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EXISTENCE AND STABILITY FOR STOCHASTIC IMPULSIVE NEUTRAL PARTIAL DIFFERENTIAL EQUATIONS DRIVEN BY ROSENBLATT PROCESS WITH DELAY AND POISSON JUMPS

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Abstract. This paper is concerned with the existence, uniqueness and stability of mild solutions to impulsive stochastic neutral functional differential equations with finite delays driven simultaneously by a Rosenblatt process and Poisson process in a Hilbert space. Sufficient conditions for the existence of solutions are derived by means of the Banach fixed point principle. Finally, an illustrative example is given to demonstrate the effectiveness of the obtained result.

1. Introduction

In this paper, we study the existence, uniqueness and asymptotic behavior of mild solutions for a class of impulsive stochastic neutral functional differential equations with delays described in the form

$$\begin{cases} d[x(t)+g(t,x(t-r(t)))] = [Ax(t)+f(t,x(t-\rho(t)))]dt+\sigma(t)dZ_H(t) \\ +\int_{\mathcal{U}} h(t,x(t-\theta(t)),y)\widetilde{N}(dt,dy),\ t\geq 0,\ t\neq t_k \\ \Delta x(t_k)=x(t_k^+)-x(t_k^-)=I_k x(t_k), \quad t=t_k,\ k=1,2,...., \\ x(t)=\varphi(t),\ -\tau\leq t\leq 0, \end{cases} \tag{1.1}$$

where A is the infinitesimal generator of an analytic semigroup of bounded linear operators, $(S(t))_{t\geq 0}$, in a Hilbert space X, Z_H is a Rosenblatt process on a real and separable Hilbert space Y, $r,\ \rho,\ \theta : [0,+\infty)\to[0,\tau]$ $(\tau>0)$ are continuous and $f,g:[0,+\infty)\times X\to X,\ \sigma:[0,+\infty)\to\mathcal{L}_2^0(Y,X),\quad h:[0,+\infty)\times X\times\mathcal{U}\to X$ are appropriate functions. Here $\mathcal{L}_2^0(Y,X)$ denotes the space of all Q-Hilbert-Schmidt operators from Y into X (see section 2 below). Moreover, the fixed moments of time t_k satisfy $0<t_1<t_2<...<t_k<...$ and $\lim_{k\longrightarrow\infty} t_k=\infty$, $x(t_k^-)$ and $x(t_k^+)$ represent the left and right limits of $x(t)$ at time $t_k,\ k=1,2,...$ respectively. $\Delta x(t_k)$ denotes the jump in the state x at time t_k with $I_k(.):X\longrightarrow X$ determining the size of the jump. The initial data $\varphi\in\mathcal{D}:=D([-\tau,0],X)$ the

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space of càdlàg functions from $[-\tau, 0]$ into X equipped with the supremum norm $|\varphi|_{\mathcal{D}} = \sup_{s\in[-\tau,0]} \|\varphi(s)\|_X$, and φ has finite second moment.

Stochastic differential equations arise in many areas of science and engineering, wherein, quite often the future state of such systems depends not only on present state but also on its history leading to stochastic functional differential equations with delays rather than SDEs. However, many stochastic dynamical systems depend not only on present and past states, but also contain the derivatives with delays. Neutral stochastic differential equations with delays are often used to describe such systems (see, e.g., [6, 9]). On the other hand, the stability of impulsive differential equations has been discussed by several authors (see, e.g. [1, 18, 23, 27]). In addition, the study of neutral SFDEs driven by jumps process also have begun to gain attention and strong growth in recent years. To be more precise, we refer [8, 12, 15, 21].

In recent years the stochastic functional differential equations driven by a fractional Brownian motion (fBm) have attracted the attention of many authors and many valuable results on existence, uniqueness and the stability of the solution have been established, see [2, 11, 13, 10]. For example, using the Riemann-Stieltjes integral, Boufoussi et al. [4] proved the existence and uniqueness of a mild solution and studied the dependence of the solution on the initial condition in infinite dimensional space. Very recently, Caraballo et al. [5] and Boufoussi and Hajji [3] have discussed the existence, uniqueness and exponential asymptotic behavior of mild solutions to stochastic evolution differential equations by using Wiener integral.

On the other hand, the very large utilization of the fractional Brownian motion in practice are due to its self-similarity, stationarity of increments and long-range dependence; one prefers in general fBm before other processes because it is Gaussian and the calculus for it is easier; but in concrete situations when the gaussianity is not plausible for the model, one can use for example the Rosenblatt process. Although defined during the 60s and 70s [22, 25] due to their appearance in the Non-Central Limit Theorem, the systematic analysis of Rosenblatt processes has only been developed during the last ten years, motivated by their nice properties (self-similarity, stationarity of the increments, long-range dependence). Since they are non-Gaussian and self-similar with stationary increments, the Rosenblatt processes can also be an input in models where self-similarity is observed in empirical data which appears to be non-Gaussian. There exists a consistent literature that focuses on different theoretical aspects of the Rosenblatt processes. Let us recall some of these works. For example, the rate of convergence to the Rosenblatt process in the Non Central Limit Theorem has been given by Leonenko and Ahn [14]. Tudor [26] studied the analysis of the Rosenblatt process. The distribution of the Rosenblatt process has been given in [17].

On the other hand, to the best of our knowledge, there is no paper which investigates the study of impulsive stochastic neutral functional differential equations with delays driven both by Rosenblatt process and by Poisson point processes. Thus, we will make the first attempt to study such problem in this paper. Our results are inspired by the one in [3] where the existence and uniqueness of mild solutions to model (1.1) is studied, with fractional Brownian motion and $h = 0$, as well as some results on the asymptotic behavior.

The rest of this paper is organized as follows, In Section 2, we summarize several important working tools on Rosenblatt process, Poisson point processes and we recall some preliminary results about analytic semi-groups and fractional power associated to its generator that will be used to develop our results. In section 3, by the Banach fixed point theorem we consider a sufficient condition for the existence, uniqueness and exponential decay to zero in mean square for mild solutions of equation (1.1). In Section 4, we give an example to illustrate the efficiency of the obtained result.

2. Preliminaries

In this section, we collect some definitions and lemmas on Wiener integrals with respect to an infinite dimensional Rosenblatt process and we recall some basic results about analytical semi-groups and fractional powers of their infinitesimal generators, which will be used throughout the whole of this paper. For details of this section, we refer the reader to [26, 19] and references therein.

Let $(\mathcal{U}, \mathcal{E}, \nu(du))$ be a σ-finite measurable space. Given a stationary Poisson point process $(p_t)_{t>0}$, which is defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with values in $\mathcal{U}$ and with characteristic measure ν (see [7]). We will denote by $N(t, du)$ be the counting measure of p_t such that $\widehat{N}(t, A) := \mathbb{E}(N(t, A)) = t\nu(A)$ for $A \in \mathcal{E}$. Define $\widetilde{N}(t, du) := N(t, du) - t\nu(du)$, the Poisson martingale measure generated by p_t.

2.1. Rosenblatt process. Selfsimilar processes are invariant in distribution under suitable scaling. They are of considerable interest in practice since aspects of the selfsimilarity appear in different phenomena like telecommunications, turbulence, hydrology or economics. A self-similar processes can be defined as limits that appear in the so-called Non-Central Limit Theorem (see [25]). We briefly recall the Rosenblatt process as well as the Wiener integral with respect to it.

