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BIPOLAR INTERVAL-VALUED FUZZY GRAPHS

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Abstract: In literature, there exist various classifications of bipolar fuzzy graphs which describe imprecise information of the system properly, but for the adequate and more accurate description of uncertainty involved in the system we define bipolar interval-valued fuzzy graphs and obtained some classifications to materialized the concept properly. In this paper, we established the notion of regular and irregular bipolar interval-valued fuzzy graphs, neighbourly irregular bipolar interval-valued fuzzy graphs, highly irregular bipolar interval-valued fuzzy graphs and obtained some fundamental theorems related to the proclaimed graphs.

Key Words: *Bipolar fuzzy graph, regular fuzzy graph, irregular fuzzy graph and interval-valued fuzzy graph.*

I. INTRODUCTION

In 1994, Zhang [18] introduced the concept of bipolar fuzzy sets which were the generalization of fuzzy sets introduced by Zadeh [17] in 1965. For a bipolar fuzzy set the membership values of an element range in [-1, 1]. In a bipolar fuzzy set if an element admit the membership value within (0, 1] then we consider element somewhat satisfy the property of the set and known as positive membership of an element, and the membership value within [-1, 0) indicates the element somewhat satisfies the implicit counter property of the set and called negative membership value of an element. For neutral behaviour, we assign 0 membership value to it. Whereas in a fuzzy set, element possesses membership value within [0, 1] assuming element only satisfy the property of a set. In 1975 Rosenfeld [13] defined fuzzy relation and introduced the concept of a fuzzy graph. Afterward, many researchers like Mordeson and Peng [10], Sunitha and Vijayakumar [14], Bhutani and Rosenfeld [6], Pramanik, Samanta and Pal [11,12] etc., obtained various properties on fuzzy graphs and applied it to several areas of computer science and engineering to solve the problems bearing imprecise information.

Simultaneously, the work continued towards the extension of the concept of fuzzy graphs to obtain more accurate and exact approximation of the system having imprecise information. In 2011 Akram and Dudek [1] introduced interval-valued fuzzy graph and many other researchers, Ismayil and Ali [9], Talebi and Rashmanlou [16], Debnath [7,8] obtained analogs of several graph theoretical concepts and also obtained many promiscuous real life applications of an interval-valued fuzzy graph. Akram [2-5] introduced the concept of a bipolar fuzzy graph, regular bipolar fuzzy graph, irregular and highly, totally, neighbourly irregular fuzzy graphs described many properties of bipolar fuzzy graphs. Simultaneously, S. Samanta and M. Pal [15] also contributed several significant properties on bipolar fuzzy graphs. In this paper, we enhanced the idea of a bipolar fuzzy graph to bipolar interval-valued fuzzy graph and obtained some properties over it, because in real life many objects bear an interval-valued membership value. So, bipolar interval-valued fuzzy graph theoretical approach becomes more significant to describe the relational behavior of the objects having an interval-valued membership.

II. PRELIMINARIES

Definition 2.1: Let X be a nonempty set. A bipolar fuzzy set B in X is an object having the form B= $\{(x, \mu^P(x),\ \mu^N(x)) \mid x \in X\}$ where $\mu^P(x): X \to [0,1]$ and $\mu^N(x): X \to [-1,0]$ are mappings.

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Here $\mu^P(x)$ is the positive membership value which denotes the satisfaction degree of an element $x \in B$ and $\mu^N(x)$ is the negative membership value which denotes the satisfaction degree to some implicit counter-property of an element$x \in B$. If for any$x \in B$, $\mu^P(x) \neq 0$ and $\mu^N(x) = 0$, it is the situation that x has only positive satisfaction for B. If for any $x \in B$, $\mu^P(x) = 0$ and $\mu^N(x) \neq 0$, then it is the situation that x does not satisfy the property of B but somewhat satisfies the counter property of B. It is possible for an element x for which $\mu^P(x) = 0$ and $\mu^N(x) = 0$ then we say the satisfaction property of an element overlaps that of its counter satisfaction property over some portion. We shall use the symbol $B = (\mu^P, \mu^N)$ for the bipolar fuzzy set $B = \{(x, \mu^P(x), \mu^N(x)) | x \in X\}$.

