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APPROXIMATION OF COMMON FIXED POINT OF FINITE FAMILY OF NONSELF AND NONEXPANSIVE MAPPINGS IN HILBERT SPACE

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Abstract

Let $T_1, T_2, \dots T_N : K \to H$ be finite family of nonself, nonexpansive and inward mappings on a non empty,closed and strictly convex subset K of a real Hilbert space H with $F = \bigcap_{k=1}^{N} F(T_k)$ is non empty. Let $T_n = T_{n(ModN)}$, $h_k : K \to \Re$ be defined by $h_k(x) = inf\{\lambda \ge 0 : \lambda x + (1-\lambda)T_k x \in K\}$. Then for each $x_1 \in K$, $\alpha_1 = max\{\alpha, h_1(x_1)\}, \alpha > 0$, we define the Krasnoselskii-Mann type algorithm by $x_{n+1} = \alpha_n x_n + (1-\alpha_n)T_n x_n$, where $\alpha_{n+1} = max\{\alpha_n, h_{n+1}(x_{n+1})\}$, $n = 1,2,\dots$ and we prove weak and strong convergence of the sequence $\{x_n\}$ to a common fixed point of the family, $\{T_1, T_2, \dots, T_N\}$ .We also extend our main result to the class of quasi-nonexpansive, fixed point closed and inward mappings.

Keywords and Phrases: *Fixed points; nonexpansive; nonself mapping; uniformly convex Banach space; uniformly smooth Banach space.*

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1. INTRODUCTION

In various areas of mathematical problems, finding a solution of the mathematical model for the real problem is equivalent to finding a fixed point for a certain map or family of mappings. Finding a fixed point has been therefore, very important for role in several areas of Mathematics and other sciences.

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Let K be a nonempty subset of a real normed linear space E and $T: K \to K$ be a map. Then a point $x \in K$ is said to be a fixed point of T if

$$Tx = x. \tag{1.1}$$

For example, we consider the differential equation or systems of differential equations,

$$\frac{du}{dt} + Au(t) = 0 \tag{1.2}$$

which describes an evolution system where A is called an accretive or monotone operator from a Hilbert space H in to itself if it satisfies

$$A: H \to H \text{ Satisfies} \langle Ax - Ay, x - y \rangle \geq 0 \text{ for all } x, y \in H \tag{1.3}$$

At the equilibrium state $\frac{du}{dt} = 0$, so a solution of $Au = 0$ describes the equilibrium or stable state of the system. This is applicable in many areas, for example, in dynamical system, Economics, Physics, Optimization to mention a few, consequently, considerable research efforts have been devoted to methods of solving the equation $Au = 0$ when A is accretive or monotone. Since generally A is nonlinear, there is no closed-form solution of this equation. The standard technique is to introduce an operator T defined by $T = I - A$ where I is the identity map on H. Such a T is called a pseudo contraction (or is called pseudo-contractive) where $T: K \to K$ on a non empty subset of a real Hilbert space H, satisfies

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2 \text{ for all } x, y \in K. \quad (1.4).$$

It is then clear that any zero of A is a fixed point of T and vice versa, as a result of this, the study of fixed point theory for pseudo contractive maps has attracted the interest of numerous researchers and has become an interesting area of research, especially within the last 40 years for numerous mathematicians. An important subclass of the class of pseudo contractive mappings is that of the class of nonexpansive mappings, where $T: K \to K$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \text{ for arbitrary } x, y \in K. \tag{1.5}$$

If we want to consider another class of pseudo contractive mappings, the so called strictly pseudo contractive mappings, where $T: K \to K$ is called*α-strictly* pseudo contractive if and only if there exists $\alpha \in [0,1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \alpha\|(I - T)x - (I - T)y\|^2 \ x, y \in K. \qquad (1.6)$$

Thus, it can be shown that every non expansive mapping is an α-strictly pseudo contractive mapping, and every α-strictly pseudo contractive mapping is a pseudo contractive mapping, and hence a class of pseudo contractive mappings is a more general class of mappings.

A number of research efforts have been made for finding fixed point of such a mapping (see, for example [3, 13,16-17,19-20,23-24 &27] and their references).