Let us recall the notion of Hermite rank. Denote by $H_j(x)$ the Hermite polynomial of degree j given by $H_j(x) = (-1)^j e^{\frac{x^2}{2}} \frac{d^j}{dx^j} e^{\frac{-x^2}{2}}$ and let g be a function on $\mathbb{R}$ such that $\mathbb{E}[g(\zeta_0)] = 0$ and $\mathbb{E}[g(\zeta_0)^2] < \infty$. Assume that g has the following expansion in Hermite polynomials

$$g(x) = \sum_{j \geq 0} c_j H_j(x),$$

where $c_j = \frac{1}{j!}\mathbb{E}(g(\zeta_0 H_j(\zeta_0)))$. The Hermite rank of g is defined by

$$k = \min\{j | c_j \neq 0\}.$$

Consider $(\zeta_n)_{n\in\mathbb{Z}}$ a stationary Gaussian sequence with mean zero and variance 1 which exhibits long range dependence in the sense that the correlation function satisfies

$$r(n) = \mathbb{E}(\zeta_0 \zeta_n) = n^{\frac{2H-2}{k}} L(n),$$

with $H \in (\frac{1}{2}, 1)$ and L is a slowly varying function at infinity. Since $\mathbb{E}[g(\zeta_0)] = 0$, we have $k \geq 1$. Then the following family of stochastic processes

$$\frac{1}{n^H} \sum_{j=1}^{[nt]} g(\zeta_j)$$

converges as $n \longrightarrow \infty$, in the sense of finite dimensional distributions, to the selfsimilar stochastic process with stationary increments

$$Z_H^k(t) = c(H,k) \int_{\mathbb{R}^k} \left(\int_0^t \prod_{j=1}^k (s - y_j)_+^{-(\frac{1}{2} + \frac{1-H}{k})} ds \right) dB(y_1)...dB(y_k), \qquad (2.1)$$

where $x_+ = max(x, 0)$. The above integral is a Wiener-Itô multiple integral of order k with respect to the standard Brownian motion $(B(y))_{y \in \mathbb{R}}$ and the constant $c(H,k)$ is a normalizing constant that ensures $\mathbb{E}(Z_H^k(1))^2 = 1$. The process $(Z_H^k(t))_{t \geq 0}$ is called the Hermite process. When $k = 1$ the process given by (2.1) is nothing else that the fractional Brownian motion (fBm) with Hurst parameter $H \in (\frac{1}{2}, 1)$. For $k = 2$ the process is not Gaussian. If $k = 2$ then the process (2.1) is known as the Rosenblatt process. It was introduced by Rosenblatt in [22] and was given its name by Taqqu in [24]. The Rosenblatt process is of course the most studied process in the class of Hermite processes due to its significant importance in modelling. A stochastic calculus with respect to it has been intensively developed in the last decade. The Rosenblatt process is, after fBm, the most well known Hermite process.

We also recall the following properties of the Rosenblatt process:

- The process Z_H^k is H-selfsimilar in the sense that for any $c > 0$,

  $$(Z_H^k(ct)) =^{(d)} (c^H Z_H^k(t)),$$

  where " $=^{(d)}$ " means equivalence of all finite dimensional distributions. It has stationary increments and all moments are finite.
- From the stationarity of increments and the self-similarity, it follows that, for any $p \geq 1$

  $$\mathbb{E}|Z_H(t) - Z_H(s)|^p \leq |\mathbb{E}(Z_H(1))|^p |t - s|^{pH}.$$

  As a consequence the Rosenblatt process has Hölder continuous paths of order γ with $0 < \gamma < H$.

Self-similarity and long-range dependence make this process a useful driving noise in models arising in physics, telecommunication networks, finance and other fields.

Consider a time interval $[0, T]$ with arbitrary fixed horizon T and let $\{Z_H(t), t \in [0, T]\}$ the one-dimensional Rosenblatt process with parameter $H \in (1/2, 1)$. By Tudor [26], it is well known that Z_H has the following integral representation:

$$Z_H(t) = d(H) \int_0^t \int_0^t \left[\int_{y_1 \vee y_2}^t \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) du \right] dB(y_1) dB(y_2), \quad (2.2)$$

where $B=\{B(t): \ t\in[0,T]\}$ is a Wiener process, $H'=\frac{H+1}{2}$, $d(H)=\frac{1}{H+1}\sqrt{\frac{H}{2(2H-1)}}$ is a normalizing constant, and $K^H(t,s)$ is the kernel given by

$$K^H(t,s)=c_H s^{\frac{1}{2}-H}\int_s^t (u-s)^{H-\frac{3}{2}}u^{H-\frac{1}{2}}du,$$

for $t>s$, where $c_H=\sqrt{\frac{H(2H-1)}{\beta(2-2H,H-\frac{1}{2})}}$ and $\beta(.,.)$ denotes the Beta function. We put $K^H(t,s)=0$ if $t\leq s$.

The covariance of the Rosenblatt process $\{Z_H(t), t\in[0,T]\}$ satisfies, for every $s,t\geq 0$,

$$R_H(s,t):=\mathbb{E}(Z_H(t)Z_H(s))=\frac{1}{2}(t^{2H}+s^{2H}-|t-s|^{2H}).$$

The basic observation is the fact that the covariance structure of the Rosenblatt process is similar to the one of the Rosenblatt process and this allows the use of the same classes of deterministic integrands as in the Rosenblatt process case whose properties are known.

Now, we introduce Wiener integrals with respect to the Rosenblatt process. We refer to [26] for additional details on the Rosenblatt process . By formula (2.2) we can write

$$Z_H(t)=\int_0^t\int_0^t I(\mathbf{1}_{[0,t]})(y_1,y_2)dB(y_1)dB(y_2),$$

where by I we denote the mapping on the set of functions $f:[0,T]\longrightarrow\mathbb{R}$ to the set of functions $f:[0,T]^2\longrightarrow\mathbb{R}$

$$I(f)(y_1,y_2)=d(H)\int_{y_1\vee y_2}^T f(u)\frac{\partial K^{H'}}{\partial u}(u,y_1)\frac{\partial K^{H'}}{\partial u}(u,y_2)du.$$

Let us denote by $\mathcal{E}$ the class of elementary functions on R of the form

$$f(.)=\sum_{j=1}^n a_j\mathbf{1}_{(t_j,t_{j+1}]}(.),\qquad 0\leq t_j<t_{j+1}\leq T,\quad a_j\in\mathbb{R},\quad j=1,...,n.$$

For $f\in\mathcal{E}$ as above, it is natural to define its Wiener integral with respect to the Rosenblatt process Z_H by

$$\int_0^T f(s)dZ_H(s):=\sum_{j=1}^n a_j\left[Z_H(t_{j+1})-Z_H(t_j)\right]=\int_0^T\int_0^T I(f)(y_1,y_2)dB(y_1)dB(y_2).$$

Let $\mathcal{H}$ be the set of functions f such that

$$\mathcal{H}=\left\{f:[0,T]\longrightarrow\mathbb{R}:\quad \|f\|_{\mathcal{H}}:=\int_0^T\int_0^T\left(I(f)(y_1,y_2)\right)^2dy_1dy_2<\infty\right\}.$$

It hold that (see Maejima and Tudor [16])

$$\|f\|_{\mathcal{H}}=H(2H-1)\int_0^T\int_0^T f(u)f(v)|u-v|^{2H-2}dudv,$$

and, the mapping

$$f\longrightarrow\int_0^T f(u)dZ_H(u)\tag{2.3}$$

provides an isometry from $\mathcal{E}$ to $L^2(\Omega)$. On the other hand, it has been proved in [20] that the set of elementary functions $\mathcal{E}$ is dense in $\mathcal{H}$. As a consequence the mapping (2.3) can be extended to an isometry from $\mathcal{H}$ to $L^2(\Omega)$. We call this extension as the Wiener integral of $f \in \mathcal{H}$ with respect to Z_H.

