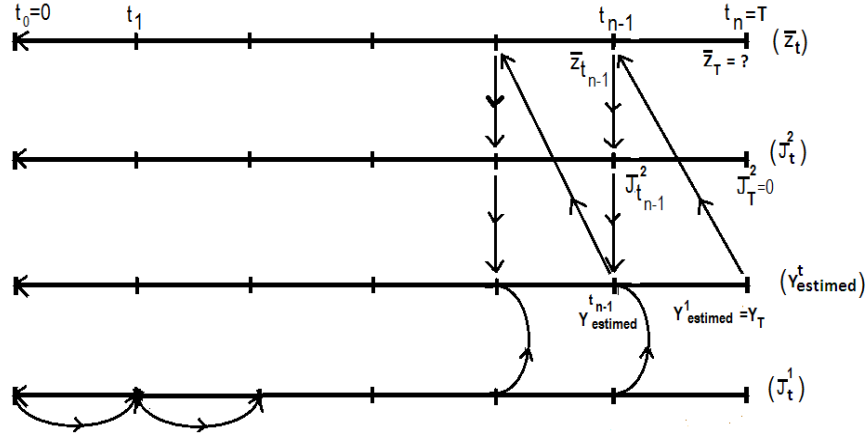
$$\bar{z}_{t_i} = \mathbb{E} \left(Y_{\text{estimated}}^{t_{i+1}} \frac{\Delta W_{t_i}}{\Delta t_i} \right),$$

and then, we apply the Monte Carlo simulation.

- (3) J_t^1 is obtained directly from W_t by forward scheme and the Monte Carlo simulation.
- (4) Finally, J_t^2 is obtained by the backward Euler scheme:

$$\begin{cases} J_{t_i}^2 = J_{t_{i+1}}^2 + \left(g(t_i, \bar{z}_{t_i}) - \bar{z}_{t_i} \frac{\partial g(t_i, \bar{z}_{t_i})}{\partial z} \right) \Delta t_i, & i \leq n, \\ J_T^2 = 0. \end{cases}$$

We summarize the steps of this algorithm in the following scheme:



Application (with $\Delta t_i = \frac{T}{n}$): Consider the classical example of BSDE :

$$Y_t = \xi - \frac{1}{2} \int_t^T Z_s^2 ds + \int_t^T Z_s dW_s, \quad t \in [0, T].$$

The changes of variables $P_t = e^{Y_t}$ and $Q_t = Z_t e^{Y_t}$ with the Itô's formula, leads to the equation:

$$P_t = e^\xi - \int_t^T Q_s dW_s,$$

and the exact solution is

$$Y_{\text{real}}^t = \ln \mathbb{E}(e^\xi | \mathcal{F}_t).$$

We take $\xi = W_T$, then

$$Y_{\text{real}}^t = W_t + \frac{(T-t)}{2}.$$

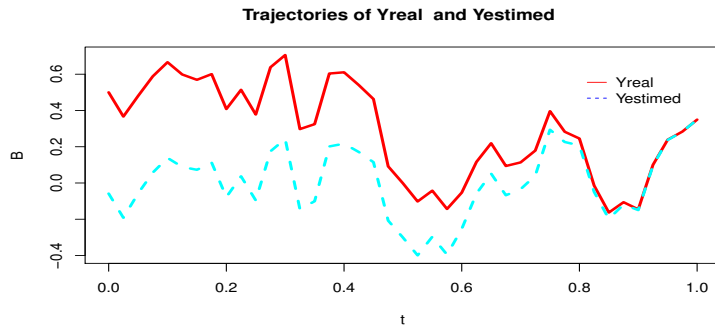
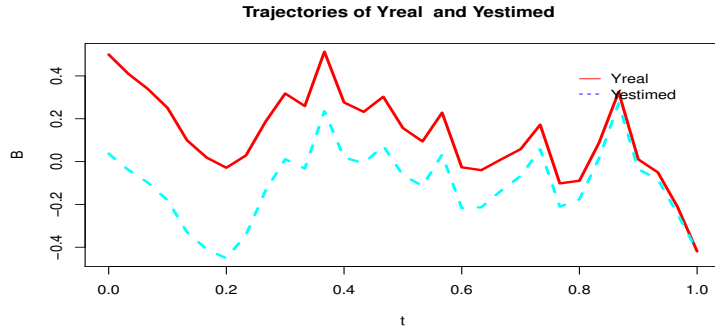
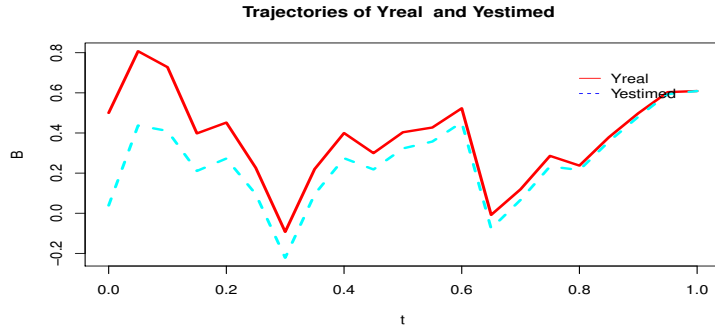
In this example, we have:

$$g(t, \bar{z}_t) = \frac{1}{2} \bar{z}_t^2 \quad \text{and} \quad \tilde{g}(t, \bar{z}_t) = 0,$$

$$\tilde{b}_t = \bar{z}_t \quad \text{and} \quad \tilde{c}_t = -\bar{z}_t^2.$$

Then

$$Y^t_{\text{estimated}} = W_t + \int_t^T \left(\bar{z}_s ds - \frac{1}{2} \bar{z}_s^2 \right) ds.$$



Now we deal with the second type of BSDE:

$$Y_t = \xi + \int_t^T h(s, Y_s) ds - \int_t^T Z_s dW_s, \quad t \in [0, T], \quad (3.5)$$

with

$$\begin{cases} \bar{z}_t = \mathbb{E}(Z_t), \\ \bar{y}_t = \mathbb{E}(\xi) + \int_t^T \mathbb{E}\left(h(s, \bar{y}_s)\right) ds, \quad t \in [0, T]. \end{cases} \quad (3.6)$$

Clearly (\bar{y}, \bar{z}) exist by the Lipschitz hypothesis of the generator.

Combining (3.5) and (3.6), we obtain the following dynamics of $\tilde{Y}_t = Y_t - \bar{y}_t$:

$$\tilde{Y}_t = \tilde{\xi} + \int_t^T \tilde{h}(s, Y_s) ds - \int_t^T Z_s dW_s, \quad t \in [0, T], \quad (3.7)$$

where $\tilde{\xi} = \xi - \mathbb{E}(\xi)$ and $\tilde{h}(s, Y_s) = h(s, Y_s) - \mathbb{E}\left(h(s, \bar{y}_s)\right)$.

To obtain a linear approximation of the BSDE (3.7), we make a Taylor series expansion of $\tilde{h}(s, Y_s)$ around \bar{y}_s :

$$\tilde{h}(s, Y_s) \approx \tilde{h}(s, \bar{y}_s) + \tilde{Y}_s \frac{\partial h(s, \bar{y}_s)}{\partial y}.$$

Finally, we obtain the approximate linear model:

$$\tilde{Y}_t = \tilde{\xi} + \int_t^T \left(\tilde{a}_s Y_s + \tilde{c}_s \right) ds - \int_t^T Z_s dW_s, \quad t \in [0, T], \quad (3.8)$$

with

$$\tilde{a}_s = \frac{\partial h(s, \bar{y}_s)}{\partial y} \quad \text{and} \quad \tilde{c}_s = \tilde{h}(s, \bar{y}_s).$$

Now, we suppose that the processes \tilde{a} and \tilde{c} satisfies the assumptions (\mathcal{H}_2) , then by Proposition 2.1, we have

$$\Gamma_t = \exp\left(\int_0^t \tilde{a}_s ds\right) \quad \text{and} \quad \tilde{Y}_t = \Gamma_t^{-1} \mathbb{E}\left(\Gamma_T \tilde{\xi} + \int_t^T \tilde{c}_s \Gamma_s | \mathcal{F}_t\right).$$

Finally, the second approximation result of this section reads.

Theorem 3.3. *If the processes \tilde{a} and \tilde{c} satisfies the assumptions (\mathcal{H}_2) , we get the following approximation of the solution Y of BSDE (3.5):*

$$Y_{estimated}^t = \bar{y}_t + \Gamma_t^{-1} \mathbb{E}\left(\Gamma_T \left(\xi - \mathbb{E}(\xi)\right) + \int_t^T \tilde{c}_s \Gamma_s | \mathcal{F}_t\right).$$

Remark 3.4.

