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A Survey: List Coloring Problem

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Abstract : Graph coloring is an assignment of integers to the vertices of a graph so that no two adjacent vertices are assigned the same number. A list coloring of a graph is an assignment of integers to the vertices of a graph with the restriction that the integers must come from specific lists of available colors at each vertex. This paper presents a survey of last three decades of research in the field of colorings. Also, the goal of this survey is to show the path to mathematicians to bring new innovative ideas in the subject of list colorings.

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1. INTRODUCTION

One of the most popular and useful area of graph theory is graph colorings. This problem frequently arises in scheduling and channel assignment applications. A list coloring of a graph is an assignment of integers to the vertices of a graph as before with the restriction that the integers must come from specific lists of available colors at each vertex. For real life applications of this problem, consider a wireless network. Due to hardware restrictions, each radio has a limited set of frequencies through which it can communicate, and radios within a certain distance with each other cannot operate on the same frequency without interfering. This situation could be converted as graph by representing the wireless radios by vertices and assigning a list to each vertex according to its available frequencies. This survey provides an overview of the research on list coloring that has been carried out by the researchers for the past three decades. This article begins with the introduction of the list coloring problem as defined by Erdos, Rubin and Taylor in [3]. Second, it continues with the study of various parameters of the problem that includes cases when all the list colorings have the same length. Third, it briefly mentions the list colorings and overview of restricted list colors such as partial list colorings, connected list colorings and chromatic polynomial.

2. LIST COLORINGS OF GRAPHS

Let G be a simple graph with vertex set V(G) and edge set E(G). The number of vertices and edges are denoted by |V(G)| and |E(G)| respectively. A proper coloring of a graph is a function $f : V(G) \rightarrow Z^{+}$ such that $\forall\ u,v \in V(G)$, $f(u) \neq f(v)$ if $u,v \in E(G)$. The chromatic number of G is denoted by $\chi(G)$, to be the least positive integers r such that G has a proper coloring assigning the integers $\{1,2,3,\ldots r\}$ to V(G). Let C be the set of colors and, for each $v \in V(G)$, let $L:V(G) \rightarrow 2^{C}$ be a function assigning each vertex $v \in V(G)$, a list of color $L(V) \subseteq C$. If there is a function $f : V(G) \rightarrow C$ such that $f(v) \in L(v)$for all $v \in V(G)$ and $f(u) \neq f(v)$ for $u, v \in E(G)$ then G is said to be L-colorable. This is called list coloring of G. If r is a positive integer, the function L is such that $|L(v)| = r$ for all $v \in V(G)$, and the graph G has a proper list coloring, then G is called r-colorable and it is defined the choice number, $\chi_L^{(G)}$ to be the minimum such r, so that G has a proper list coloring. List colorings were introduced in [3] by Erdos, Rubin and Taylor. The objective of this review article is introduced to the reader to do the research in the area of list colorings.

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Theorem 2.1.

(Erdos, Rubin, Taylor [3]) There is no bound on how much $\chi_L^{(G)}$ can exceed $\chi(G)$ number of vertices increases.

As noted in [3], since the usual graph colorings are special cases of list colorings, we have $\chi_L^{(G)} \leq \chi(G)$for all graphs G. A natural and most common question that could be asked as `Given a graph G, for what values of r in G, *r* – choosable? 'To answer this question is quite difficult but Rubin proved in [3] that finding a list coloring for a graph is NP-hard. In [16], Zeithlofer and Wess describe using list coloring, how to determine register assignments for computing processes .They discussed a situation with multiple functional units, some capable of addition and multiplication, and some capable of only addition. List coloring is then used for instruction scheduling. Since, the list coloring problem is NP-hard, Zeithlofer and Wess take advantage of certain properties of the graphs emerging from their registry assignment problem in order to find a coloring. Ramachandran et.al. discussed in [8], wireless networks near each other often interfere. Thus, to limit the interference and satisfy the hardware requirements, one must limit the frequencies available to a router. This situation was modeled as list coloring problem. They described assigning frequencies to a wireless mesh network built on a mixture of multi-radio and single radio routers.

3. R-CHOOSABILITY OF LIST COLORINGS OF GRAPHS

If G has adjacency vertices *u* and *v* such that *u* and *v* have more than one edge between them, *r*-choosable means that the list on each vertex has length *r*, and that from any such set of lists, the graph G may be properly colored. The following theorem is due to Rubin and may be found in [3]. It is noted that G is only 2-choosable.