Family of nonself and nonexpansive mappings arises when the mappings are defined on K which is a proper subset of H; that is, family of mappings appear from a proper subset K of H in to H. Thus, it is the purpose of this paper to find approximation method fora common fixed point (if it exists) for family of non self and nonexpansive mappings in a real Hilbert space with the assumption that the mappings to be inward;

Definition 1.1: A mapping $T : K \to H$ is said to be inward (or to satisfy the inward condition) if for any $x \in K$, $Tx \in IK(x) = \{x + c(u - x) : c \ge 1 \,\&\, u \in K\}$ and T is said to satisfy weakly inward condition if $Tx \in \overline{IK(x)}$ (the closure of $IK(x)$).

Definition 1.2: A uniformly convex space is a normed space E for which, for every $0 < \varepsilon < 2$ there is a $\delta > 0$ such that for all $x, y \in S = \{x \in E : \|x\| = 1\}$, if

$$\|x - y\| > \varepsilon (x \ne y) \text{,then } \left\|\frac{x+y}{2}\right\| \le 1 - \delta. \qquad (1.7)$$

Let E be a normed space with dimension, $\dim E \ge 2$ .The modulus of convexity of E is a function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(t) = \inf\left\{1 - \frac{\|x+y\|}{2}, \|x\| = \|y\| = 1 \,\&\, \|x - y\| = t\right\} \ \textit{for all } t \ , 0 < t \le 2 \quad (1.8)$$

Hilbert spaces, the Lebesgue L_p, the sequence l_p and the Sobolev W_p^m spaces, for $p \in (1,\infty)$ are examples of uniformly convex Banach spaces.

A Banach Space E is said to be uniformly convex if $\delta_E(t) > 0$ for all $0 < t \le 2$.

Definition 1.3: A uniformly smooth space E is a normed space in which for every $\varepsilon > 0$ there is $\delta > 0$ such that for all

$$x, y \in E, \|x\| = 1, \|y\| \leq \delta, \ \|x + y\| + \|x - y\| \leq 2 + \varepsilon\|y\|. \tag{1.9}$$

Let E be a normed space with $\dim E \geq 2$. The modulus smoothness of E is a function $\rho_E : [0, \infty) \to [0, \infty)$ defined by,

$$\rho_E(t) = Sup\left\{\frac{\|x + y\| + \|x - y\|}{2} - 1, \|x\| = 1, \|y\| = t\right\} \text{ for } t \geq 0 \tag{1.10}$$

and E is uniformly smooth if and only if $\lim_{t \to 0} \frac{\rho_X(t)}{t} = 0$. (1.11)

Let $p, q > 1$ be real numbers. Then E is said to be p-uniformly convex (respectively q-uniformly smooth) if there is a constant $c > 0$ such that

$$\delta_E(t) \geq ct^p \text{ (respectively } \rho_E(t) \leq ct^q). \tag{1.12}$$

Hilbert spaces, the Lebesgue L_p, the sequence l_p and the Sobolev W_p^m spaces, for $p \in (1, \infty)$ are examples of uniformly smooth Banach spaces. (see, for example in[3])

Definition 1.4: A subset K of E is said to be strictly convex if for any $x, y \in \partial K, x \neq y, 0 < t < 1, tx + (1-t)y \in \text{int}(K)$, that is, no line segment joining any two points of K totally lies on the boundary of K.

Definition 1.5: Let E be a real Banach space. Then

(a) a subset K of E is said to be a retract of E if there exists a continuous map $P : E \to K$ such that $P(x) = x$ for all $x \in K$.

(b) a map $P : E \to K$ is said to be a retraction if $P^2 = P$, It follows that if a map P is a retraction, then $P(y) = y$ for all y in the range of P.

(c) a map $P : E \to K$ is said to be sunny if $P(Px + t(x - Px)) = Px$ for all $x \in E$ and $t \geq 0$.

(d) a subset K of E is said to be a sunny nonexpansive retract of E if there exists a sunny nonexpansive retraction of E onto K and it is said to be a nonexpansive retract of E if there exists a nonexpansive retraction of E onto K.

For example in a Hilbert space H, the metric projection P_K is a sunny nonexpansive retraction from H to any closed convex subset K of H (see, for example, in [3,8])

We denote the set of all fixed points of the mapping T by F(T) and the set of all common fixed points of the family $\{T_1, T_2, ...T_N\}$ by $F = \bigcap_{i=1}^{N} F(T_i)$.

A number of research efforts have been made to find iterative methods for approximating a fixed point or common fixed point (when it exists) for nonexpansive and pseudo contractive mapping or a family of nonexpansive and Pseudo contractive mappings respectively.

