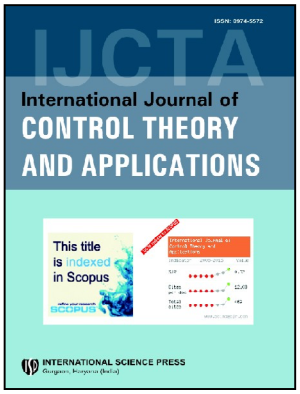

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$L\ (h_1, h_2,..., h_m)$ -labeling problems on interval graphs

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Abstract: $L(h,k)$ -labeling problem is a well studied problem due to its wide applications, specially in frequency assignment in (mobile) communication system, coding theory, *X*-ray crystallography, radar, astronomy, circuit design etc. An extension of $L(h,k)$ -labeling problem, i.e., $L(h_1,h_2,\ldots,h_m)$ -labeling problem is now becomes a well studied problem due to its application. Motivated from this point of view, we consider $L(h_1,h_2,\ldots,h_m)$ -labeling problem for interval graph, where interval graph is a very important subclass of intersection graph.

An $L(h_1,h_2,\ldots,h_m)$ -labeling of a graph $G=(V,E)$ is defined as the function $f:V\rightarrow Z^*$, the set of non-negative integers such that $|f(x)-f(y)|\geq h_i$, if $d(x,y)=i$, for $i=1,2,\ldots,m$, where $d(x,y)$ represents the distance (minimum number of edges) between the vertices x and y. The $L(h_1,h_2,\ldots,h_m)$ -labeling numbers of a graph G, is denoted by $\lambda_{h_1,h_2,\ldots,h_m}(G)$ is the difference between the highest and lowest label used in $L(h_1,h_2,\ldots,h_m)$ -labeling.

In this paper, we have proved that $\lambda_{h_1,h_2,\ldots,h_m}(G)\leq(\Delta-1)\sum_{i=1}^{m}h_i+h_1$ for interval graph G, where Δ is the degree of the graph and also we design an algorithm to label an interval graph by $L(h_1,h_2,\ldots,h_m)$ -labeling. The time complexity of the algorithm is also computed.

Keywords: $L(h_1,h_2,\ldots,h_m)$ -labeling, frequency assignment, interval graphs, network.

Mathematics subject classification: 05*C*85, 68*R*10.

1. INTRODUCTION

To find an efficient way, safe transmissions are needed in areas such as Wi-Fi, Cellular telephony, Security systems and many more. It is unpleasant being on the phone and getting someone else on the same line. This inconvenience is given by interferences caused by unconstrained simultaneous transmissions. Two close enough channels can interfere or resonate thereby damaging communications. The interference can be avoided by means of a suitable channel assignment. The channel assignment problem (CAP) is a problem where the task is to assign a channel (non-negative integer) to a given group of televisions or radio transmitters such that there is no interference between channels and the span of assign channel is minimize. A channel assignment problem is motivated from the distance labeling problem of graph where the vertices are the radio transmitters and adjacency indicate possible interference.

Hale [12] introduced a graph theory model of channel assignment problem which is known as vertex coloring problem of graph. The notion of $L(h,k)$-labeling was introduced by Griggs and Yeh [11] in the special case where $h=2$ and $k=1$ in connection with the problem of assigning frequencies in a multihop radio network. The formal definition of the general graph labeling problem i.e., $L(h_1,h_2,\ldots,h_m)$-labeling is given as follows.

Defination 1 $L(h_1,h_2,\ldots,h_m)$ **-labeling:** *For* $h_i \geq 1$ *for* $i=1,2,\ldots,m$, an $L(h_1,h_2,\ldots,h_m)$-labeling of a graph $G=(V,E)$ is an assignment f from the vertex set V to the set of non-negative integers $\{0,1,\ldots,\lambda\}$ such that $|f(x)-f(y)| \geq h_i$, if $d(x,y)=i$, for $i=1,2,\ldots,m$, where $d(x,y)$ represents the distance (minimum number of edges) between the vertices x and y. The minimum value of λ for which G has $L(h_1,h_2,\ldots,h_m)$-labeling is denoted by $\lambda_{h_1,h_2,\ldots,h_m}(G)$.