Let us consider the operator K_H^* from $\mathcal{E}$ to $\mathbb{L}^2([0,T])$ defined by

$$(K_H^*\varphi)(y_1, y_2) = \int_{y_1 \vee y_2}^T \varphi(r) \frac{\partial K}{\partial r}(r, y_1, y_2) dr,$$

where $K(.,.,.)$ is the kernel of Rosenblatt process in representation (2.2)

$$K(r, y_1, y_2) = \mathbf{1}_{[0,t]}(y_1)\mathbf{1}_{[0,t]}(y_2) \int_{y_1 \vee y_2}^t \frac{\partial K^{H'}}{\partial u}(u, y_1) \frac{\partial K^{H'}}{\partial u}(u, y_2) du.$$

We refer to [26] for the proof of the fact that K_H^* is an isometry between $\mathcal{H}$ and $L^2([0,T])$. It follows from [26] that $\mathcal{H}$ contains not only functions but its elements could be also distributions. In order to obtain a space of functions contained in $\mathcal{H}$, we consider the linear space $|\mathcal{H}|$ generated by the measurable functions ψ such that

$$\|\psi\|_{|\mathcal{H}|}^2 := \alpha_H \int_0^T \int_0^T |\psi(s)||\psi(t)||s-t|^{2H-2} ds dt < \infty,$$

where $\alpha_H = H(2H-1)$. The space $|\mathcal{H}|$ is a Banach space with the norm $\|\psi\|_{|\mathcal{H}|}$ and we have the following inclusions (see [26]).

Lemma 2.1.

$$\mathbb{L}^2([0,T]) \subseteq \mathbb{L}^{1/H}([0,T]) \subseteq |\mathcal{H}| \subseteq \mathcal{H},$$

and for any $\psi \in \mathbb{L}^2([0,T])$, *we have*

$$\|\psi\|_{|\mathcal{H}|}^2 \leq 2HT^{2H-1} \int_0^T |\psi(s)|^2 ds.$$

Let X and Y be two real, separable Hilbert spaces and let $\mathcal{L}(Y,X)$ be the space of bounded linear operator from Y to X. For the sake of convenience, we shall use the same notation to denote the norms in X, Y and $\mathcal{L}(Y,X)$. Let $Q \in \mathcal{L}(Y,Y)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite trace $trQ = \sum_{n=1}^{\infty} \lambda_n < \infty$. where $\lambda_n \geq 0$ $(n = 1, 2...)$ are non-negative real numbers and $\{e_n\}$ $(n = 1, 2...)$ is a complete orthonormal basis in Y. We define the infinite dimensional $Q-$Rosenblatt process on Y as

$$Z_H(t) = Z_Q(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n z_n(t), \tag{2.4}$$

where $(z_n)_{n \geq 0}$ is a family of real independent Rosenblatt process. Note that the series (2.4) is convergent in $L^2(\Omega)$ for every $t \in [0,T]$, since

$$\mathbb{E}|Z_Q(t)|^2 = \sum_{n=1}^{\infty} \lambda_n \mathbb{E}(z_n(t))^2 = t^{2H} \sum_{n=1}^{\infty} \lambda_n < \infty.$$

Note also that Z_Q has covariance function in the sense that

$$E\langle Z_Q(t), x\rangle \langle Z_Q(s), y\rangle = R(s,t)\langle Q(x), y\rangle \text{ for all } x, y \in Y \text{ and } t, s \in [0,T].$$

In order to define Wiener integrals with respect to the Q-Rosenblatt process, we introduce the space $\mathcal{L}_2^0 := \mathcal{L}_2^0(Y, X)$ of all Q-Hilbert-Schmidt operators $\psi : Y \to X$. We recall that $\psi \in \mathcal{L}(Y, X)$ is called a Q-Hilbert-Schmidt operator, if

$$\|\psi\|^2_{\mathcal{L}_2^0} := \sum_{n=1}^{\infty} \|\sqrt{\lambda_n}\psi e_n\|^2 < \infty,$$

and that the space $\mathcal{L}_2^0$ equipped with the inner product $\langle \varphi, \psi \rangle_{\mathcal{L}_2^0} = \sum_{n=1}^{\infty} \langle \varphi e_n, \psi e_n \rangle$ is a separable Hilbert space.

Now, let $\phi(s)$; $s \in [0, T]$ be a function with values in $\mathcal{L}_2^0(Y, X)$, such that $\sum_{n=1}^{\infty} \|K^* \phi Q^{\frac{1}{2}} e_n\|^2_{\mathcal{L}_2^0} < \infty$. The Wiener integral of ϕ with respect to Z_Q is defined by

$$\begin{aligned} \int_0^t \phi(s) dZ_Q(s) &= \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n} \phi(s) e_n dz_n(s) \\ &= \sum_{n=1}^{\infty} \int_0^t \int_0^t \sqrt{\lambda_n} K_H^*(\phi e_n)(y_1, y_2) dB(y_1) dB(y_2). \end{aligned}$$

Now, we end this subsection by stating the following result which is fundamental to prove our result.

Lemma 2.2. *If $\psi : [0, T] \to \mathcal{L}_2^0(Y, X)$ satisfies $\int_0^T \|\psi(s)\|^2_{\mathcal{L}_2^0} ds < \infty$ then the above sum in* (2.1) *is well defined as a X-valued random variable and we have*

$$\mathbb{E}\| \int_0^t \psi(s) dZ_H(s) \|^2 \leq 2Ht^{2H-1} \int_0^t \|\psi(s)\|^2_{\mathcal{L}_2^0} ds.$$

Proof. By Lemma 2.1, we have

$$\begin{aligned} \mathbb{E}\| \int_0^t \psi(s) dZ_H(s) \|^2 &= \sum_{n=1}^{\infty} \mathbb{E}\| \int_0^t \int_0^t \sqrt{\lambda_n} K_H^*(\psi e_n)(y_1, y_2) dB(y_1) dB(y_2) \|^2 \\ &\leq \sum_{n=1}^{\infty} 2Ht^{2H-1} \int_0^t \lambda_n \|\psi(s) e_n\|^2 ds \\ &= 2Ht^{2H-1} \int_0^t \|\psi(s)\|^2_{\mathcal{L}_2^0} ds. \end{aligned}$$

□

Now we turn to state some notations and basic facts about the theory of semigroups and fractional power operators. Let $A : D(A) \to X$ be the infinitesimal generator of an analytic semigroup, $(S(t))_{t \geq 0}$, of bounded linear operators on X. For the theory of strongly continuous semigroup, we refer to [19]. We will point out here some notations and properties that will be used in this work. It is well known that there exist $M \geq 1$ and $\nu \in \mathbb{R}$ such that $\|S(t)\| \leq Me^{\nu t}$ for every $t \geq 0$. If $(S(t))_{t \geq 0}$ is a uniformly bounded and analytic semigroup such that $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of A, then it is possible to define the fractional power $(-A)^\alpha$ for $0 < \alpha \leq 1$, as a closed linear operator on its domain $D(-A)^\alpha$. Furthermore, the subspace $D(-A)^\alpha$ is dense in X, and the expression

$$\|h\|_\alpha = \|(-A)^\alpha h\|$$

defines a norm in $D(-A)^\alpha$. If X_α represents the space $D(-A)^\alpha$ endowed with the norm $\|.\|_\alpha$, then the following properties are well known (cf. [19], p. 74).