Definition 2.2: For every two bipolar fuzzy sets $A = (\mu_A^P, \mu_A^N)$ and $B = (\mu_B^P, \mu_B^N)$ in X, we define

- $(A \cap B)(x) = \left(min(\mu_A^P(x), \mu_B^P(x)), max\left(\mu_A^N(x), \mu_B^N(x)\right)\right)$, for all $x \in X$
- $(A \cup B)(x) = \left(max\left(\mu_A^P(x), \mu_B^P(x)\right), min\left(\mu_A^N(x), \mu_B^N(x)\right)\right)$, for all $x \in X$

Definition 2.3: Let $A = (\mu_A^P, \mu_A^N)$ and $B = (\mu_B^P, \mu_B^N)$ be two bipolar fuzzy set on X. If $C = A \times B$ is any relation on X, then $C = (\mu_C^P, \mu_C^N)$ is called a bipolar fuzzy relation from $A = (\mu_A^P, \mu_A^N)$ on $B = (\mu_B^P, \mu_B^N)$, where $\mu_C^P(x,y) \leq min\{\mu_A^P(x), \mu_B^P(y)\}$ and $\mu_C^N(x,y) \geq max\{\mu_A^N(x), \mu_B^N(y)\}$ for all $x \in A$ and $y \in B$.

Throughout this paper, G is a crisp graph, $\tilde{G}$ is a fuzzy graph, $\breve{G}$ is bipolar fuzzy graph, G^* is an interval-valued fuzzy graph and $\tilde{\breve{G}}$ is bipolar interval-valued fuzzy graph.

Definition 2.3: (see [1]) By an interval-valued fuzzy graph of a graph G we mean a pair $G^* = (A, B)$, where $\$A = [\mu_A^-, \mu_A^+]$ is an interval-valued fuzzy set on V and $B = [\mu_B^-, \mu_B^+]$ is an interval-valued fuzzy relation on E such that $\mu_B^-(xy) \leq min\left(\mu_A^-(x), \mu_A^-(y)\right)$, $\mu_B^+(xy) \leq min\left(\mu_A^+(x), \mu_A^+(y)\right)$, for all $xy \in E$.

Definition 2.4: By a bipolar fuzzy graph $\breve{G}$ of a graph G=(V, E) we mean a pair (S, Q), where $S = (\mu_S^P, \mu_S^N)$ is a bipolar fuzzy set on V and $Q = (\mu_Q^P, \mu_Q^N)$ is a bipolar fuzzy relation on $E \subseteq V \times V$ such that $\mu_Q^P(xy) \leq min\{\mu_S^P(x), \mu_S^P(y)\}$ and $\mu_Q^N(xy) \geq max\{\mu_S^N(x), \mu_S^N(y)\}$ for all $x, y \in V$ and $(xy) \in E$.

Definition 2.5: Let $\breve{G} = (S, Q)$ is a bipolar fuzzy graph on G then, degree of any vertex $x \in V$ is $(d^P(x), d^N(x))$ where $d^P(x) = \sum_{x \neq y, x \in V} \mu_Q^P(xy)$ and $d^N(x) = \sum_{x \neq y, x \in V} \mu_Q^N(xy)$. If $(d^P(x), d^N(x)) = (k_1, k_2)$ where k_1, k_2 are any real number then $\breve{G}$is said to be (k_1, k_2) $-$regular bipolar fuzzy graph if every vertex of $\breve{G}$ has degree (k_1, k_2).

III. BIPOLAR INTERVAL-VALUED FUZZY GRAPH

Throughout the paper, we assume D [0, 1] be the set of all closed sub-intervals of the interval [0, 1] and H [-1, 0] the set of all closed sub-intervals of the interval [-1, 0].