The notion of $L(h_1,h_2,\ldots,h_m)$-labeling has attracted a lot of attention both for its motivation by channel assignment problem, and also for its interesting graph theoretic properties. In the past, there are many papers that study the problem of graph labeling for several classes of graphs [1, 2, 6, 7, 11, 12, 13, 18, 19]. If the reader is interested in this field, please see the update surveys[4]. Different bounds for $\lambda_{h_1,h_2,\ldots,h_m}(G)$ were obtained for various type of graphs. The upper bound of $\lambda_{p,1}(G)$ of any graph G is $\Delta^2-(p-1)\Delta-2$ [4], where Δ is the degree of the graph. In [9], Clipperton *et al*. showed that $\lambda_{3,2,1}(G) \leq \Delta^3+\Delta^2+3\Delta$ for any graph. Later Chai *et al*. [8] improved this upper bound and showed that $\lambda_{3,2,1}(G) \leq \Delta^3+2\Delta$ for any graph. In [14], Lui and Shao studied the $L(3,2,1)$-labeling of planer graph and showed that $\lambda_{3,2,1}(G) \leq 15(\Delta^2-\Delta+1)$. Later Calamoneri studied $L(\delta 1,\delta 2,1)$-labeling of eight grids [5]. Again, Clipperton [10] shown that $\lambda_{4,3,2,1}(G) \leq \Delta^3+2\Delta^2+6\Delta$ for any graph G. In [17], Paul *et al*. showed that $\lambda_{2,1}(G) \leq \Delta+w$ for interval graph and they also shown that $\lambda_{2,1}(G) \leq \Delta+3w$ for circular-arc graph, where w represents the size of the maximum clique, also in [16] they have shown that the value of $\lambda_{0,1}$ can be computed for interval graph using polynomial time. In [3], Calamoneri *et al*. have been studied a lot of result about $L(h,k)$-labeling of co-comparability graphs, interval graphs and circular-arc graphs. They have shown that $\lambda_{h,k}(G) \leq max(h,2k)2\Delta+k$ for co-comparability graphs, $\lambda_{h,k}(G) \leq max(h,2k)\Delta$ for interval graphs. Also they have proved $\lambda_{h,k}(G) \leq max(h,2k)\Delta+hw$ for circular-arc graphs. Also in [20], Amanathulla and Pal shown that $\lambda_{0,1}(G) \leq \Delta$ and $\lambda_{1,1}(G) \leq 2\Delta$ for circular-arc graphs and they also shown that $\lambda_{3,2,1}(G) \leq 9\Delta - 6$ and $\lambda_{4,3,2,1}(G) \leq 16\Delta - 12$ for circular-arc graphs [21], they also studied $L(3,2,1)$- and $L(4,3,2,1)$ -labeling problems of interval graph [22]. In the present paper we focus on a very important subclass of intersection graph namely interval graph. Although, Bertossi *et al*. have been studied approximate $L(\delta 1, \delta 2, \ldots, \delta t)$ -labeling of interval graphs [1], but our proposed algorithm is the first upper bound in terms of Δ, which is essential to labeling of graph.

The remaining part of the paper is organized as follows. Some notations and definitions are presented in Section 2. In Section 3, some lemmas related to our work and an algorithm to $L(h_1, h_2, \ldots, h_m)$ -label an interval graph are presented. In Section 4, a conclusion is made.

2. PRELIMINARIES AND NOTATIONS

All graphs considered in this paper will be finite, connected, simple, having no self loop or multiple edges and undirected. A graph $G = (V, E)$ is said to be an intersection graph for a finite family F of a non-empty set if there exists a one-to-one correspondence between F and V such that two sets in the family F have non-empty intersection if and only if there corresponding vertices in V are adjacent to each other. The set F is called an intersection model of the graph G. Depends on the nature of the set F one gets different intersection graphs. If any one is interested about intersection graph he can see the survey on intersection graph [15]. Interval graph is a very important subclass of intersection graph.

Defination 2 (Interval Graph) An undirected graph $G = (V, E)$ is an interval graph if the vertex set V can be put into one to one correspondence with a set of intervals I on the real line R such that two vertices are adjacent in G iff their corresponding intervals have non empty intersection.

Let $I = \{I_1, I_2, \ldots, I_n\}$, where $I_j = [a_j, b_j]$, $j = 1,2,\ldots n$ be a set of intervals on a real line, a_j and b_j are respectively the left and right end points of the interval I_j. We draw a vertex v_j for the interval I_j, $j = 1,2,\ldots,n$. Two vertices v_i and v_j are connected by an edge if and only if there corresponding intervals have non-empty intersection. Thus, an undirected graph $G = (V, E)$ is said to be an interval graph if the vertex set V can be put into one-to-one correspondence with a set I of intervals on the real line such that two vertices are adjacent in G if and only if there corresponding intervals have non empty intersection, i.e. there is a bijection $f: V \to I$. The set I is called an interval representation of G and G is referred to as the interval graph of I. Also, it is observed that an interval I_j of I and a vertex v_j of V are one and same thing. An interval graph and its corresponding interval representation are shown in Figure 1.

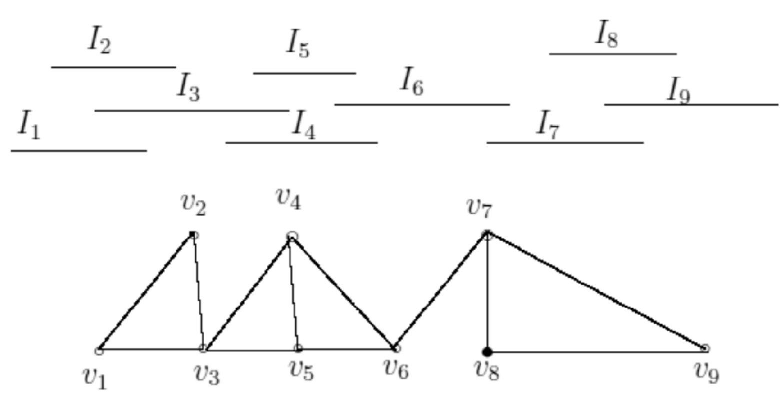

Figure 1: An interval representation and its corresponding interval graph

Notations: For any interval graph *G* and its interval representation *I*, we define the following objects:

1. $L(I_k)$: the set of used $L(h_1, h_2, \ldots, h_m)$ -labels before labeling the interval I_k , for any $I_k \in I$.