Lemma 2.3. *Suppose that the preceding conditions are satisfied.*
(1) Let $0 < \alpha \leq 1$. Then X_α is a Banach space.
(2) If $0 < \beta \leq \alpha$ then the injection $X_\alpha \hookrightarrow X_\beta$ is continuous.
(3) For every $0 < \alpha \leq 1$ there exists $M_\alpha > 0$ such that

$$\|(-A)^\alpha S(t)\| \leq M_\alpha t^{-\alpha} e^{-\nu t}, \quad t > 0, \;\; \nu > 0.$$

3. Main Results

In this section, we consider existence, uniqueness and exponential stability of mild solution to Equation (1.1). Our main method is the Banach fixed point principle. First we define the space S_φ of the càdlàg processes $x(t)$ as follows:

Definition 3.1. Let the space S_φ denote the set of all càdlàg processes $x(t)$ such that $x(t) = \varphi(t)$ $t \in [-\tau, 0]$ and there exist some constants $N = N(\varphi, a) > 0$ and $a > 0$

$$\mathbb{E}\|x(t)\|^2 \leq N e^{-at}, \qquad \forall\, t \geq 0.$$

Definition 3.2. $\|.\|_{S_\varphi}$ denotes the norm in S_φ which is defined by

$$\|x\|_{S_\varphi} := \sup_{t \geq 0} \mathbb{E}\|x(t)\|_X^2 \qquad for\ x \in S_\varphi.$$

Remark 3.3. It is routine to check that S_φ is a Banach space endowed with the norm $\|.\|_{S_\varphi}$.

In order to obtain our main result, we assume that the following conditions hold.

($\mathcal{H}$.1) A is the infinitesimal generator of an analytic semigroup, $(S(t))_{t \geq 0}$, of bounded linear operators on X. Further, to avoid unnecessary notations, we suppose that $0 \in \rho(A)$, and that

$$\|S(t)\| \leq M e^{-\nu t}$$

for some constants M, ν and every $t \in [0, T]$.

($\mathcal{H}$.2) There exists a positive constant $K_1 > 0$ such that, for all $t \in [0, T]$ and $x, y \in X$

$$\|f(t, x) - f(t, y)\|^2 \leq K_1 \|x - y\|^2.$$

($\mathcal{H}$.3) There exist constants $0 < \beta < 1$, $K_2 > 0$ such that the function g is X_β-valued and satisfies for all $t \in [0, T]$ and $x, y \in X$

$$\|(-A)^\beta g(t, x) - (-A)^\beta g(t, y)\|^2 \leq K_2 \|x - y\|^2.$$

($\mathcal{H}$.4) The function $(-A)^\beta g$ is continuous in the quadratic mean sense:

$$\text{For all } x \in D([0,T], \mathbb{L}^2(\Omega, X)), \;\; \lim_{t \to s} \mathbb{E}\|(-A)^\beta g(t, x(t)) - (-A)^\beta g(s, x(s))\|^2 = 0.$$

($\mathcal{H}$.5) There exists some $\gamma > 0$ such that the function $\sigma : [0, +\infty) \to \mathcal{L}_2^0(Y, X)$ satisfies

$$\int_0^\infty e^{2\gamma s} \|\sigma(s)\|^2_{\mathcal{L}_2^0} ds < \infty.$$

($\mathcal{H}$.6) There exists a positive constant $K_3 > 0$ such that, for all $t \in [0,T]$ and $x, y \in X$

$$\int_{\mathcal{U}} \|h(t,x,z) - h(t,y,z)\|_X^2 \nu(dz) \leq K_3 \|x-y\|_X^2.$$

We further assume that $g(t,0) = f(t,0) = h(t,0,z) = 0$ for all $t \geq 0$ and $z \in \mathcal{U}$.

($\mathcal{H}$.7) $I_k \in C(X,X)$ and there exists a positive constant h_k such that $\|I_k(x) - I_k(y)\| \leq h_k\|x-y\|$ and $I_k(0) = 0$, k=1,2,..., for each $x, y \in X$ and $\sum_{k=1}^{\infty} h_k < \infty$.

Similar to the deterministic situation we give the following definition of mild solutions for equation (1.1).

Definition 3.4. An X-valued stochastic process $\{x(t),\ t \in [-\tau, T]\}$, is called a *mild solution* of equation (1.1) if

i) $x(.)$ has càdlàg path, and $\int_0^T \|x(t)\|^2 dt < \infty$ almost surely;
ii) $x(t) = \varphi(t)$, $-\tau \leq t \leq 0$.
iii) For arbitrary $t \in [0,T]$, $x(t)$ satisfies the following integral equation

$$\begin{cases} x(t) = & S(t)(\varphi(0) + g(0, \varphi(-r(0)))) - g(t, x(t-r(t))) \\ & - \int_0^t AS(t-s)g(s, x(s-r(s)))ds + \int_0^t S(t-s)f(s, x(s-\rho(s)))ds \\ & + \int_0^t S(t-s)\sigma(s)dZ_H(s) \\ & + \int_0^t \int_{\mathcal{U}} S(t-s)h(s, x(s-\theta(s)), y)\widetilde{N}(ds, dy) \\ & + \sum_{0<t_k<t} S(t-t_k)I_k x(t_k), \quad \mathbb{P}-a.s. \end{cases}$$

The main result of this paper is given in the next theorem.

Theorem 3.5. *Suppose that* ($\mathcal{H}$.1) – ($\mathcal{H}$.7) *hold and that*

$$K_2\|(-A)^{-\beta}\|^2 + K_2 M_{1-\beta}^2 \nu^{-2\beta}\Gamma(\beta)^2 + K_1 M^2 \nu^{-2} + M^2 K_3 (2\nu)^{-1} + M^2 (\sum_{k=1}^{\infty} h_k)^2 < \frac{1}{5},$$

where $\Gamma(.)$ *is the Gamma function,* $M_{1-\beta}$ *is the corresponding constant in Lemma 2.3. If the initial value* $\varphi(t)$ *satisfies*

$$\mathbb{E}\|\varphi(t)\|^2 \leq M_0 \mathbb{E}|\varphi|_{\mathcal{D}}^2 e^{-at}, \quad t \in [-\tau, 0],$$

for some $M_0 > 0$*,* $a > 0$*; then, for all* $T > 0$*, the equation (1.1) has a unique mild solution on* $[-\tau, T]$ *and is exponential decay to zero in mean square, i.e., there exists a pair of positive constants* $a > 0$ *and* $N = N(\varphi, a)$ *such that*

$$\mathbb{E}\|x(t)\|^2 \leq Ne^{-at},\ \forall t \geq 0.$$

Proof. Define the mapping Ψ on S_φ as follows:

$$\Psi(x)(t) := \varphi(t), \qquad t \in [-\tau, 0],$$

and for $t \in [0, T]$

$$\begin{aligned}\Psi(x)(t) &= S(t)(\varphi(0) + g(0, \varphi(-r(0)))) - g(t, x(t - r(t))) \\ &- \int_0^t AS(t-s)g(s, x(s - r(s)))ds + \int_0^t S(t-s)f(s, x(s - \rho(s))ds \\ &+ \int_0^t S(t-s)\sigma(s)dZ_H(s) + \int_0^t \int_{\mathcal{U}} S(t-s)h(s, x(s - \theta(s)), y)\widetilde{N}(ds, dy) \\ &+ \sum_{0<t_k<t} S(t - t_k)I_k x(t_k).\end{aligned}$$

Then it is clear that to prove the existence of mild solutions to equation (1.1) is equivalent to find a fixed point for the operator Ψ.