Definition 3.1: A bipolar interval-valued fuzzy graph $\tilde{\breve{G}}$ with underlying graph G=(V, E) is defined to be a pair (A, B) where:

1. The functions $\chi_A : V \to D[0,1]$ and $\eta_A : V \to H[-1,0]$ denote satisfaction degree interval and the satisfaction degree interval to some implicit counter-property of an element $x \in A$ respectively.
2. The functions $\chi_B : E \subset V \times V \to D[0,1]$ and $\eta_B : E \subset V \times V \to H[-1,0]$ are defined by:

   $\chi_{BL}(x, y) \leq min(\chi_{AL}(x), \chi_{AL}(y))$ and $\eta_{BL}(x, y) \geq max(\eta_{AL}(x), \eta_{AL}(y))$,
   $\chi_{BU}(x, y) \leq min(\chi_{AU}(x), \chi_{AU}(y))$ and $\eta_{BU}(x, y) \geq max(\eta_{AU}(x), \eta_{AU}(y))$.

Definition 3.2: Let $\tilde{G}$ is a bipolar interval-valued fuzzy graph with underlying graph G then degree of any vertex $x \in V$ is $(d^P(x),\ d^N(x))$ where $d^P(x) = \sum_{x \neq y, xy \in B} \frac{\chi_{BL}(xy)+\chi_{BU}(xy)}{2}$ and $d^N(x) = \sum_{x \neq y, xy \in B} \frac{\eta_{BL}(xy)+\eta_{BU}(xy)}{2}$.

Definition 3.3: A bipolar interval-valued fuzzy graph $\tilde{G}$=(A, B) is said to be complete bipolar interval-valued fuzzy graph if

$\chi_{BL}(x,\ y) = min(\chi_{AL}(x),\ \chi_{AL}(y))$ and $\eta_{BL}(x,\ y) = max(\eta_{AL}(x),\ \eta_{AL}(y))$,
$\chi_{BU}(x,\ y) = min(\chi_{AU}(x),\ \chi_{AU}(y))$ and $\eta_{BU}(x,\ y) = max(\eta_{AU}(x),\ \eta_{AU}(y))$.

Example 3.1: Consider a bipolar interval-valued fuzzy graph $\tilde{G}$ where A={(u, [.3, .7], [-.6, -.4]), (v, [.4, .6], [-.8, -.2]), (w, [.6, .9], [-.7, -.3])} then the corresponding complete bipolar interval-valued fuzzy graph is shown in figure 3.

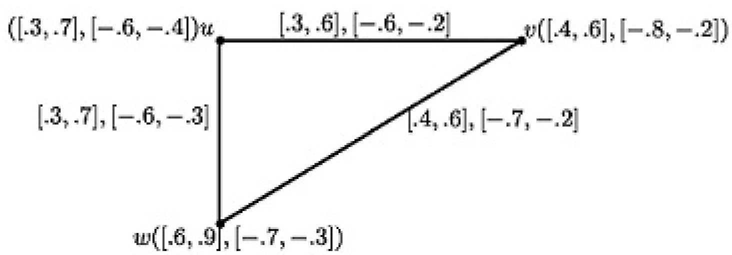

Figure 1: A complete bipolar interval-valued fuzzy graph

Definition 3.4: Let $\widetilde{G_1} = (A_1,\ B_1)$ and $\widetilde{G_2} = (A_2,\ B_2)$ be two bipolar interval-valued fuzzy graph underlying set V_1 and V_2 respectively then the intersection of $\widetilde{G_1}$ *and* $\widetilde{G_2}$ is denoted by $\widetilde{G_1} \cap \widetilde{G_2} = (A_1 \cap A_2,\ B_1 \cap B_2)$ and defined as follows:

$$\chi_{(A_1 \cap A_2)L}(x) = min\{\chi_{A_1L}(x),\ \chi_{A_2L}(x)\}\ and\ \chi_{(A_1 \cap A_2)U}(x) = min\ \{\chi_{A_1U}(x), \chi_{A_2U}(x)\}$$

$$\eta_{(A_1 \cap A_2)L}(x) = max\ \{\eta_{A_1L}(x), \eta_{A_2L}(x)\}\ and\ \eta_{(A_1 \cap A_2)U}(x) = max\{\eta_{A_1U}(x), \eta_{A_2U}(x)\}$$

$$\chi_{(B_1 \cap B_2)L}(x) = min\{\chi_{B_1L}(x), \chi_{B_2L}(x)\}\ and\ \chi_{(B_1 \cap B_2)U}(x) = min\{\chi_{B_1U}(x), \chi_{B_2U}(x)\}$$

$$\eta_{(B_1 \cap B_2)L}(x) = max\{\eta_{B_1L}(x), \eta_{B_2L}(x)\}\ and\ \eta_{(B_1 \cap B_2)U}(x) = max\{\eta_{B_1U}(x), \eta_{B_2U}(x)\}$$

Proposition 3.1: Let $\widetilde{G_1}$=$(A_1,\ B_1)$ and $\widetilde{G_2} = (A_2,\ B_2)$ be two complete bipolar interval-valued fuzzy graphs then, $\widetilde{G_1} \cap \widetilde{G_2}$ is also a complete bipolar interval-valued fuzzy graph.