2. $L^i(I_k)$: the set of used labels at distance i ($i = 1,2,\ldots,m$) from the interval I_k , before labeling the interval I_k , for any $I_k \in I$.

3. $L^{vl}(i, I_k)$: the set of all valid labels to label the interval I_k before labeling I_k , satisfying the condition of distance $1,2,\ldots,k$, for $k = 1,2,\ldots,m$ of $L(h_1, h_2, \ldots, h_m)$ -labeling, for any $I_k \in I$.

4. f_j : the label of the interval I_j , for any $I_j \in I$.

5. *L*: the label set, i.e. the set of labels used after completion of $L(h_1, h_2, \ldots, h_m)$ -labeling of the interval graph *G*.

3. $L(h_1, h_2, ..., h_m)$-LABELING OF INTERVAL GRAPHS

In this section, we present some lemmas related to our work and upper bound of $L(h_1, h_2, \ldots, h_m)$ -labeling of interval graph. Also, we designed an algorithm to $L(h_1, h_2, \ldots, h_m)$ -labeling of interval graph. The time complexity of the algorithm is also calculated.

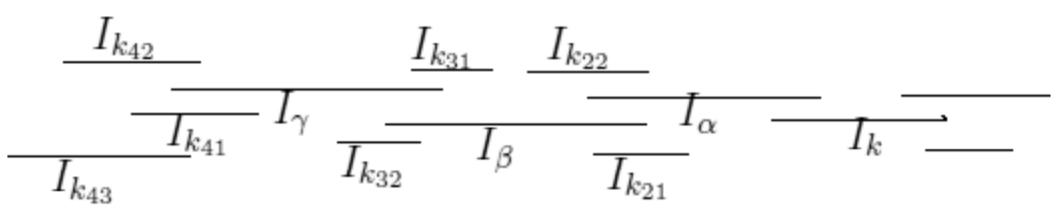

Figure 2: An interval graph

Lemma 1: For any interval graph G , $|L^i(I_k)| \leq \Delta - 1, i = 2,3,\ldots,m$ for any interval $I_k \in I$ of G .

Proof. Let G be an interval graph with n vertices. we label the graph starting from the leftmost interval. Let the vertex v_k be corresponds to the interval I_k of the interval graph G . We consider a stage in which the intervals $I_1, I_2, \ldots, I_{k-1}$ (for $k = 2,3,\ldots,n$) are already labeled by $L(h_1, h_2, \ldots, h_m)$ -labeling and the remanning intervals are not label. In this stage we consider the following cases:

Case 1: $i = 2$

Let $|L^2(I_k)| = p$. This means that the number of distinct $L(h_1, h_2, \ldots, h_m)$ -labels are used to label the intervals at distance two from the interval I_k, before labeling the interval I_k is *p*.

Since, Δ is the degree of the interval graph *G*, so there exists an interval I_α (in Figure 2) which are adjacent to at most Δ intervals of G . In Figure 3, I_α is adjacent to $I_k, I_\beta, I_{k_{21}}, I_{k_{22}}$. Among the intervals, some intervals

($I_\beta, I_{k_{21}}, I_{k_{22}}$ in Figure 2) are of distance two from I_k and among these intervals at least one interval (I_k in Figure 2) is not distance two from I_k. Therefore, $p \le \Delta - 1$, i.e. $|L^2(I_k)| \le \Delta - 1$.

Case 2: $i = 3$

Let $|L^3(I_k)| = q$. This implies that q distinct labels are used to label the intervals at distance three from the interval I_k, before labeling the interval I_k by $L(h_1, h_2, \ldots, h_m)$ -labeling. Since, G is an interval graph, so among the intervals at distance two from I_k, there exists an interval I_β (in Figure 2) which is adjacent to at most Δ intervals ($I_\alpha, I_{k_{21}}, I_{k_{22}}, I_{k_{31}}, I_{k_{32}}, I_\gamma$) of G. Among these intervals some intervals ($I_\gamma, I_{k_{31}}, I_{k_{32}}$ in Figure 2) are of distance three from the interval I_k. Also among the intervals which are adjacent to I_β there exists at least one interval (I_α in Figure 2) which is not at distance three from I_k. So, $q \le \Delta - 1$. Hence, $|L^3(I_k)| \le \Delta - 1$.