First, Bauschke in [2] introduced Halpern typeiterative method (see, *e.g.*, [13]) for approximating a common fixed point for a finite family of nonexpansive and self mappings and he proved the following theorem;

Theorem 1.1: *BSK* Let K be a nonempty, closed and convex subset of a Hilbert space H, $T_i, i = 1,2,3,...N$ be a finite family of nonexpansive mappings of K into itself with $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ and $F = F(T_1 T_2....T_N) = F(T_N T_{N-1}....T_1)$ $= F(T_{N-1} T_{N-2},....T_1 T_N)$ where $F(T)$ is the set of all fixed points of T. Let $u, x_0 \in K$ be given. Let $\{x_n\}$ be generated by $x_{n+1} = \alpha_{n+1} u + (1 - \alpha_{n+1}) T_{n+1} x_n$, $n \geq 0$. Let $T_n = T_n (\text{mod } N)$, $\alpha_n \in (0,1)$satisfies $\sum_{n \geq 1} |\alpha_{n+r} - \alpha_n| < \infty$.Then $\{x_n\}$ converges strongly to $P_F u$, where $P : H \to F$ is the metric projection.

Similarly, various researchers have studied iterative methods in more general Banach spaces, and they proved convergence of the iterative method using restrictions on the sequence $\{\alpha_n\}$ (see, for example, Colao *et al in*[10], Yao in [26], Takahashi and Takahashiin[25], Plubtieng and Punpaeng in [22], Ceng *et al in*[6] and their references).

Many author shave also studied iterative methods for a family of α -strictly pseudo contractive mappings (see, *e.g.*, [11, 21, 28-31, & 33-34] and their references).

All the above results are for self mappings only. However the mapping in many practical cases can be nonself;

For example, let $H = l_2(\Re^2)$ and $K = \left\{ (x, y), \frac{-1}{2} \leq x, y \leq 1 \right\}$, thus K is nonempty, closed and convex subset of H. Let $T : K \to H$ be defined by

$T(x, y) = (-x, y)$ is nonself and nonexpansive mapping. Indeed, for $X = (x, y), X' = (x', y') \in K$, we have

$$\|TX - TX'\| = \|(x, y) - (x', y')\| = \sqrt{(x'-x)^2 + (y - y')^2} = \|X - X'\| .$$

Thus, T is nonexpansive and nonself mapping.

We also see that T is inward mapping, since

$$\forall (x, y) \in K ,\ T(x, y) = (-x, y) = (x, y) + 4\left(\left(\frac{x}{2}, y\right) - (x, y)\right) \text{ holds.}$$

As a result, in 2005, Chidume *et al* in [7] constructed an iterative method for common fixed point of family of nonself and nonexpansive mappings in reflexive Banach space with the assumption that, every nonempty, closed, bounded and convex subset of K has the fixed point property for nonexpansive mappings. They proved strong convergence of the following theorem.

Theorem 1.5: CZS ([7]Let K be a nonempty, closed and convex subset of a reflexive real Banach space E, which has a uniformly Gateaux differentiable norm. Assume that K is a sunny nonexpansive retract of E with Q as the sunny nonexpansive retraction. Assume that every nonempty closed, bounded and convex subset of K has the fixed point property for nonexpansive mappings. Let $T_1, T_2, \ldots T_N : K \to E$ be a finite family of nonexpansive and weakly inward mappings with $F = \bigcap_{k=1}^{N} F(T_k)$ is nonempty, $T_n = T_n (\text{mod } N)$, $F = F(T_1 T_2 \ldots . T_N)$ $F(T_N T_{N-1} \ldots . T_1) = F(T_{N-1} T_{N-2}, \ldots . T_1 T_N)$ .Given points $u, x_0 \in K$, let $\{x_n\}$be the sequence generated by the algorithm, $x_{n+1} = \alpha_{n+1} u + (1 - \alpha_{n+1}) Q T_{n+1} x_n, n \ge 0$, $\{a_n\}$is a real sequence which satisfies the following conditions:

(i) $\lim_{n\to\infty} a_n = 0$; (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$ and either (iii) $\sum_{n\ge 1} |\alpha_{n+N} - \alpha_n| < \infty$ or

(iii) $\lim_{n\to\infty} \frac{a_{n+N} - a_n}{a_{n+N}} = 0$. Then $\{x_n\}$converges strongly to a common fixed point of the family $\{T_1, T_2, \ldots T_N\}$. Further, if $Pu = \lim_{n\to\infty} x_n$ for each $u \in K$, then P is a sunny nonexpansive retraction of K onto F.