- $Y_{estimated}^T = Y_T = \xi$.
- If the generator h is deterministic (non random), then $\tilde{h}(s, \bar{z}_s) = 0$ and we have:

$$Y_{estimated}^t = \bar{y}_t + \exp\left(\int_t^T \frac{\partial h(s, \bar{y}_s)}{\partial y} ds\right) \left(\mathbb{E}(\xi | \mathcal{F}_t) - \mathbb{E}(\xi)\right).$$

- If ξ has the form: $\xi = l_1(W_T^Q)$, then by the Markov property, we have

$$Y_{\text{estimated}}^t = \bar{y}_t + \exp\left(\int_t^T \frac{\partial h(s, \bar{y}_s)}{\partial y} ds\right) \left(l_2(W_t) - \mathbb{E}(\xi)\right) := \bar{y}_t + J_t^1 \times J_t^2,$$

and we obtain the discretization scheme for the process $Y_{\text{estimated}}^t$ as follows:

- (1) The deterministic nominal reference trajectory \bar{y} is given by the backward Euler scheme in (3.6):

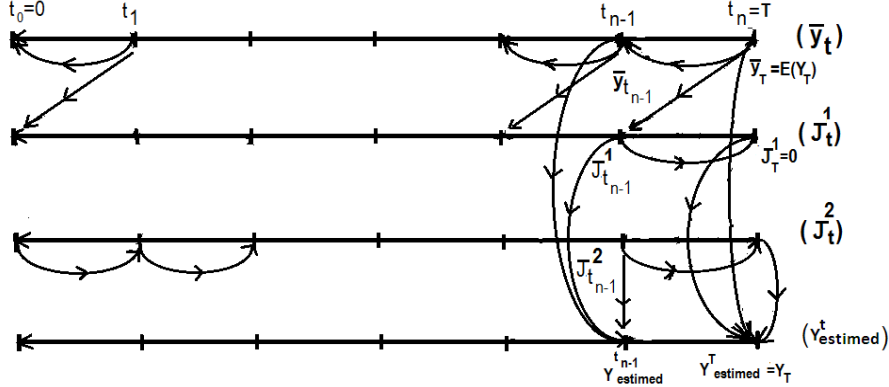
$$\begin{cases} \bar{y}_t \approx \bar{y}_{t_{i+1}} + h(t_{i+1}, \bar{y}_{t_{i+1}}) \Delta t_i, & i < n, \\ \bar{y}_T = \mathbb{E}(\xi), & \text{(obtained by Monte Carlo simulation).} \end{cases}$$

- (2) We put: $J_t^1 = \exp(I_t^1)$, where I_t^1 is obtained by the backward Euler scheme:

$$\begin{cases} I_t^1 \approx I_{t_{i+1}}^1 + \frac{\partial h(t_{i+1}, \bar{y}_{t_{i+1}})}{\partial y} \Delta t_i, & i < n, \\ I_T^1 = 0. \end{cases}$$

- (3) Finally, we use easily the Monte Carlo simulation for J_t^2 .

We summarize the steps of this algorithm in the following scheme:



Application (with $\Delta t_i = \frac{T}{n}$): Consider the BSDE :

$$Y_t = \xi + \int_t^T \frac{Y_s}{2} (1 - Y_s)(2Y_s - 1) ds - \int_t^T Z_s dW_s,$$

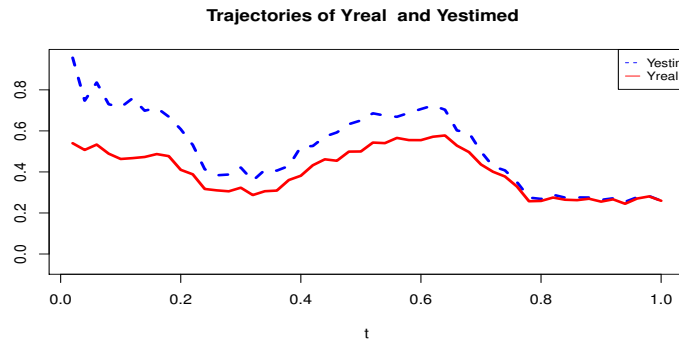
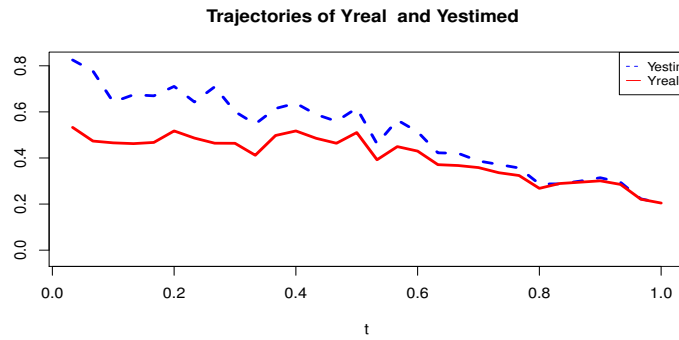
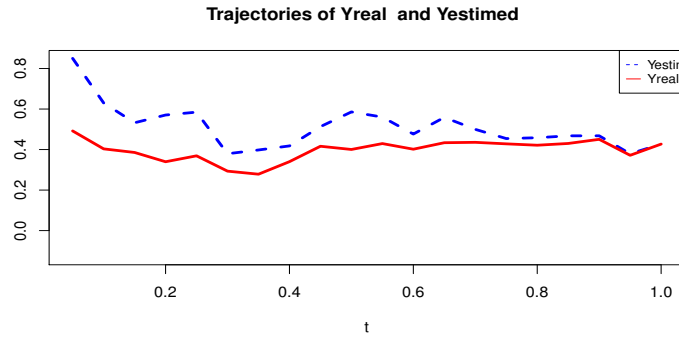
with the terminal condition :