Case 3: $i = 4$

Let $|L^4(I_k)| = r$. This means that r distinct labels are used to $L(h_1, h_2, \ldots, h_m)$ -label the intervals at distance four from the interval I_k, before labeling the interval I_k. Since, *G* is an interval graph with maximum degree Δ, so among the intervals at distance three from I_k, there exists an interval I_γ (in Figure 2) which is adjacent to at most Δ intervals ($I_\beta, I_{k_{31}}, I_{k_{32}}, I_{k_{41}}, I_{k_{42}}, I_{k_{43}}$) of the graph *G*. Among these intervals some intervals $I_{k_{41}}, I_{k_{42}}, I_{k_{43}}$ (in Figure 2) are of distance four from the interval I_k. Also among the intervals which are adjacent to I_γ, there exists at least one interval (I_β in Figure 2) which is not at distance three from I_k. Hence, $r \le \Delta - 1$. Hence, $|L^4(I_k)| \le \Delta - 1$.

Case 4: $i = 5, 6, \ldots, m$

Proceeding with similar manner we can prove that $|L^i(I_k)| \le \Delta - 1$ for $i = 5, 6, \ldots, m$. Hence $|L^i(I_k)| \le \Delta - 1$ for $i = 2, 3, \ldots, m$.

Lemma 2 *For any interval graph* G, $L^i(I_k) \subseteq L(I_k)$, *for any interval* $I_k \in I$, $i = 1, 2, \ldots, m$.

Proof. According to the definition, $L(I_k)$ is the set of used $L(h_1, h_2, \ldots, h_m)$ -labels before labeling the interval I_k. So any label $l \in L^i(I_k)$ implies $l \in L(I_k)$, for $i = 1, 2, \ldots, m$. Hence $L^i(I_k) \subseteq L(I_k)$.

Theorem 1 *For any interval graph* G, $\lambda_{h_1, h_2, \ldots, h_m}(G) \le (\Delta - 1)\sum_{i=1}^{m} h_i + h_1$, *where* Δ *is the degree of the graph G.*

Proof. Let *G* be an interval graph with n vertices and let $I = \{I_1, I_2, \ldots, I_n\}$, where I_k is the interval corresponding to the vertex v_k of the graph *G*. We shall label the intervals by $L(h_1, h_2, \ldots, h_m)$ -labeling in ascending order of the subscripts of the intervals.

Let $L(I_k) = \{0,1,\ldots,(\Delta-1)\sum_{i=1}^{m} h_i + h_1\}$, where $I_k \in I$.

Then $|L(I_k)| = (\Delta-1)\sum_{i=1}^{m} h_i + h_1 + 1$. If we can $L(h_1, h_2, \ldots, h_m)$-label all the vertices of the graph G by using only the label of the set $L(I_k)$, then we can say that $\lambda_{h_1,h_2,\ldots,h_m}(G) \le (\Delta-1)\sum_{i=1}^{m} h_i + h_1$. At the time of labeling we consider a circumstances in which the intervals $I_1, I_2, \ldots, I_{k-1}$, for $k = 2,3\ldots,n$ are already labeled by $L(h_1, h_2, \ldots, h_m)$-labeling and $I_k, I_{k+1}, \ldots, I_n$ are not labeled. In this case, we want to label the interval I_k by $L(h_1, h_2, \ldots, h_m)$-labeling. We know that $|L^1(I_k)| \le \Delta$ and $|L^i(I_k)| \le \Delta - 1$ for $i = 2,3,\ldots,m$ (By Lemma 1). So, in the extreme unfavorable case $[(\Delta-1)\sum_{i=1}^{m} h_i + h_1 + 1] - h_1\Delta - \sum_{i=2}^{m} h_i(\Delta-1) = 1$ label is available, which satisfies $L(h_1, h_2, \ldots, h_m)$-labeling condition. Since I_k is arbitrary so we can label any vertex corresponding to the interval of the interval graph G by $L(h_1, h_2, \ldots, h_m)$-labeling by using only the label of the set $L(I_k)$. Hence, $\lambda_{h_1,h_2,\ldots,h_m}(G) \le (\Delta-1)\sum_{i=1}^{m} h_i + h_1$.

3.1. Algorithm for $L(h_1, h_2, \ldots, h_m)$-labeling

In this section, we design an algorithm to compute the set $L^{vl}(k, I_j)$ for $j = 2,3,\ldots,n$ and $k = 1,2,\ldots,m$ and also we design an algorithm to $L(h_1, h_2, \ldots, h_m)$-label an interval graph. We consider a situation in which some intervals (the intervals I_k's with index $k < j$) are labeled by $L(h_1, h_2, \ldots, h_m)$-labeling and some intervals (the intervals I_k's with index $k \ge j$) are not labeled.

Algorithm *VLIG*

Input: I_j, $j = 2,3,\ldots,n$.

Output: $L^{vl}(k, I_j)$ for $k = 1,2,\ldots,m$, $j = 2,3,\ldots,n$.

Step 1: Compute $L^i(I_j)$ and $L(I_j)$ for $i = 1,2,\ldots,m$

for $i = 0$ to r, where $r = max\{L(I_j)\} + h_1$

for $k = 1$ to $|L^1(I_j)|$

if $|i - l_k| \ge h_1$, then i to $L^{vl}(1, I_j)$ //where $l_k \in L^1(I_j)$//

end for;

end for;

Step 2: for $k = 1$ to $m - 1$

for $p = 1$ to $|L^{vl}(k, I_j)|$

for $q = 1$ to $|L^{k+1}(I_j)|$

if $|l_p - s_q| \geq h_{k+1}$, then lp to $L^{vl}(k+1, I_j)$

//where $l_p \in L^{vl}(k, I_j)$'s and $s_q \in L^{k+1}(I_j)$ //

end for;

end for;

end for;

end $VLIG$.

