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AN ANALYSIS OF SOME PROPERTIES OF CLASS OF ELLIPTIC PARTIAL DIFFERENTIAL OPERATORS: A SOLUTION OF SOME PROBLEMS

Abstract: We describe a fast and robust method for solving the large sparse linear systems that arise upon the discretization of elliptic partial differential equations such as Laplace's equation and the Helmholtz equation at low frequencies. While most existing fast schemes for this task rely on so called \iterative" solvers, the method described here solves the linear sys- tem directly We prove Schwarz-Pick type estimates and coefficient estimates for a class of functions induced by the elliptic partial differential operators. Then we apply these results to obtain a Landau type theorem.

INTRODUCTION

This paper describes a method for rapidly solving large systems of linear equations with sparse coefficient matrices. It is capable of handling the equations arising from the finite element or finite difference discretization of elliptic partial differential equations such as Laplace's equation, as well as the systems associated with heat conduction and random walks on certain networks. While most existing fast schemes for such problems rely on iterative solvers, the method described here solves the linear system directly (to within a preset computational accuracy). This obviates the need for customized pre -conditioners, improves robustness in the handling of ill-conditioned matrices, and leads to dramatic speed-ups in environments in which several linear systems with the same coefficient matrix are to be solved.

The scheme is described for the case of equations defined on a uniform square grid. Extensions to more general grids, including those associated with complicated geometries and local mesh refinements are possible.

For a system matrix of size NxN (corresponding to a $\sqrt{N} \times \sqrt{N}$ grid), the scheme requires $O(N \log^2 N)$ arithmetic operations. For a single solve, only $O(\sqrt{N} \log N)$

storage is required. Moreover, for problems loaded on the boundary only, any solves beyond the first require only $O(\sqrt{N}\ \log N)$ arithmetic operations provided that only the solution on the boundary is sought. For problems loaded on the entire domain, it is still possible to perform very fast subsequent solves, but this requires O(N log AT) storage. Numerical experiments indicate that the constants in these asymptotic estimates are quite moderate. For instance, to directly solve a system involving a 1 000 000 x 1 000 000 matrix to seven digits of accuracy takes about four minutes on a 2.8GHz desktop PC with 512Mb of memory. Additional solves beyond the first can be performed in 0.03 seconds (provided that only boundary data is involved).

The proposed scheme is conceptually similar to a couple of recently developed methods for accelerating domain decomposition methods such as nested dissection. The original nested dissection algorithm reduces a problem defined on a two dimensional domain in the plane to a sequence of problems defined on one dimensional domains. These problems involve dense coefficient matrices, but the reduction in dimensionality results in a decrease in the cost of a direct solve from $O(N^3)$ to $O(N^{3/2})$ fora grid containing A^r points. Observed that these dense matrices in fact have internal structure, and that by exploiting this structure, it is possible to further reduce the computational cost. The scheme proposed here is similar to the schemes in that it relies on a combination of a dimension-reduction technique, and fast algorithms for structured matrices to solve the resulting sequence of dense problems. However, it uses a different technique for dimension reduction, and a much simpler format for working with structured matrices than previous schemes. Its principal advantage over previous work is that what it actually computes is a sequence of Schur complements for successively larger parts of the computational domain. As a consequence, the scheme directly computes the solution operator that maps a boundary load to the solution on the boundary. Having access to this operator enables very fast solves in environments where a sequence of equations on the same computational grid are to be solved for a number of different boundary loads. The technique of hierarchical computation of Schur complements also appears to lead to improvements in robustness over competing methods.

ELLIPTIC EQUATIONS OF SECOND ORDER

Here we consider linear elliptic equations of second order, mainly the Laplace equation

$$\triangle u = 0.$$

Solutions of the Laplace equation are called potential functions or harmonic functions. The Laplace equation is called also potential equation.