Proof: Proof is obvious as we defined above the intersection of two bipolar interval-valued fuzzy graph.

IV. REGULAR BIPOLAR INTERVAL-VALUED FUZZY GRAPH

Definition 4.1: A bipolar interval-valued fuzzy graph $\tilde{G}$ is said to be $(k_1,\ k_2)$-regular bipolar interval-valued fuzzy graph if degree of each vertex remain constant $(k_1,\ k_2)$ for all $v \in G$ where k_1 and k_2 are any real numbers and $k_1 = d^P(x)$ and $k_2 = d^N(x)\ \forall x \in V$.

Example 4.1: Let $\tilde{G}$ be a bipolar interval-valued fuzzy graph underlying $G = (V, E)$ where $V = \{u, v, w\}$ and $E = \{uv, vw, uw\}$. Define $\tilde{G}(A, B)$ as shown in figure 2.

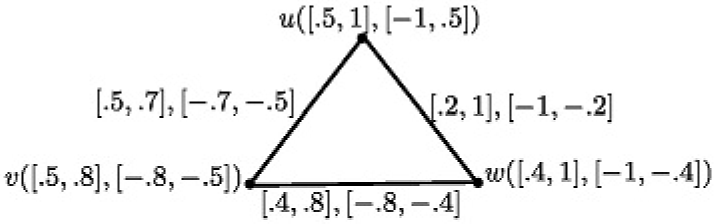

Figure 2. A regular bipolar interval-valued fuzzy graph

Here in figure 2 degree of u, v *and* w are

$$(d^P(u), d^N(u)) = (\frac{(.5+.7)+(.2+1)}{2}, \frac{(-.7+(-.5))+(-1+(-.2))}{2}) = (1.2, -1.2), ,$$

$$\left(d^P(v), d^N(v)\right) = \left(\frac{(.5+.7)+(.4+.8)}{2}, \frac{(-.7+(-.5))+(-.8+(-.4))}{2}\right) = (1.2, -1.2)$$

And $\left(d^P(w), d^N(w)\right) = \frac{(.4+.8)+(.2+1)}{2}, \frac{(-.8+(-.4))+(-1+(-.2))}{2}) = (1.2, -1.2)$ respectively. Since degree of all vertices are equal thus $\tilde{G}$ is $(1.2, -1.2)$-regular bipolar interval-valued fuzzy graph.

Theorem 4.1: A bipolar interval-valued fuzzy graph $\tilde{G}$ is regular if all the vertices of G induced from $\tilde{G}$ have the same degree and $\chi_{BL} + \chi_{BU}$, $\eta_{BL} + \eta_{BU}$ are constant for all $xy \in B$.

Proof: As we know a bipolar interval-valued fuzzy graph $\tilde{G}$ is regular iff all the vertices have constant degree i.e., (k_1, k_2) where $k_1 = \sum_{x \neq y, xy \in B} \frac{\chi_{BL}(xy)+\chi_{BU}(xy)}{2}$ and $k_2 = \sum_{x \neq y, xy \in B} \frac{\eta_{BL}(xy)+\eta_{BU}(xy)}{2}$. So, if the sum of the upper and lower limits of the satisfaction degree interval and the satisfaction degree interval to some implicit counter-property remains constant for all edges of $\tilde{G}$ then the degree of each vertex of $\tilde{G}$ depends only upon the degree of the vertices of the underlying graph G. Thus, if degree of all vertices of G also becomes constant then $\tilde{G}$ must get a constant degree for each node.