Lemma 3: The set $L^{vl}(k, I_j)$ is correctly computed by Algorithm $VLIG$, for $j = 2,3,\ldots,n$ and $k = 1,2,\ldots,m$ and the time complexity for this algorithm is $O(m\Delta^2)$.

Proof. According to this algorithm for each $l_p \in L^1(I_k)$, every element $i \in L^{vl}(1, I_j)$ is differ from l_p by at least h_1. So, $|i - l_p| \geq h_1$ for all $i \in L^{vl}(1, I_j)$ and for all $l_p \in L^1(I_k)$. So the algorithm correctly computes $L^{vl}(1, I_j)$.

Again, from the above algorithm we see that $L_p \in L^{vl}(k+1, I_j)$ for $k = 1,2,\ldots,m-1$ implies that $|l_p - l_q| \geq h_{k+1}$ for all $l_q \in L^{k+1}(I_j)$. Hence the algorithm correctly computes $L^{vl}(k, I_j)$ for $k = 1,2,\ldots m$.

Since $|L|$ be the cardinality of the label set L, so, $|L^i(I_j)| \leq |L|$ for $i = 1,2,\ldots,m$ and $I^j \in I$ and also $r \leq (\Delta-1)\sum_{i=1}^m h_i + 2h_1$, where $r = max\{L(I_j)\} + h_1$. So $L^{vl}(1, I_j)$ is computed by using at most $|L|((\Delta-1)\sum_{i=1}^m h_i + 2h_1)$ times, i.e. using $O(\Delta|L|)$ times. Also, $|L^{vl}(k, I_j)| \leq (\Delta-1)\sum_{i=1}^m h_i + h_1 + 1$, for $k = 1,2,\ldots,m$. So for each $k = 1,2,\ldots,m, L^{vl}(k+1, I_j)$ can be computed using at most $|L|((\Delta-1)\sum_{i=1}^m h_i + h_1 + 1)$ times, i.e. using $O(\Delta|L|)$ times. Since k runs for 1 to $m-1$, so the total time required for this algorithm is $O(m\Delta|L|)$ i.e. $O(m\Delta^2)$, since $|L| \leq (\Delta-1)\sum_{i=1}^m h_i + h_1 + 1$.

Lemma 4: For any $I_j \in I$ of an interval graph G, $L^{vl}(k, I_j)$ is the maximal non empty set satisfying the condition of distance $1,2,\ldots,k$ for $k = 1,2,\ldots,m$ of $L(h_1, h_2,\ldots,h_m)$ -labeling, where $l \leq u$ for all $l \in L^{vl}(k, I_j)$ and $u = max\{L(I_j)\} + h_1$, for any $I_j \in I$ and $k = 1,2,\ldots,m$.

Proof. Since, $L^i(I_j) \subseteq L(I_j)$ for $i = 1,2,\ldots,m$ (by Lemma 2) and $u = max\{L(I_j)\} + h_1$, so $|u - l_q| \geq h_1$ for any $l_q \in L^i(I_j), i = 1,2,\ldots,m$. Therefore, $u \in L^{vl}(k, I_j)$ for $k = 1,2,\ldots,m$. Hence, $L^{vl}(k, I_j)$ is non empty set of labels for any $k = 1,2,\ldots,m$.

Again, let P be an arbitrary set of labels which satisfies the condition of distance $1,2,\ldots,k$ for $k = 1,2,\ldots,m$ of $L(h_1, h_2,\ldots,h_m)$ -labeling, where $l \leq u$ for all $l \in P$. Also, let $p \in P$. Then $|p - l_q| \geq h_i$ for all $l_q \in L^i(I_j)$

and for $i=1,2,\ldots,m$. Thus, for each $k=1,2,\ldots,m$, $p \in L^{vl}(k,I_j)$. So, $p \in P$ implies $p \in L^{vl}(k,I_j)$, for $k=1,2,\ldots,m$. So, $P \subseteq L^{vl}(k,I_j)$, for $k=1,2,\ldots,m$. Since p is arbitrary, so for any $k=1,2,\ldots,m$, $L^{vl}(k,I_j)$ is the maximal non-empty set of labels satisfying the condition of distance $1,2,\ldots,k$ for $k=1,2,\ldots,m$ of $L(h_1,h_2,\ldots,h_m)$ -labeling, where $l \le u$ and for all $l \in L^{vl}(k,I_j)$.

Algorithm LIG

Input: I_j and $L^{vl}(k,I_j)$ for $j=1,2,\ldots,n$ and $k=1,2,\ldots,m$

Output: f_j, the $L(h_1,h_2,\ldots,h_m)$ -label of I_j, $j=1,2,\ldots,n$.