$$\xi = \frac{e^{W_T}}{1 + e^{W_T}}.$$

Clearly the generator h and ξ satisfies the assumptions (\mathcal{H}_1) , and $\frac{\partial h}{\partial y}$ is bounded.

Using the Itô's formula, we have the unique adapted solutions :

$$Y_t = \frac{e^{W_t}}{1 + e^{W_t}} \quad \text{and} \quad Z_t = \frac{e^{W_t}}{(1 + e^{W_t})^2}.$$



Remark 3.5.

- We believe that our result is new and is valid for the multidimensional case.
- We should point out that in this paper we only study the case where the generator f depend on one of the two variables y and z . This is enough for the purpose of this study. We will study the general case in future work and apply this idea to the Markovian case and the Feynman-Kac's formula for PDEs.

Acknowledgment. The authors are very grateful to the anonymous referee whose remarks and suggestions greatly improve the presentation of this paper.

References

1. Bally, V. and Pagés, G.: A quantization algorithm for solving discrete time multidimensional optimal stopping problems, *Bernoulli* **9** (2002), no. 6, 1003–1049.
2. Bismut, J. M.: Conjugate convex functions in optimal stochastic control, *J. Math. Anal. Appl* **44** (1973), 384–404.
3. Bouchard, B. and Touzi, N.: Discrete-time approximation and Monte-Carlo simulation of backward stochastic differential equations, *Stochastic Processes and their Applications* **111** (2004), no. 2, 175–206.
4. El Karoui, N., Peng, S. and Quenez, M. C.: Backward stochastic differential equations in finance, *Mathematical Finance* **7** (1997), no. 1, 1–71.
5. Gobet, E., Lemor, J. P. and Warin, X.: Rate of convergence of empirical regression method for solving generalized BSDE, *Bernoulli* **12** (2006), 889–916.
6. Kobylanski, M.: Backward stochastic differential equations and partial differential equations with quadratic growth, *Annals of probability* **28** (2000), no. 2, 558–602.
7. Oksendal, B.: *Stochastic Differential Equations*, Springer-Verlag Heidelberg, New York, 1998.
8. Pardoux, E. and Peng, S.: Adapted solution of a backward stochastic differential equation, *Systems. Control. Lett.*, **14** (1990), p. 55–61.
9. Pardoux, E. and Peng, S.: Backward stochastic differential equations and quasilinear parabolic partial differential equations, *Lecture Notes in CIS, Springer Berlin* **176** (1992), 200–217.
10. Pham, H.: *Continuous-time stochastic control and optimization with financial applications*, Springer-Verlag Heidelberg. Berlin, 2009.

AISSA SGHIR: LABORATOIRE DE MODÉLISATION STOCHASTIQUE ET DÉTERMINISTE ET URAC 04, DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ MOHAMMED PREMIER, FACULTÉ DES SCIENCES OUJDA, BP, 717, MAROC

E-mail address: sghir.aissa@gmail.com

DRISS SEGHIR: ÉQUIPE EDP ET CALCUL SCIENTIFIQUE, DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ MOULAY ISMAIL, FACULTÉ DES SCIENCES MEKNÈS, 11201, ZITOUNE, MAROC

E-mail address: seghirdriss@yahoo.fr

SOUKAINA HADIRI: ÉQUIPE EDP ET CALCUL SCIENTIFIQUE, DÉPARTEMENT DE MATHÉMATIQUES ET INFORMATIQUE, FACULTÉ DES SCIENCES MEKNÈS, UNIVERSITÉ MOULAY ISMAIL, BP, 11201, ZITOUNE, MAROC

E-mail address: sokaina.hadiri@gmail.com