Theorem 4.2: A bipolar interval-valued fuzzy cycle is regular then for any $xy \in B$ have at most $\max\{\lfloor \frac{k_1}{2} \rfloor, \lfloor \frac{k_2}{2} \rfloor\}$ number of distinct satisfaction degree interval or the satisfaction degree interval to some implicit counter-property assuming $k_1 \text{and } k_2$ as the positive integer.

Proof: Let $\tilde{C}_n$ be a bipolar interval-valued fuzzy cycle underlying cycle C_n. Suppose, $\lambda_1 and \lambda_2$ be the sum of lower and the upper limit of χ_B and η_B of $\tilde{C}_n$ for each edge. Thus for regular $\tilde{C}_n$, $\lambda_1 and \lambda_2$ have to be same for all edges. We know that $k_1 = \sum_{xy \in B} \frac{\lambda_1}{2} and k_2 = \sum_{xy \in B} \frac{\lambda_2}{2}$ thus constant $k_1 and k_2$ is only possible when the satisfaction degree interval and the satisfaction degree interval to some implicit counter-property are made up of from bipartition of $k_1 and k_2$. As we know bipartition for any number $n = \lfloor \frac{n}{2} \rfloor$ thus, the partition of $k_1 and k_2$ are $\lfloor \frac{k_1}{2} \rfloor$ and $\lfloor \frac{k_1}{2} \rfloor$ respectively if $k_1 and k_2$ are positive integers. As in the problem k_1, k_2 are always float so, we have to ignore the decimal for the bipartition of $k_1 and k_2$. Now the distinct interval for any number is equal to the number of possible bipartition. Thus, the number of distinct pair of the interval for two numbers are almost maximum of bipartition of that two numbers. Hence the theorem.

Example 4.2: Let $\widetilde{C_n}$ is (1.1, -1.3)-regular bipolar interval-valued fuzzy cycle. In this example, we obtained the maximum length (1.1, -1.3)-regular bipolar interval-valued fuzzy cycle having distinct positive or negative membership interval.

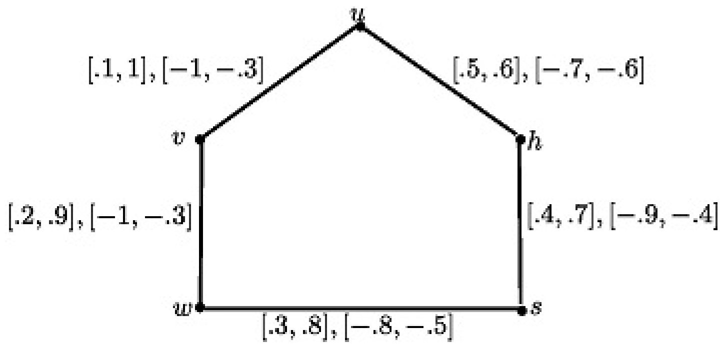

Figure 3. Regular bipolar interval-valued fuzzy cycle

In this cycle $\widetilde{C_n}$ shown in figure 3 every vertices have the degree (1.1, -1.3) this is possible if every edge has membership interval as follows:

Either positive membership intervals are from [.1, 1], [.5, .6], [.4, .7], [.3, .8], [.2, .9] or the negative membership intervals are from [-1, -.3], [-.7, -.6], [-.9, -.4], [-.8, -.5] in both the cases distinct positive membership interval or distinct negative membership interval are either 5 or 4 which is less then $max\ \{\lfloor\frac{11}{2}\rfloor, \lfloor\frac{13}{2}\rfloor\} = 6$. Hence the maximum length for the cycle is 5 for distinct positive membership interval.

V. IRREGULAR BIPOLAR INTERVAL-VALUED FUZZY GRAPH

Definition 5.1: A bipolar interval-valued fuzzy graph $\tilde{G}$ is said to be irregular bipolar interval-valued fuzzy graph if the degree of some vertices are different than other.

Definition 5.2: A connected bipolar interval-valued fuzzy graph $\tilde{G}$ is said to be a neighbourly irregular bipolar interval-valued fuzzy graph if every two adjacent vertices of G or $\tilde{G}$ have the distinct degree.