Step 1: (Initialization)

$f_1 = 0$;

$L(I_2) = \{0\}$;

Step 2: for $j = 2$ to $n-1$

$f_j = min\{L^{vl}(m,I_j)\}$;

$L(I_{j+1}) = L(I_j) \bigcup \{f_j\}$;

end for;

Step 3: $f_n = min\{L^{vl}(m,I_n)\}$;

Step 4: $L = L(I_n) \bigcup \{f_n\}$;

$\lambda = maxL$.

end *LIG*.

Theorem 2: An interval graph G is correctly labeled by Algorithm LIG satisfying $L(h_1,h_2,\ldots,h_m)$ - labeling condition.

Proof. Let G be an interval graph with n vertices. Let $I=\{I_1,I_2,\ldots,I_n\}$ and also let $f_1=0$, $L(I_2)=\{0\}$. If G is a graph with single vertex then the graph can be completely label by using the label 0 only. So, $\lambda_{h_1,h_2,\ldots,h_m}(G)=0$.

If the graph has more than one vertex, then we can not label the graph completely by $L(h_1,h_2,\ldots,h_m)$ - labeling, by using the label of the set $L(I_2)$ only, because in this case more than one label is required and $L(I_2)$ contain only one label.

We consider a situation in which the intervals $I_1,I_2,\ldots,I_{j-1}$ of the interval graph G are already labeled by $L(h_1,h_2,\ldots,h_m)$ -labeling for $j=2,3,\ldots,n$. In this situation we want to label the interval I_j by $L(h_1,h_2,\ldots,h_m)$ -labeling. By Lemma 4, we know that $L^{vl}(k,I_j)$ is the maximal non-empty set satisfying the condition of distance $1,2,\ldots,k$ for $k=1,2,\ldots,m$ of $L(h_1,h_2,\ldots,h_m)$ -labeling, where $l \le u$ for all $l \in L^{vl}(k,I_j)$ and $u = max\{L(I_j)\}+h_1$ for any $I_j \in I$ and $k=1,2,\ldots,m$. Again, no label $l \le u$ and $l \notin L^{vl}(m,I_j)$ satisfying $L(h_1,h_2,\ldots,h_m)$ -labeling. So the labels on the set $L^{vl}(m,I_j)$ is the only valid label for I_j, which is less than or equal to u and satisfying $L(h_1,h_2,\ldots,h_m)$ -labeling. So, $f_j = p$, where $p = min\{L^{vl}(m,I_j)\}$. Then p is the least label for I_j, since no label less than p satisfies $L(h_1,h_2,\ldots,h_m)$ -labeling condition. Since, I_j is arbitrary so by Algorithm LIG, the graph G can be labeled using minimum number of labels satisfying $L(h_1,h_2,\ldots,h_m)$ -labeling condition and $\lambda_{h_1,h_2,\ldots,h_m}(G) = max\{L(I_n)\bigcup\{f_n\}\} = \lambda$.

Theorem 3: The running time for Algorithm LIG is $O(mn\Delta^2)$, where n is the number of vertices of the graph and Δ is the degree of the graph and m is the number of conditions of the labeling.

Proof. According to our proposed algorithm it is clear that if $L^{vl}(m,I_j)$ is computed then f_j is computed. Now Algorithm $VLIG$ can take $O(m\Delta^2)$ time to compute $L^{vl}(m,I_j)$ (by Lemma 3). Since we need to find $L^{vl}(m,I_j)$ for $j=2,3,\ldots,n$, so the time complexity of Algorithm LIG is $O((n-1)m\Delta^2)$, i.e. $O(mn\Delta^2)$.

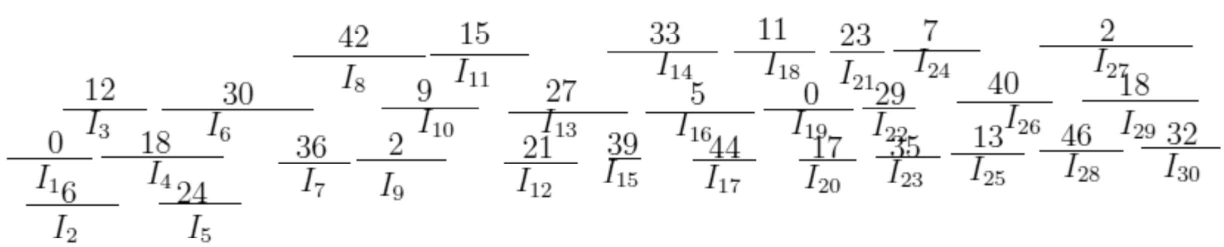

Figure 3: The interval graph G, the number above the interval represents the label of the corresponding vertices

To illustrate the above algorithm we consider an interval graph with 30 vertices (see Figure 3) and label this graph by Algorithm LIG. For this graph,

$I = \{I_1, I_2, \ldots, I_{30}\}$ and $\Delta = 4$. Also let $h_i = 7-i$ for $i=1,2,3,4,5,6$, i.e. we label the graph by $L(6,5,4,3,2,1)$ -labeling, by using our proposed algorithm.

f_j, the label of the interval I_j, for $j=1,2,\ldots,30$.