The general elliptic equation for a scalar function $u(x)$, $x \in \Omega \subset \mathbb{R}^n$, is

$$Lu := \sum_{i,j=1}^{n} a^{ij}(x) u_{x_i x_j} + \sum_{j=1}^{n} b^j(x) u_{x_j} + c(x)u = f(x),$$

where the matrix $A = (a^{ij})$ is real, symmetric and positive definite. If A is a constant matrix, then a transform to principal axis and stretching of axis leads to

$$\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} = \triangle v,$$

Fundamental solution

Here we consider particular solutions of the Laplace equation in $\mathbb{R}^n$ of the type

$$u(x) = f(|x - y|),$$

where $y \in \mathbb{R}^n$ is fixed and / is a function which we will determine such that u defines a solution if the Laplace equation. Set $r = |x - y|$, then

$$\begin{aligned} u_{x_i} &= f'(r)\frac{x_i - y_i}{r} \\ u_{x_i x_i} &= f''(r)\frac{(x_i - y_i)^2}{r^2} + f'(r)\left(\frac{1}{r} - \frac{(x_i - y_i)^2}{r^3}\right) \\ \triangle u &= f''(r) + \frac{n-1}{r} f'(r). \end{aligned}$$

Thus a solution of $\triangle u = 0$ is given by

$$f(r) = \begin{cases} c_1 \ln r + c_2 & : \quad n = 2 \\ c_1 r^{2-n} + c_2 & : \quad n \geq 3 \end{cases}$$

with constants c_1, c_2

Definition. Set $r = |x - y|$. The function

$$s(r) := \begin{cases} -\frac{1}{2\pi} \ln r & : \quad n = 2 \\ \frac{r^{2-n}}{(n-2)\omega_n} & : \quad n \geq 3 \end{cases}$$

is called singularity function associated to the Laplace equation. Here is ω_n the area of the n-dimensional unit sphere which is given by $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$, where

$$\Gamma(t) := \int_0^\infty e^{-\rho} \rho^{t-1} \, d\rho, \quad t > 0,$$

is the Gamma function. Definition. A function

$$\gamma(x, y) = s(r) + \phi(x, y)$$

is called fundamental solution associated to the Laplace equation if $\phi \in C^2(\Omega)$ and $\triangle_x \phi = 0$ for each fixed $y \in \Omega$

Remark. The fundamental solution 7 satisfies for each fixed $y \in \Omega$ the relation

$$-\int_\Omega \gamma(x, y) \triangle_x \Phi(x) \, dx = \Phi(y) \quad \text{for all } \Phi \in C_0^2(\Omega),$$

In the language of distribution, this relation can be written by definition as

$$-\triangle_x \gamma(x, y) = \delta(x - y),$$

where δ is the Dirac distribution, which is called J-function.

Boundary value problems

Assume $\Omega \subset \mathbb{R}^n$ is a connected domain.

Dirichlet problem - The Dirichlet problem (first boundary value problem) is to find a solution

$u \in C^2(\Omega) \cap C(\overline{\Omega})$ of

$$\triangle u = 0 \text{ in } \Omega \tag{1}$$

$$u = \Phi \text{ on } \partial\Omega, \tag{2}$$

where Φ is given and continuous on $\partial\Omega$

Proposition 1. Assume Ω is bounded, then a solution to the Dirichlet problem is uniquely determined.

Proof.Maximum principle.

Remark. The previous result fails if we take away in the boundary condition (2) one point from the theboundary as the following example shows.

Let $\Omega \subset \mathbb{R}^2$ be the domain $\Omega = \{x \in B_1(0) : x_2 > 0\}$,

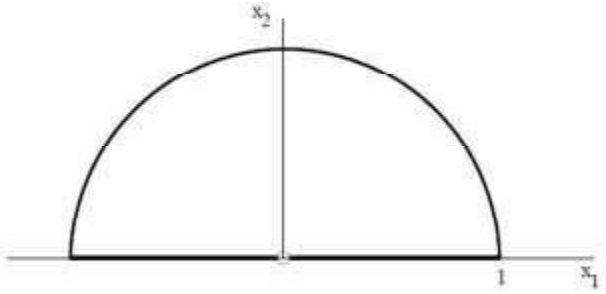

Figure 1: Counterexample

Assume $u \in C^2(\Omega) \cap C(\overline{\Omega} \setminus \{0\})$ is a solution of

$$\begin{aligned} \triangle u &= 0 \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \setminus \{0\}. \end{aligned}$$

This problem has solutions $u \equiv 0$ and $u = \mathrm{Im}(z + z^{-1})$, where $z = x_1 + ix_2$.

Another example see an exercise.