Example 5.1: Consider a graph G=(V, E) such that V={u, v, w, x} and E={uv, uw, vw, wx, xu}. Let A be a bipolar interval-valued subset of V and B is a bipolar interval-valued subset of $E = V \times V$. Bipolar interval-valued fuzzy graph $\tilde{G}$ corresponding to G are shown in figure 4.

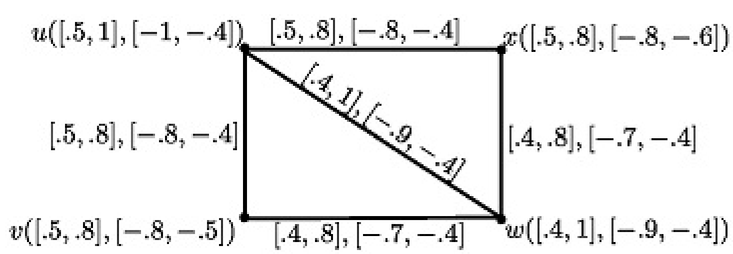

Figure 4. .Neighbourly irregular bipolar interval-valued fuzzy graph

Here in this graph all adjacent vertices u-v, v-w, w-x, x-u, u-w have distinct degree because degree of u is (2, -1.85), degree of v is (1.25, -1.15), degree of w is (1.9, -1.75) and degree of x is (1.25, -1.15). So, the graph is neighbourly irregular bipolar interval-valued fuzzy graph.

Definition 5.3: A connected bipolar interval-valued fuzzy graph $\tilde{G}$ is said to be a highly irregular bipolar interval-valued fuzzy graph if any two vertices of $\tilde{G}$ have the distinct degree.

Example 5.2: Consider a graph G=(V, E) such that $V = \{u, v, w, x\}$ and $E = \{uv,\ vw,\ wx,\ xu\}$. Let A be a bipolar interval-valued subset of V and B is a bipolar interval-valued subset of $E = V \times V$. Bipolar interval-valued fuzzy graph $\tilde{G}$ corresponding to G are shown in figure 5.

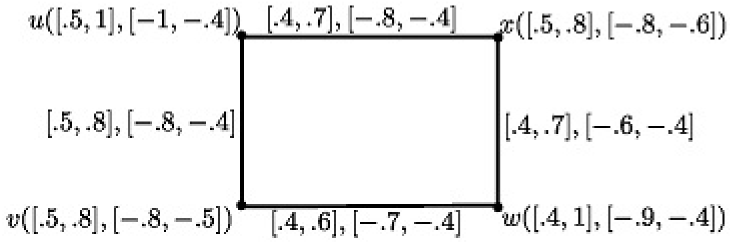

Figure 5. Highly irregular bipolar interval-valued fuzzy graph

Here in this graph all adjacent vertices u-v, v-w, w-x, x-u as well as non adjacent vertices have distinct degree because degree of u is (1.2, -1.2), degree of v is (1.15, -1.15), degree of w is (1.05, -1.05) and degree of x is (1.1, -1.1). So, the graph is highly irregular bipolar interval-valued fuzzy graph.

Theorem 5.1: A neighbourly irregular bipolar interval-valued fuzzy graph $\tilde{G}$ is always a highly irregular bipolar interval-valued fuzzy graph if and only if the degrees of all the vertices of $\tilde{G}$ are distinct.

Proof: As for neighbourly irregular bipolar interval-valued fuzzy graph $\tilde{G}$ any two adjacent vertices must have the distinct degree. So distinct degree for any two adjacent vertices is necessary for both the neighbourly and highly irregular bipolar interval-valued fuzzy graph. But for highly irregular bipolar interval-valued fuzzy graph all other non-adjacent vertices must have the distinct degree. Thus if $\tilde{G}$ have all distinct degrees then it suffices the condition for highly irregular bipolar interval-valued fuzzy graph. Hence the theorem.

VI. Conclusion

An interval-valued fuzzy graph has numerous application in the modeling of real life systems where the level of information inherited in the system varies with respect to time and have the different level of precision. Most of the actions in real life are time dependent, symbolic models used in the expert system are more effective than traditional one. In this paper, we introduced the concept of the bipolar interval-valued fuzzy graph and obtained many properties like regularity and irregularity of the bipolar interval-valued fuzzy graphs. In future, we extend this concept to hypergraphs and in some more areas of graph theory.

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