First we label the interval graph G.

$f_1 = 0$, $L(I_2) = \{0\}$.

Iteration 1: For $j=2$.

$L^1(I_2) = \{0\}$, $L^i(I_2) = \phi$, for $i=2,3,4,5,6$.

$L^{vl}(1,I_2)=\{6\}$, $L^{vl}(k,I_2)=\{6\}$, for $k=2,3,4,5,6$.

Therefore, $f_2=\min\{L^{vl}(6,I_2)\}=6$ and $L(I_3)=L(I_2)\cup\{f_2\}=\{0\}\cup\{6\}=\{0,6\}$.

Iteration 2: For $j=3$.

$L^1(I_3)=\{0,6\}$, $L^i(I_3)=\phi$, for $i=2,3,4,5,6$.

$L^{vl}(1,I_3)=\{12\}$, $L^{vl}(k,I_3)=\{12\}$, for $k=2,3,4,5,6$.

Therefore, $f_3=\min\{L^{vl}(6,I_3)\}=12$ and $L(I_4)=L(I_3)\cup\{f_3\}=\{0,6\}\cup\{12\}=\{0,6,12\}$

Iteration 3: For $j=4$.

$L^1(I_4)=\{6,12\}$, $L^2(I_4)=\{0\}$, $L^i(I_4)=\phi$, for $i=3,4,5,6$.

$L^{vl}(1,I_4)=\{0,18\}$, $L^{vl}(2,I_4)=\{18\}$, $L^{vl}(k,I_4)=\{18\}$, for $k=3,4,5,6$.

Therefore,

$f_4=\min\{L^{vl}(6,I_4)\}=18$ and $L(I_5)=L(I_4)\cup\{f_4\}=\{0,6,12\}\cup\{18\}=\{0,6,12,18\}$.

Iteration 4: For $j=5$.

$L^1(I_5)=\{18\}$, $L^2(I_5)=\{6,12\}$, $L^3(I_5)=\{0\}$, $L^i(I_5)=\phi$, for $i=4,5,6$.

$L^{vl}(1,I_5)=\{0,1,\ldots,12,24\}$, $L^{vl}(2,I_5)=\{0,1,24\}$, $L^{vl}(k,I_5)=\{24\}$, for $k=3,4,5,6$.

Therefore,

$f_5=\min\{L^{vl}(6,I_5)\}=24$ and $L(I_6)=L(I_5)\cup\{f_5\}=\{0,6,12,18\}\cup\{24\}=\{0,6,12,18,24\}$.

Iteration 5: For $j=6$.

$L^1(I_6)=\{18,24\}$, $L^2(I_6)=\{6,12\}$, $L^3(I_6)=\{0\}$, $L^i(I_6)=\phi$, for $i=4,5,6$.

$L^{vl}(1,I_6)=\{0,1,\ldots,12,30\}$, $L^{vl}(2,I_6)=\{0,1,30\}$, $L^{vl}(k,I_6)=\{30\}$, for $k=3,4,5,6$.

Therefore,

$f_6=\min\{L^{vl}(6,I_6)\}=30$ *and*

$L(I_7)=L(I_6)\cup\{f_6\}=\{0,6,12,18,24\}\cup\{30\}=\{0,6,12,18,24,30\}$.

Iteration 6: For $j=7$.

$L^1(I_7)=\{30\}$, $L^2(I_7)=\{18,24\}$, $L^3(I_7)=\{6,12\}$, $L^4(I_7)=\{0\}$, $L^i(I_7)=\phi$, for $i=5,6$.

$L^{vl}(1,I_7)=\{0,1,\ldots,24,36\}$, $L^{vl}(2,I_7)=\{0,1,\ldots,13,36\}$, $L^{vl}(3,I_7)=\{0,1,2,36\}$,

$L^{vl}(k,I_7)=\{36\}$, for $k=4,5,6$.

Therefore,

$f_7 = \min\{L^{vl}(6, I_7)\} = 36$

and $L(I_8) = L(I_7) \cup \{f_7\} = \{0,6,12,18,24,30\} \cup \{36\} = \{0,6,12,18,24,30,36\}$.

Iteration 7: For $j = 8$.

$L^1(I_8) = \{30,36\}$, $L^2(I_8) = \{18,24\}$, $L^3(I_8) = \{6,12\}$, $L^4(I_8) = \{0\}$, $L^i(I_8) = \phi$,

for $i = 5,6$.

$L^{vl}(1, I_8) = \{0,1,\ldots,24,42\}$, $L^{vl}(2, I_8) = \{0,1,\ldots,13,42\}$, $L^{vl}(3, I_8) = \{0,1,2,42\}$,

$L^{vl}(k, I_8) = \{42\}$, for $k = 4,5,6$.

Therefore,

$f_8 = \min\{L^{vl}(6, I_8)\} = 42$

and $L(I_9) = L(I_8) \cup \{f_8\} = \{0,6,12,18,24,30,36\} \cup \{42\} = \{0,6,12,18,24,30,36,42\}$.

Iteration 8: For $j = 9$.