In contrast to this behaviour of the Laplace equation, one has uniqueness if $\triangle u = 0$ is replaced by the minimal surface equation $\frac{\partial}{\partial x_1}\left(\frac{u_{x_1}}{\sqrt{1+|\nabla u|^2}}\right) + \frac{\partial}{\partial x_2}\left(\frac{u_{x_2}}{\sqrt{1+|\nabla u|^2}}\right) = 0.$

Mixed boundary value problem - The Mixed boundary value problem (third boundary value problem) is to find a solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ of

$$\triangle u = 0 \text{ in } \Omega \tag{3}$$

$$\frac{\partial u}{\partial n} + hu = \Phi \text{ on } \partial\Omega, \tag{4}$$

where Φ and h are given and continuous on $\partial\Omega$.e $\Phi$and h are given and continuous on $\partial\Omega$

Proposition 2. Assume ft is bounded and sufficiently regular, then a solution to the mixed problem is uniquely determined in the class $u \in C^2(\overline{\Omega})$ provided $h(x) \geq 0$on $\partial\Omega$ and $h(x) > 0$for at least one point $x \in \partial\Omega$

Proof Exercise. Hint: Multiply the differential equation $\triangle w = 0$by w and integrate the result over Ω

SCHWARZ-PICK TYPE ESTIMATES AND COEFFICIENT ESTIMATES

We consider the Dirichlet boundary value problem of distributional sense as follows

$$\begin{cases} T_\alpha(f) = 0 & \text{in } \mathbb{D}, \\ f = f^* & \text{on } \partial\mathbb{D}. \end{cases} \tag{5}$$

Here, the boundary data $f^* \in \mathfrak{D}'(\partial\mathbb{D})$ is a distribution on the boundary $\partial\mathbb{D}$ of $\mathbb{D}$, and the boundary condition in (5) is interpreted in the distributional sense that $f_r \to f^*$in $\mathfrak{D}'(\partial\mathbb{D})$ as $r \to 1-$, where

$$f_r(e^{i\theta}) = f(re^{i\theta}),\ e^{i\theta} \in \partial\mathbb{D}, \tag{6}$$

for $r \in [0, 1)$.

Olofsson (2014) proved that, for parameter values $\alpha > -1$, a function $f \in C^2(\mathbb{D})$ satisfies (5) if and only if it has the form of a Poisson type integral

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} K_\alpha(ze^{-i\tau}) f^*(e^{i\tau}) d\tau, \text{ for } z \in \mathbb{D}, \tag{7}$$

where $$K_\alpha(z) = c_\alpha \frac{(1-|z|^2)^{\alpha+1}}{|1-z|^{\alpha+2}},$$

$$c_\alpha = \left(\Gamma(\alpha/2+1)\right)^2/\Gamma(1+\alpha)$$
$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt \quad s>0$$

Gamma function. If we take $\alpha = 2(n-1)$, then / is polyharmonic (or n-harmonic), where $n \in \{1,2,\ldots\}$.

Furthermore, Borichev and Hedenmalm (2014) proved that

$$(1-|z|^2)^n \Delta^n = 4(1-|z|^2)T_0 \circ 4(1-|z|^2)^2 T_2 \circ \cdots \circ 4(1-|z|^2)^n T_{2(n-1)}.$$

In particular, if $\alpha = 0$, then / is harmonic.

For $a, b, c \in \mathbb{R} \quad c \neq 0, -1, -2, \ldots$, the hyper geometric function is defined by the power series

$$F(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}, \ |x| < 1$$

where (a) o = 1 and for n = 1,2,... are the Pochhammer symbols. Obviously, for. In particular, for and, we have

$$F(a,b;c;1) = \lim_{x\to 1} F(a,b;c;x) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} < \infty. \tag{8}$$

For $\alpha = 0$, Heinz (1959) and Colonna(1989) proved the following Schwarz-Pick type estimates on planar harmonic functions, which are the following.

An exact $O(N^2)$ direct solver

In this section we describe a method for directly solving the linear system that relies on the sparsity pattern of the matrix only. In the absence of rounding errors, it would be exact. When the matrix A is of size $N \times N$. the method requires $O(N^2)$ floating point operations and 0(N) memory. This makes the scheme significantly slower

than well- $O(N^{3/2})$ known schemes. (We mention that $O(N^{3/2})$ is optimal in this environment.) The merit of the scheme presented in this section is simply that it can straight-forwardly be accelerated to an $O(N \log^2 N)$ or possibly even $O(N)$ scheme.