We will show by using Banach fixed point theorem that Ψ has a unique fixed point. First we show that $\Psi(S_\varphi) \subset S_\varphi$. Let $x(t) \in S_\varphi$, then we have

$$\begin{aligned}&\mathbb{E}\|\Psi(x)(t)\|^2 \leq 7\mathbb{E}\|S(t)(\varphi(0) + g(0, \varphi(-r(0))))\|^2 \\ &+7\mathbb{E}\|g(t, x(t - r(t)))\|^2 + 7\mathbb{E}\|\int_0^t AS(t-s)g(s, x(s - r(s)))ds\|^2 \\ &+7\mathbb{E}\|\int_0^t S(t-s)f(s, x(s - \rho(s))ds\|^2 + 7\mathbb{E}\|\int_0^t S(t-s)\sigma(s)dZ_H(s)\|^2 \\ &+7\mathbb{E}\|\int_0^t \int_{\mathcal{U}} S(t-s)h(s, x(s - \theta(s)), y)\widetilde{N}(ds, dy)\|^2 \\ &+7\mathbb{E}\|\sum_{0<t_k<t} S(t - t_k)I_k x(t_k)\|^2 \\ &:= 7\sum_{i=1}^{7} I_i.\end{aligned} \tag{3.1}$$

Now, let us estimate the terms on the right of the inequality (3.1). Let $N = N(\varphi, a) > 0$ and $a > 0$ such that

$$\mathbb{E}\|x(t)\|^2 \leq Ne^{-at}, \quad \forall\, t \geq 0.$$

Without loss of generality we may assume that $0 < a < \nu$. Then, by assumption $(\mathcal{H}.1)$ we have

$$I_1 \leq M^2\mathbb{E}\|\varphi(0) + g(0, \varphi(-r(0)))\|^2 e^{-\nu t} \leq C_1 e^{-\nu t}, \tag{3.2}$$

where $C_1 = M^2\mathbb{E}\|\varphi(0) + g(0, \varphi(-r(0)))\|^2 < +\infty$.

By using assumption $(\mathcal{H}.3)$ and the fact that the operator $(-A)^{-\beta}$ is bounded, we obtain that

$$\begin{aligned}I_2 &\leq \|(-A)^{-\beta}\|^2\mathbb{E}\|(-A)^{\beta}g(t, x(t - r(t))) - (-A)^{\beta}g(t, 0)\|^2 \\ &\leq K_2\|(-A)^{-\beta}\|^2\mathbb{E}\|x(t - r(t))\|^2 \\ &\leq K_2\|(-A)^{-\beta}\|^2(Ne^{-a(t-r(t))} + \mathbb{E}\|\varphi(t - r(t))\|^2) \\ &\leq K_2\|(-A)^{-\beta}\|^2(N + M_0\mathbb{E}|\varphi|^2_{\mathcal{D}})e^{-a(t-r(t))} \\ &\leq K_2\|(-A)^{-\beta}|^2(N + M_0\mathbb{E}|\varphi|^2_{\mathcal{D}})e^{-at}e^{a\tau} \\ &\leq C_2 e^{-at},\end{aligned} \tag{3.3}$$

where $C_2 = K_2\|(-A)^{-\beta}\|^2(N + M_0\mathbb{E}|\varphi|^2_{\mathcal{D}})e^{a\tau} < +\infty$.

To estimate I_3, we use the trivial identity

$$c^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1} e^{-ct} \quad , \forall\, c > 0. \tag{3.4}$$

Using Hölder's inequality, Lemma 2.3 together with assumption $(\mathcal{H}.3)$ and the identity (3.4), we get

$$\begin{aligned}
I_3 &\leq \mathbb{E}\| \int_0^t AS(t-s) g(s, x(s-r(s)))ds\|^2 \\
&\leq \int_0^t \|(-A)^{1-\beta} S(t-s)\| ds \int_0^t \|(-A)^{1-\beta} S(t-s)\| \\
&\quad \times \mathbb{E}\|(-A)^{\beta} g(s, x(s-r(s)))\|^2 ds \\
&\leq M_{1-\beta}^2 K_2 \int_0^t (t-s)^{\beta-1} e^{-\nu(t-s)} ds \\
&\quad \times \int_0^t (t-s)^{\beta-1} e^{-\nu(t-s)} \mathbb{E}\|x(s-r(s))\|^2 ds \\
&\leq M_{1-\beta}^2 K_2 \nu^{-\beta} \Gamma(\beta) \int_0^t (t-s)^{\beta-1} e^{-\nu(t-s)} (N + M_0 \mathbb{E}|\varphi|_{\mathcal{D}}^2) e^{-as} e^{a\tau} ds \\
&\leq M_{1-\beta}^2 K_2 \nu^{-\beta} \Gamma(\beta)(N + M_0 \mathbb{E}|\varphi|_{\mathcal{D}}^2) e^{-at} e^{a\tau} \int_0^t (t-s)^{\beta-1} e^{(a-\nu)(t-s)} ds \\
&\leq M_{1-\beta}^2 K_2 \nu^{-\beta} \Gamma^2(\beta)(\nu - a)^{-1}(N + M_0 \mathbb{E}|\varphi|_{\mathcal{D}}^2) e^{a\tau} e^{-at} \\
&\leq C_3 e^{-at},
\end{aligned} \tag{3.5}$$

where $C_3 = M_{1-\beta}^2 K_2 \nu^{-\beta} \Gamma^2(\beta)(\nu - a)^{-1}(N + M_0 \mathbb{E}|\varphi|_{D}^2) e^{a\tau} < +\infty$.

Similar computations can be used to estimate the term I_4.

$$\begin{aligned}
I_4 &\leq \mathbb{E}\| \int_0^t S(t-s) f(s, x(s-\rho(s)))ds\|^2 \\
&\leq M^2 K_1 \int_0^t e^{-\nu(t-s)} ds \int_0^t e^{-\nu(t-s)} \mathbb{E}\|x(s-\rho(s))\|^2 ds \\
&\leq M^2 K_1 \nu^{-1} \int_0^t e^{-\nu(t-s)} (N + M_0 \mathbb{E}|\varphi|_{\mathcal{D}}^2) e^{-as} e^{a\tau} ds \\
&\leq M^2 K_1 \nu^{-1} (N + M_0 \mathbb{E}|\varphi|_{\mathcal{D}}^2) e^{-at} e^{a\tau} \int_0^t e^{(a-\nu)(t-s)} ds \\
&\leq M^2 K_1 \nu^{-1} (\nu - a)^{-1} (N + M_0 \mathbb{E}|\varphi|_{\mathcal{D}}^2) e^{-at} e^{a\tau} \\
&\leq C_4 e^{-at}.
\end{aligned} \tag{3.6}$$

where $C_4 = M^2 K_1 \nu^{-1} (\nu - a)^{-1} (N + M_0 \mathbb{E}|\varphi|_{\mathcal{D}}^2) e^{a\tau}$.