$L^1(I_9) = \{42\}$, $L^2(I_9) = \{30,36\}$, $L^3(I_9) = \{18,24\}$, $L^4(I_9) = \{6,12\}$, $L^5(I_9) = \{0\}$,

$L^6(I_9) = \phi$.

$L^{vl}(1, I_9) = \{0,1,\ldots,36,48\}$, $L^{vl}(2, I_9) = \{0,1,\ldots,25,48\}$, $L^{vl}(3, I_9) = \{0,1,\ldots,14,48\}$,

$L^{vl}(4, I_9) = \{0,1,2,3,9,48\}$, $L^{vl}(5, I_9) = \{2,3,9,48\}$, $L^{vl}(6, I_9) = \{2,3,9,48\}$.

Therefore, $f_9 = \min\{L^{vl}(6, I_9)\} = 2$

and $L(I_{10}) = L(I_9) \cup \{f_9\} = \{0,6,12,18,24,30,36,42\} \cup \{2\} = \{0,2,6,12,18,24,30,36,42\}$.

Iteration 9: For $j = 10$.

$L^1(I_{10}) = \{2,42\}$, $L^2(I_{10}) = \{30,36\}$, $L^3(I_{10}) = \{18,24\}$, $L^4(I_{10}) = \{6,12\}$, $L^5(I_{10}) = \{0\}$,

$L^6(I_{10}) = \phi$.

$L^{vl}(1, I_{10}) = \{8,9,\ldots,36,48\}$, $L^{vl}(2, I_{10}) = \{8,9,\ldots,25,48\}$, $L^{vl}(3, I_{10}) = \{8,9,\ldots,14,48\}$,

$L^{vl}(4, I_{10}) = \{9,48\}$, $L^{vl}(5, I_{10}) = \{9,48\}$, $L^{vl}(6, I_{10}) = \{9,48\}$.

Therefore,

$f_{10} = \min\{L^{vl}(6, I_{10})\} = 9$

And

$L(I_{11}) = L(I_{10}) \cup \{f_{10}\} = \{0,2,6,12,18,24,30,36,42\} \cup \{9\} = \{0,2,6,9,12,18,24,30,36,42\}$.

Iteration 10: For $j = 11$.

$L^1(I_{11}) = \{2,9\}$, $L^2(I_{11}) = \{42\}$, $L^3(I_{11}) = \{30,36\}$, $L^4(I_{11}) = \{18,24\}$, $L^5(I_{11}) = \{6,12\}$,

$L^6(I_{11}) = \{0\}$.

$L^{vl}(1, I_{11}) = \{15,16,\ldots,48\}$, $L^{vl}(2, I_{11}) = \{15,16,\ldots,37,47,48\}$,

$L^{vl}(3, I_{11}) = \{15,16,\ldots,26,47,48\}$, $L^{vl}(4, I_{11}) = \{15,21,47,48\}$,

$L^{vl}(5, I_{11}) = \{15,21,47,48\}$, $L^{vl}(6, I_{11}) = \{15,21,47,48\}$.

Therefore,

$f_{11} = \min\{L^{vl}(6, I_{11})\} = 15$

and

$L(I_{12}) = L(I_{11}) \cup \{f_{11}\} = \{0,2,6,9,12,18,24,30,36,42\} \cup \{15\} = \{0,2,6,9,12,15,18,24,30,36,42\}$.

Proceeding in this way we have, $f_{12} = 21$, $f_{13} = 27$, $f_{14} = 33$, $f_{15} = 39$, $f_{16} = 5$,

$f_{17} = 44$, $f_{18} = 11$, $f_{19} = 0$, $f_{20} = 17$, $f_{21} = 23$, $f_{22} = 29$, $f_{23} = 35$, $f_{24} = 7$,

$f_{25} = 13$, $f_{26} = 40$, $f_{27} = 2$, $f_{28} = 46$, $f_{29} = 18$, $f_{30} = 32$. So $\lambda = maxL = 46$ for the interval graph in figure 3.

Hence, $L(6,5,4,3,2,1)$-labeling number for the circular-arc graph in figure 3 is 46.

4. CONCLUSION

Although, $L(h,k)$-labeling problem has been widely studied in the past, there are only a few classes of graphs for which the result about $L(h,k)$-labeling is available. So,the generalization of $L(h,k)$-labeling i.e. $L(h_1, h_2, \ldots, h_m)$-labeling and the upper bound of $L(h_1, h_2, \ldots, h_m)$-labeling is clearly welcome. In this paper, we determine the upper bounds for $\lambda_{h_1,h_2,\ldots,h_m}$ for interval graph and have shown that $\lambda_{h_1,h_2,\ldots,h_m}(G) \leq (\Delta - 1)\sum_{i=1}^{m} h_i + h_1$ for interval graph. Although, Bertossi *et al*. have been studied approximate $L(\delta_1, \delta_2, \ldots, \delta_t)$-labeling of interval graphs [1], but our proposed algorithm is the first upper bound in terms of Δ. Also, an efficient algorithms is designed to $L(h_1, h_2, \ldots, h_m)$-label for interval graphs.

Since the upper bound is not tight, so there is a chance for new upper bound for the problem.

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