Theorem 3.1.

If every component of G is 2- choosable. Theorem 3.1 (Rubin [3]) A graph G is 2-choosable if and only if the core of G is *r*, an even cycle, or of the form $\theta_{2,2,2r}$ where r is a positive integer. They define core of a graph G to be G with all vertices of degree one recursively removed. Also, they define a graph to be a graph with two distinct vertices, u and v with three vertex disjoint path between them.

4. LIST COLORINGS OF PLANAR GRAPHS

Planar graphs, especially, the four color problem plays an important part in graph theory. Margit Voigt [14] is concerned with the choosability of graphs generalizing the ordinary coloring and again the class of planar graphs is very interesting. It is easy to see that every planar graph is 6-choosable and Alon and Tarsi [1] showed that every planar bipartite graph is 3-choosable. This result is sharp because there are planar bipartite graphs which are 2-choosable [3]. Furthermore, there are two intriguing conjectures from Erdos, Rubin and Taylor 1979 [3]. That is,

- Every planar graph is 5-choosable.
- There are planar graphs which are not 4-choosable.

In the following way, Margit Voigt proved the second conjecture by constructing a planar graph which is not 4-choosable [14]. However, the other conjecture that every planar graph is

5-choosable remains an open problem [14]. Clearly, Erdos et.al, also conjectured in [3] that is a planar graph which is not 4-choosable, meaning that if this graph can be shown to exist, then Thomassen's bound of 5 for $\chi^L(G)$, for planar graph G is sharp [10].

Theorem 4.1.

(Thomassen [10]) Let G be near-triangulation with outer cycle C = $v_1, v_2, v_3 ... vk$, v_1 Assume that L(v) is a list of at least three colors if v is in C and at least five colors if v is in G ⁄ C without loss of generality, choose colors 1 and 2 for v_1 and v_2, respectively. Then, the coloring of v_1 and v_2, can be extended to a list coloring of G. M.Voigt constructed such a graph in [13]. But the construction involves an elaborate resting of triangulated graphs such that the final graph contains 238 vertices.

5. LIST EDGE COLORING OF GRAPHS

An edge coloring is a function C: E(G) → Z^+ such that if any two edges e_1, e_2, incident with a vertex, then $C(e_1) \neq C(e_2,)$. Clearly, note that only loop less graphs are edge colorable, since a loop is self-incident, but can only be assigned one color [2]. Edge coloring index of G is denoted by χ' (G).

Conjecture 5.1. (List Edge Coloring Conjecture)

Let G be a loop less graph, then χ' L(G) = χ' (G).

Haggkvist and Chetwynd [4] mention that it has been proven for trees, graphs which are 3 - regular but not 3 - edge colorable, 2 - regular graphs and the complete bipartite graphs $K_{3,3}$, $K_{4,4}$, $K_{6,6}$ and $K_{r,n}$ with $7r \leq 20n$. One of the most famous results in edge colorings in Vizing's theorem which states as follows:

Theorem 5.2. (Vizing, [11; 12].)

Let G be a simple graph, then $\chi'(G) \leq \Delta + 1$.

In 1994, Vizing extended his theorem to graphs with parallel edges. Then, the generalization of Vizing's theorem is as follows.

Theorem 5.3. (Vizing, [11; 12].)

$\chi'(G) \leq \Delta + \mu$. where μ is the maximum number of parallel edges between any two vertices of G. Haggkvist et.al apply Vizing's theorem to proposition 1 in [4] to obtain an upper bound on $\chi'_{L,1,r}(G)$.

Theorem 5.4. (Haggkvist and Chetwynd, [4].)

Let G be a loop less graph on n vertices and let t be the least integer such that $n < t!$. Then, taking $r = \chi'$ we have $\chi'_{L,1,r} \leq r + t$. After that Haggkvist et.al [4] generalize this theorem to prohibiting a set of q colors at each as follows.

Theorem 5.5. (Haggkvist and Chetwynd, [4].)

Let G be a loop less graph $nq^t < t!$ on n vertices and let q and t be positive integers such that $q_t < t!$. Then, taking $r = \chi'$ we have $\chi'_{L,q,r} \leq r + t + \mu - 1$. After that Haggkvist et. al [4] find the upper bound for the minimum k necessary to ensure a list coloring from lists of size k taken from the set $\{1,2,3,\ldots k + q\}$ of previous theorem.