By using Lemma 2.2, we get that

$$\begin{aligned}
I_5 &\leq \mathbb{E}\| \int_0^t S(t-s)\sigma(s) dZ_H(s)\|^2 \\
&\leq 2M^2 H t^{2H-1} \int_0^t e^{-2\nu(t-s)} \|\sigma(s)\|_{\mathcal{L}_2^0}^2 ds.
\end{aligned} \tag{3.7}$$

If $\gamma < \nu$, then the following estimate holds

$$\begin{aligned} I_5 &\leq 2M^2Ht^{2H-1}\int_0^t e^{-2\nu(t-s)}e^{-2\gamma(t-s)}e^{2\gamma(t-s)}\|\sigma(s)\|^2_{\mathcal{L}_2^0}ds \\ &\leq 2M^2Ht^{2H-1}e^{-2\gamma t}\int_0^t e^{-2(\nu-\gamma)(t-s)}e^{2\gamma s}\|\sigma(s)\|^2_{\mathcal{L}_2^0}ds \\ &\leq 2M^2HT^{2H-1}e^{-2\gamma t}\int_0^T e^{-2(\nu-\gamma)(t-s)}e^{2\gamma s}\|\sigma(s)\|^2_{\mathcal{L}_2^0}ds \\ &\leq 2M^2HT^{2H-1}e^{-2\gamma t}\int_0^T e^{2\gamma s}\|\sigma(s)\|^2_{\mathcal{L}_2^0}ds. \qquad (3.8) \end{aligned}$$

If $\gamma > \nu$, then the following estimate holds

$$I_5 \leq 2M^2HT^{2H-1}e^{-2\nu t}\int_0^T e^{2\gamma s}\|\sigma(s)\|^2_{\mathcal{L}_2^0}ds. \qquad (3.9)$$

In virtue of (3.7), (3.8) and (3.9) we obtain

$$I_5 \leq C_5 e^{-\min(\nu,\gamma)t} \qquad (3.10)$$

where $C_5 = 2M^2HT^{2H-1}\int_0^T e^{2\gamma s}\|\sigma(s)\|^2_{\mathcal{L}_2^0}ds < +\infty$.

On the other hand, by assumptions ($\mathcal{H}.1$) and ($\mathcal{H}.6$), we get

$$\begin{aligned} I_6 &\leq \mathbb{E}\|\int_0^t\int_{\mathcal{U}} S(t-s)h(s,x(s-\theta(s)),y)\widetilde{N}(ds,dy)\|^2 \\ &\leq M^2\mathbb{E}\int_0^t e^{-2\nu(t-s)}\int_{\mathcal{U}}\|h(s,x(s-\theta(s)),y)\|^2\nu(dy)ds \\ &\leq M^2K_3\int_0^t e^{-2\nu(t-s)}\mathbb{E}\|x(s-\theta(s))\|^2ds \\ &\leq M^2K_3\int_0^t e^{-2\nu(t-s)}(N+M_0\mathbb{E}|\varphi|^2_{\mathcal{D}})e^{-as}e^{a\tau}ds \\ &\leq M^2K_3(N+M_0\mathbb{E}|\varphi|^2_{\mathcal{D}})e^{-at}e^{a\tau}\int_0^t e^{(-2\nu+a)(t-s)}ds \\ &\leq M^2K_3(N+M_0\mathbb{E}|\varphi|^2_{\mathcal{D}})e^{a\tau}(2\nu-a)^{-1}e^{-at} \\ &\leq C_6e^{-at}, \qquad (3.11) \end{aligned}$$

where $C_6 = M^2K_3(N+M_0\mathbb{E}|\varphi|^2_{\mathcal{D}})e^{a\tau}(2\nu-a)^{-1} < +\infty$.

Now, we estimate the impulsive term, by assumption ($\mathcal{H}.7$), we get

$$\begin{aligned} J_7 &\leq \mathbb{E}(\sum_{0<t_k<t} Me^{-\nu(t-t_k)}h_k\|x(t_k)\|)^2 \\ &\leq M^2\mathbb{E}(\sum_{0<t_k<t} e^{-2\nu(t-t_k)}\|x(t_k)\|)^2 \\ &\leq M^2\sum_{0<t_k<t} h_k\sum_{0<t_k<t} h_ke^{-2\nu(t-t_k)}\mathbb{E}\|x(t_k)\|^2 \end{aligned}$$

$$
\begin{aligned}
&\le M^2\sum_{k=1}^{\infty}h_k\sum_{0<t_k<t}h_k e^{-2\nu(t-t_k)}Ne^{-at_k}\\
&\le M^2Ne^{-at}\sum_{k=1}^{\infty}h_k\sum_{0<t_k<t}h_k e^{(a-2\nu)(t-t_k)}\\
&\le M^2Ne^{-at}(\sum_{k=1}^{\infty}h_k)^2, \quad (0<a<\nu)\\
&\le C_7e^{-at}, \qquad (3.12)
\end{aligned}
$$

where $C_7 = M^2N(\sum_{k=1}^{\infty}h_k)^2$.

Inequalities (3.2), (3.3), (3.5), (3.6), (3.10), (3.11) and (3.12) together imply that $\mathbb{E}\|\Psi(x)(t)\|^2 \le \overline{M}e^{-\overline{a}t}$, $t \ge 0$,for some $\overline{M} > 0$ and $\overline{a} > 0$.

Next we show that $\Psi(x)(t)$ is càdlàg process on S_φ. Let $0 < t < T$ and $h > 0$ be sufficiently small. Then for any fixed $x(t) \in S_\varphi$, we have

$$
\begin{aligned}
&\mathbb{E}\|\Psi(x)(t+h)-\Psi(x)(t)\|^2\\
\le\ & 7\mathbb{E}\|(S(t+h)-S(t))(\varphi(0)+g(0,\varphi(-r(0))))\|^2\\
&+7\mathbb{E}\|g(t+h,x(t+h-r(t+h)))-g(t,x(t-r(t)))\|^2\\
&+7\mathbb{E}\|\int_0^{t+h}AS(t+h-s)g(s,x(s-r(s)))ds\\
&-\int_0^{t}AS(t-s)g(s,x(s-r(s)))ds\|^2\\
&+7\mathbb{E}\|\int_0^{t+h}S(t+h-s)f(s,x(s-\rho(s)))ds\\
&-\int_0^{t}S(t-s)f(s,x(s-\rho(s)))ds\|^2\\
&+7\mathbb{E}\|\int_0^{t+h}S(t+h-s)\sigma(s)dZ_H(s)-\int_0^{t}S(t-s)\sigma(s)dZ_H(s)\|^2\\
&+7\mathbb{E}\|\int_0^{t+h}\int_{\mathcal{U}}S(t+h-s)h(s,x(s-\theta(s)),y)\widetilde{N}(ds,dy)\\
&-\int_0^{t}\int_{\mathcal{U}}S(t-s)h(s,x(s-\theta(s)),y)\widetilde{N}(ds,dy)\|^2\\
&+7\mathbb{E}\|\sum_{0<t_k<t+h}S(t+h-t_k)I_k(x(t_k))-\sum_{0<t_k<t}S(t-t_k)I_k(x(t_k))\|^2\\
=\ & 7\sum_{1\le i\le 7}\mathbb{E}\|I_i(t+h)-I_i(t)\|^2.
\end{aligned}
$$