Ordering the N points in the grid in the spiral pattern, the matrix A in equation Ax=b has the sparsity pattern for N = 100. We next partition the grid into m concentric squares and collect the nodes into index sets $J_1, J_2, \ldots, J_m$ accordingly. In other words.

$$\begin{aligned} J_1 &= \{1, 2, 3, 4\}, \\ J_2 &= \{5, 6, \ldots, 16\}, \\ &\vdots \\ J_m &= \{(2m-2)^2 + 1, (2m-2)^2 + 2, \ldots, (2m)^2\}. \end{aligned}$$

For $\kappa, \lambda \in \{1, 2, \ldots m\}$, we let $A_{\kappa\lambda}$ denote the submatrix of A formed by the inter section of the J_κ rows with the J_λ columns. The linear system then takes on the block-tridiagonal form

$$\begin{bmatrix} A_{11} & A_{12} & 0 & 0 & \cdots & 0 \\ A_{21} & A_{22} & A_{23} & 0 & \cdots & 0 \\ 0 & A_{32} & A_{33} & A_{34} & \cdots & 0 \\ 0 & 0 & A_{43} & A_{44} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & A_{mm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ \vdots \\ b_m \end{bmatrix}$$

where x and b have been partitioned accordingly. The sparsity and block pattern of (13) for m = 5.

The blocked system of equations (13) can now easily be solved by eliminating the variables $x_1, x_2, \ldots, x_{m-1}$ one by one. Using the first, row to eliminate x_1 from the second row. we obtain the following system of equations for the variables $x_2, \ldots, x_m$:

$$\begin{bmatrix} \tilde{A}_{22} & A_{23} & 0 & \cdots & 0 \\ A_{32} & A_{33} & A_{34} & \cdots & 0 \\ 0 & A_{43} & A_{44} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & A_{mm} \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} \tilde{b}_2 \\ b_3 \\ b_4 \\ \vdots \\ b_m \end{bmatrix} \tag{14}$$

where $\tilde{A}_{22} = A_{22} - A_{21}A_{11}^{-1}A_{12}$ and $\tilde{b}_2 = b_2 - A_{21}A_{11}^{-1}b_1$.

This elimination process is continued row by row until we obtain the following equation for x_m:

$$\tilde{A}_{mm}\, x_m = \tilde{b}_m. \tag{15}$$

Equation (15) is of size $(8m-4) \times (8m-4)$ and is solved directly to obtain x_m.

Then x_{m-1} is determined by solving the equation

$$\tilde{A}_{m-1,m-1}\, x_{m-1} = \tilde{b}_{m-1} - A_{m-1,m}\, x_m.$$

The remaining x_j's are computed analogously. To summarize the entire process:

(1) $\tilde{A}_{11} = A_{11}$ and $\tilde{b}_1 = b_1$.
(2) for $\kappa = 2 : m$
(3) $\quad \tilde{A}_{\kappa\kappa} = A_{\kappa\kappa} - A_{\kappa,\kappa-1}\, \tilde{A}^{-1}_{\kappa-1,\kappa-1}\, A_{\kappa-1,\kappa}$
(4) $\quad \tilde{b}_\kappa = b_\kappa - A_{\kappa,\kappa-1}\, \tilde{A}^{-1}_{\kappa-1,\kappa-1}\, \tilde{b}_{\kappa-1}$
(5) end
(6) $x_m = \tilde{A}^{-1}_{mm}\, \tilde{b}_m$
(7) for $\kappa = (m-1) : (-1) : 1$
(8) $\quad x_\kappa = \tilde{A}^{-1}_{\kappa\kappa}\, (\tilde{b}_\kappa - A_{\kappa,\kappa+1}\, x_{\kappa+1})$
(9) end

We note that while all matrices $A_{\kappa\lambda}$ arc sparse, the matrices $\tilde{A}_{\kappa\kappa}$ arc dense. This means that the cost of inverting $A_{\kappa\kappa}$ in each step of the algorithm is $O(\kappa^3)$. (The remaining matrix-matrix operations involve matrices that are diagonal or tri-diagonal and have negligible costs in comparison to the matrix inversion.) The total cost T_{total} there fore satisfies $T_{\text{total}} \sim \sum_{\kappa=1}^{m} \kappa^3 \sim m^4 \sim N^2$.

CONCLUSION

We have presented a scheme for rapidly performing direct solves on linear systems involving the large sparse matrices arising from the discretization of elliptic PDEs such as Laplace's equation, or the Helmholtz equation at low wave-numbers. The scheme presented typically achieves an asymptotic computational complexity of $O(N \log^2 N)$, with a constant su±ciently small that a direct solve of a linear system involving a million by million sparse coe±cient matrix can be performed in about four minutes.

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