Theorem 5.6. (Haggkvist and Chetwynd [4].)

Let G be a loop less graph on n vertices and let q and t be positive integers with $nq^t < t!$ then $\chi'_{L,q,} \leq \chi' + 2t + q - 4$.

6. THE CHROMATIC POLYNOMIALS AND LIST COLORING OF GRAPHS

C. Thomassen proved that, if a graph has a list of k – available colors at every vertex, then the number of list colorings is at least the chromatic polynomial evaluated at 'k', when k is sufficiently large compared to the number of vertices of the graph. The following theorem shows that the largest order terms of the chromatic polynomial give a lower bound for the number of k - list colorings provided, some 'n' neighboring vertices have lists with large symmetric difference.

Theorem 6.1. (C. Thomassen [9])

Let G be a graph with vertices $v_1, v_2, v_3, \ldots v_n$ and let k be a natural number. Let $L(v_I)$ be a list of precisely k available colors for each $i = 1, 2, 3, \ldots n$. If $k > n^{10}$, then the number of L – Colorings is at least P (G, K), where P (G, K) is the chromatic polynomial of G.

7. LIST COLORING SQUARES OF PLANAR GRAPHS

Given a graph G, the square of G, denoted G2, is the graph with the same vertex set as G and with an edge between any two different vertices that have distance at most two in G. If maximum degree Δ, then a vertex coloring of its square will need at least Δ + 1 colors, but the greedy algorithm shows that it is always possible with $\Delta^2 + 1$ colors. Regarding the chromatic number of the square of a planar graph, Wegner [15] posed the following conjecture, suggesting that for planar graphs far less than $\Delta^2 + 1$ colors would suffice.

Conjecture 7.1. (Wegner [15]).

For a planar graph G of maximum degree Δ ≥ 8 then $\chi^{G^2} \leq \left\lceil \frac{3}{2}\Delta + 1 \right\rceil$.

Wegner also gave bounds for Δ ≤ 7. Many upper bounds of χ^{G^2} for planar graphs in terms of Δ have been obtained in the last 15 years. The asymptotically best known upper bound so far has been found by Molloy and Salavatipour [7].

Theorem 7.2. (Molloy and Salavatipour [7]

$$\chi^{G^2} \leq \left\lceil \frac{3}{2}\Delta + 1 \right\rceil$$

In this extended result, F. Havet et al. dive the following theorem.

Theorem 7.3. (F. Havet et al. [5]

The square of every planar graph G of maximum degree Δ has list chromatic number at most $(1 + O(1))\,\frac{1}{2}\,\Delta$.

Moreover, given lists of this size, there is a proper coloring in which the colors on every pair of adjacent vertices of G differ by atleast $\Delta^{1/4}$. The O(1) term here is as $\Delta \to \infty$. F.Havet et al. [5], easily extend these ideas to sets with more than one vertex.

Theorem 7.4. (F.Havet et.al [5]).

If R is a set of removable vertices of G, then there is a planar graph G_1 with vertex set V-R and maximum degree Δ such that $G^2 - R \subseteq G_1^{\,2}$.

The above theorem shows that there can be no minimal counter example to this theorem that can have a removable vertex of low degree in G^2. In particular, using a sophisticated argument due to Khan [6], F. Havet et.al [5] proved the following theorem.

Theorem 7.5. (F.Havet et.al [5]).

Suppose R is the core of a removable copy of H* in G, for some multigraph H, such that for any set X of vertices of H and corresponding set X* of the vertices of the copy of H* we have sum of the degrees in G-R of the vertices in X* exceeds the number of edges of H out of X by at most $\frac{\in|X|\Delta}{10}$.

F.Havet et.al [5] follow the approach developed by khan [6] for his proof that the list of chromatic index of a multigraph is asymptotically equal to its fractional chromatic number.

8. OPEN QUESTIONS

This study investigated the survey of the past three decades of research in the list colorings. From this review, it is observed that still more research could be done in this area. Hence, this article proposes few open problems to the young researchers to ponder over.

- Can we characterize graphs that are 3-choosable-?
- Does χ'_L (G) = χ_L (G) for all loop less graph of G?
- What is the upper bound for $\chi_{L,2}$ for the Peterson graph ?
- Can we determine $\chi_{L,2}$ for all *n*-stars?

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