By assumption $(\mathcal{H}.6)$, we have

$$
\begin{aligned}
&\mathbb{E}\|I_6(t+h)-I_6(t)\|^2\\
\le\ & 2\mathbb{E}\|\int_0^{t}\int_{\mathcal{U}}(S(t+h-s)-S(t-s))h(s,x(s-\theta(s)),y)\widetilde{N}(ds,dy)\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\mathbb{E}\|\int_t^{t+h}\int_{\mathcal{U}} S(t+h-s)h(s,x(s-\theta(s)),y)\widetilde{N}(ds,dy)\|^2 \\
\leq\;& 2M^2\|S(h)-I\|\mathbb{E}\int_0^t\int_{\mathcal{U}} e^{-2\nu(t-s)}\|h(s,x(s-\theta(s)),y)\|^2\nu(dy)ds \\
&+ 2M^2\mathbb{E}\int_t^{t+h}\int_{\mathcal{U}} e^{-2\nu(t+h-s)}\|h(s,x(s-\theta(s)),y)\|^2\nu(dy)ds, \qquad (3.13)
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{E}\int_0^t\int_{\mathcal{U}} e^{-2\nu(t-s)}\|h(s,x(s-\theta(s)),y)\|^2\nu(dy)ds \\
\leq\;& K_3\int_0^t e^{-2\nu(t-s)}\mathbb{E}\|x(s-\theta(s))\|^2ds \\
\leq\;& K_3\int_0^t e^{-2\nu(t-s)}(N+M_0\mathbb{E}|\varphi|^2_{\mathcal{D}})e^{-as}e^{a\tau}ds \\
\leq\;& K_3(N+M_0\mathbb{E}|\varphi|^2_{\mathcal{D}})e^{a\tau}(2\nu-a)^{-1}e^{-at}. \qquad (3.14)
\end{aligned}
$$

Inequality (3.14) implies that there exist a constant $B>0$ such that

$$
\mathbb{E}\int_0^t\int_{\mathcal{U}} e^{-2\nu(t-s)}\|h(s,x(s-\theta(s)),y)\|^2\nu(dy)ds \leq B. \qquad (3.15)
$$

Using the strong continuity of $S(t)$ together with inequalities (3.13) and (3.15) we obtain that $\mathbb{E}\|I_6(t+h)-I_6(t)\|^2 \to 0$ as $h\to 0$.

Similarly, we can verify that $\mathbb{E}\|I_i(t+h)-I_i(t)\|^2 \to 0$ as $h\to 0$, $i=1,...,5,7$. The above arguments show that $\Psi(x)(t)$ is càdlàg process. Then, we conclude that $\Psi(S_\varphi)\subset S_\varphi$.

Now, we are going to show that $\Psi : S_\varphi \to S_\varphi$ is a contraction mapping. For this end, fix $x,y\in S_\varphi$, we have

$$
\begin{aligned}
&\mathbb{E}\|\Psi(x)(t)-\Psi(y)(t)\|^2 \\
\leq\;& 5\mathbb{E}\|g(t,x(t-r(t)))-g(t,y(t-r(t)))\|^2 \\
&+5\mathbb{E}\|\int_0^t AS(t-s)(g(s,x(s-r(s)))-g(s,y(s-r(s))))ds\|^2 \\
&+5\mathbb{E}\|\int_0^t S(t-s)(f(s,x(s-\rho(s)))-f(s,y(s-\rho(s))))ds\|^2 \\
&+5\mathbb{E}\|\int_0^t S(t-s)\int_{\mathcal{U}} h(s,x(s-\theta(s)),z)-h(s,y(s-\theta(s)),z)\widetilde{N}(ds,dz)\|^2 \\
&+5\mathbb{E}\|\sum_{0<t_k<t} S(t-t_k)[I_k(x(t_k))-I_k(y(t_k))]\|^2 \\
:=\;& 5(J_1+J_2+J_3+J_4+J_5). \qquad (3.16)
\end{aligned}
$$

We estimate the various terms of the right hand of (3.16) separately.

For the first term, we have

$$\begin{aligned} J_1 &\leq \mathbb{E}\|g(t, x(t-r(t))) - g(t, y(t-r(t)))\|^2 \\ &\leq K_2\|(-A)^{-\beta}\|^2\mathbb{E}\|x(s-r(s)) - y(s-r(s))\|^2 \\ &\leq K_2\|(-A)^{-\beta}\|^2 \sup_{s\geq 0}\mathbb{E}\|x(s) - y(s)\|^2. \end{aligned} \tag{3.17}$$

For the second term, combing Lemma 2.3 and Hölder's inequality, we get

$$\begin{aligned} J_2 &\leq \mathbb{E}\|\int_0^t AS(t-s)(g(s, x(s-r(s))) - g(s, y(s-r(s))))ds\|^2 \\ &\leq K_2M_{1-\beta}^2\int_0^t (t-s)^{\beta-1}e^{-\nu(t-s)}ds \\ &\quad \times\int_0^t (t-s)^{\beta-1}e^{-\nu(t-s)}\mathbb{E}\|x(s-r(s)) - y(s-r(s))\|^2ds \\ &\leq K_2M_{1-\beta}^2\nu^{-\beta}\Gamma(\beta)\int_0^t (t-s)^{\beta-1}e^{-\nu(t-s)}ds(\sup_{s\geq 0}\mathbb{E}\|x(s) - y(s)\|^2) \\ &\leq K_2M_{1-\beta}^2\nu^{-2\beta}\Gamma(\beta)^2\sup_{s\geq 0}\mathbb{E}\|x(s) - y(s)\|^2. \end{aligned} \tag{3.18}$$

For the third term, by assumption $(\mathcal{H}.2)$, we get that

$$\begin{aligned} J_3 &\leq \mathbb{E}\|\int_0^t S(t-s)(f(s, x(s-\rho(s))) - f(s, y(s-\rho(s))))ds\|^2 \\ &\leq K_1M^2\int_0^t e^{-\nu(t-s)}ds\int_0^t e^{-\nu(t-s)}\mathbb{E}\|x(s-\rho(s)) - y(s-\rho(s))\|^2ds \\ &\leq K_1M^2\nu^{-2}\sup_{s\geq 0}\mathbb{E}\|x(s) - y(s)\|^2. \end{aligned} \tag{3.19}$$

For the term J_4, by using assumption $(\mathcal{H}.6)$, we get

$$\begin{aligned} J_4 &\leq \mathbb{E}\|\int_0^t S(t-s)\int_{\mathcal{U}} h(s, x(s-\theta(s)), z) - h(s, y(s-\theta(s)), z)\widetilde{N}(ds, dz)\|^2 \\ &\leq M^2\mathbb{E}\|\int_0^t e^{-2\nu(t-s)}\int_{\mathcal{U}}\|h(s, x(s-\theta(s)), z) - h(s, y(s-\theta(s)), z)\|^2\nu(dz)ds \\ &\leq M^2K_3(2\nu)^{-1}\sup_{s\geq 0}\mathbb{E}\|x(s) - y(s)\|^2. \end{aligned} \tag{3.20}$$

For the last term, we have

$$\begin{aligned} J_5 &\leq \mathbb{E}\|\sum_{0<t_k<t} S(t-t_k)[I_k(x(t_k)) - I_k(y(t_k))]\|^2 \\ &\leq M^2(\sum_{0<t_k<t} e^{-\nu(t-t_k)}h_k\mathbb{E}\|x(t) - y(t)\|)^2 \\ &\leq M^2(\sum_{k=0}^{\infty} h_k)^2(\sup_{t\geq 0}\mathbb{E}\|x(t) - y(t)\|^2). \end{aligned} \tag{3.21}$$

Thus inequalities (3.17), (3.18), (3.19), (3.20) and (3.21) together imply

$$\begin{aligned}&\sup_{t\geq 0}\mathbb{E}\|\Psi(x)(t)-\Psi(y)(t)\|^2\\ &\leq 5[K_2\|(-A)^{-\beta}\|^2+K_2M_{1-\beta}^2\nu^{-2\beta}\Gamma(\beta)^2+K_1M^2\nu^{-2}\\ &\quad +M^2K_3(2\nu)^{-1}+M^2(\sum_{k=0}^{\infty}h_k)^2](\sup_{t\geq 0}\mathbb{E}\|x(t)-y(t)\|^2).\end{aligned}$$

Therefore by the condition of the theorem it follows that Ψ is a contractive mapping. Thus by the Banach fixed point theorem Ψ has the fixed point $x(t)\in S_\varphi$, which is a unique mild solution to (1.1) satisfying $x(s)=\varphi(s)$ on $[-\tau,0]$. By the definition of the space S_φ this solution is exponentially stable in mean square. This completes the proof. □

4. Example

We consider the following impulsive neutral stochastic partial differential equation with Poisson jumps and finite delays driven by a Rosenblatt process of the form:

$$\begin{cases} d[x(t,\xi)+\frac{\alpha_1}{M_{\frac{1}{4}}\|(-A)^{\frac{3}{4}}\|}x(t-r(t),\xi)]=[\frac{\partial^2}{\partial^2\xi}x(t,\xi)+\alpha_2x(t-\rho(t),\xi)]dt+e^{-t}dZ_H(t)\\ \qquad\qquad +\int_{\mathcal{Z}}\alpha_3yx(t-\theta(t),\xi)\widetilde{N}(dt,dy),\ t\geq 0,\ t\neq t_k, 0\leq\xi\leq\pi\\ \Delta x(t_k,\xi)=I_kx(t_k,\xi)=\frac{\alpha_4}{2^k}x(t_k,\xi)),\quad t=t_k,\ k=1,2,....,\\ x(t,0)=x(t,\pi)=0,\quad t\geq 0,\quad \alpha_i>0,\ i=1,2,3,4,\\ x(s,\xi)=\varphi(s,\xi),\ \varphi(s,.)\in\mathbb{L}^2([0,\pi])\ ;-\tau\leq s\leq 0\quad a.s.,\end{cases} \tag{4.1}$$

where $M_{\frac{1}{4}}$ is the corresponding constant in Lemma 2.3, z_H is a Rosenblatt process and $\mathcal{Z}=\{z\in\mathbb{R}:0<|z|\leq c, c>0\}$. For the convenience of writing, in the following, the variable ξ of $x(t,\xi)$ is omitted.

We rewrite (4.1) into abstract form of (1.1). Let $X=L^2([0,\pi])$. Define the operator $A:D(A)\subset X\longrightarrow X$ given by $A=\frac{\partial^2}{\partial^2\xi}$ with

$$D(A)=\{y\in X:\ y' \text{ is absolutely continuous}, y''\in X,\quad y(0)=y(\pi)=0\},$$

then we get

$$Ax=\sum_{n=1}^{\infty}n^2<x,e_n>_X e_n,\quad x\in D(A),$$

where $e_n:=\sqrt{\frac{2}{\pi}}\sin nx,\ n=1,2,....$ is an orthogonal set of eigenvector of $-A$.

The bounded linear operator $(-A)^{\frac{3}{4}}$ is given by

$$(-A)^{\frac{3}{4}}x=\sum_{n=1}^{\infty}n^{\frac{3}{2}}<x,e_n>_X e_n,$$

with domain

$$D((-A)^{\frac{3}{4}}) = X_{\frac{3}{4}} = \{x \in X, \sum_{n=1}^{\infty} n^{\frac{3}{2}} < x, e_n >_X e_n \in X\}.$$

It is well known that A is the infinitesimal generator of an analytic semigroup $\{S(t)\}_{t\geq 0}$ in X, and is given by (see [19])

$$S(t)x = \sum_{n=1}^{\infty} e^{-n^2 t} < x, e_n > e_n$$

for $x \in X$ and $t \geq 0$, that satisfies $\|S(t)\| \leq e^{-\pi^2 t}$ for every $t \geq 0$.

Let

$$g(t, x(t - r(t))) = \frac{\alpha_1}{M_{\frac{1}{4}} \|(-A)^{\frac{3}{4}}\|} x(t - r(t)),$$

$$f(t, x(t - \rho(t))) = \alpha_2 x(t - \rho(t)),$$

and

$$h(t, x(t - \theta(t), y)) = \alpha_3 y x(t - \theta(t)).$$

It is obvious that all the assumptions are satisfied with $\nu = \pi^2$, $M = 1$, $K_1 = \alpha_2^2$, $K_2 = \frac{\alpha_1^2}{M_{\frac{1}{4}}^2}$, $K_3 = \alpha_3^2 \int_Z y^2 \nu(dy)$, $h_k = \frac{\alpha_4}{2^k}$, $k \in \mathbb{N}$, $\|(-A)^{\frac{3}{4}}\| = 1$, $\beta = \frac{3}{4}$, $\gamma = \frac{1}{2}$ and $\|(-A)^{-\frac{3}{4}}\| \leq \frac{1}{\Gamma(\frac{3}{4})} \int_0^{\infty} t^{\frac{-1}{4}} \|S(t)\| dt \leq \frac{1}{\pi^{\frac{3}{2}}}$.

Thus, by Theorem 3.5, if the initial value $\varphi(t)$ satisfies

$$\mathbb{E}\|\varphi(s)\|^2 \leq M_0 \mathbb{E}|\varphi|_{\mathcal{D}}^2 e^{-as}, \quad s \in [-\tau, 0],$$

for some $M_0 > 0$, $a > 0$; then, the equation (4.1) has one unique mild solution and is exponential stable in mean square provided that the following inequality

$$\frac{\alpha_1^2}{M_{\frac{1}{4}}^2 \pi} + \frac{\Gamma^2(\frac{3}{4})\alpha_1^2}{\pi} + \frac{\alpha_2^2}{\pi^2} + \frac{\alpha_3^2 \int_Z y^2 \nu(dy)}{2} + \frac{4\alpha_4^2}{\pi^2} < \frac{\pi^2}{5}$$

holds.

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