Recently, in 2007, Hukmi *et al*[14] and in 2008,Kiziltunc and Yildirim [18] also constructed iterative methods by using metric projection, and they proved convergence with the assumption of opial's condition. However, the computation for the metric projection for sunny nonexpansive retractio n is expensive, even in Hilbert spaces, it requires another approximation. Thus, it is our purpose in this paper to introduce an iterative method without the computation for metric projection.

More recently, In 2015, Colao and Marino in [9] introduced Krasnoselskii-Mann iterative method for approximating a fixed point of nonself, nonexpansive and inward mapping, and they proved the following weak and strong convergence theorem;

Theorem 1.6: *CM* [9]) Let K be a convex, closed and nonempty subset of a Hilbert space H and $T : K \to H$ be a non-self mapping and let for any given $x \in K$, $h(x) = \inf\{\lambda \geq 0 : \lambda x + (1-\lambda)Tx \in K\}$.Then the algorithm defined by

$$\begin{cases} x_0 \in K \\ \alpha_0 = \max\left\{\frac{1}{2}, h(x_0)\right\}, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n \\ \alpha_{n+1} = \max\{\alpha_n, h(x_{n+1})\} \end{cases}$$

is well-defined and assume that; K is strictly convex set, T is nonexpansive, nonself and Inward mapping with F(T) is non empty, then $\{x_n\}$converges weakly to $p \in F = F(T)$. Moreover, if $\sum_{n=1} (1-\alpha_n) < \infty$, then the convergence is strong.

Meanwhile, they proposed the possibility to have an approximation for a common fixed point of a countable family of nonself and nonexpansive mappings.

Motivated by the result of Colao and Marino and their open question for the convergence of approximation for a common fixed point of family of nonself mappings, our concern is that; can we construct iterative method for approximating a common fixed point of a finite family of nonself, nonexpansive and inward mappings in a real Hilbert space?

Thus, our purpose in this paper is to define an algorithm which approximates a common fixed point of a finite family of nonself, nonexpansive and inward mappings and prove the weak and strong convergence results of the algorithm in Hilbert spaces which is a positive answer for our concern.

2. PRELIMINARY

In this paper, we made the following assumptions which will be used in our main result:

Definition 2.1: (See, for example, BAUSCHKE [1]) Let K be a non empty subset of a Hilbert space. *Then*, a sequence $\{x_n\}$ in K is said to be Fejer monotone with respect to a subset F of K if $\forall x \in F, \|x_{n+1} - x\| \leq \|x_n - x\|, \forall n$.

Lemma 2.1: (see, for example, (Zegeye and Shahzad [32, lemma 1.1) Let H be a Hilbert space. Then, for all $\lambda \in [0,1]$, $\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$, for all $x, y \in H$.

Definition 2.2: A Banach space E is said to satisfy opial's condition. If E^* is the weak limit of a weakly convergent sequence $\{x_n\}$, then $\limsup_{n\to\infty}\|x_n - x^*\| < \limsup_{n\to\infty}\|x_n - x\|$ for all $x \neq x^*$.

Remark: Every Hilbert space satisfies opial's condition.

Definition 2.2: Let K be a closed subset of a Banach space E. Then a mapping $T: K \to E$ is said to be demi closed at v if $\{x_n\}$ is a sequence in K, which converges weakly to $u \in K$ and if $\{Tx_n\}$ converges strongly to $v \in E$, then $Tu = v$.

T is said to satisfy demi-Closedness principle (be demi closed) if I-T is demi closed at 0.

Theorem 2.1: (See, for example, Browder[4]) Thedemi Closedness principle for nonexpansive mappings holds in a Banach space which is either uniformly convex or satisfy Opial's condition.

Lemma 2.2: (See, for example, Browder [5]). Let E be a uniformly convex Banach space, K a nonempty, closed and convex subset of E and $T: K \to E$ a nonexpansive mapping. Then I − T is demi closed at zero.

Lemma 2.3: (See, for example, Browder [4], Ferreira-Oliveira [12]). Let E be a complete metric space and K is a nonempty subset of E. If $\{x_n\}$ in E is Fejer monotone with respect to K, then $\{x_n\}$ is bounded. Furthermore, if a cluster point x of $\{x_n\}$ belongs to K then $\{x_n\}$ converges strongly to x.

In particular, in Hilbert space, given the set of all weakly cluster points of $\{x_n\}$ $\omega_w(x_n) = \{x : \exists x_{n_k} \to x, \text{weakely}\}$, then $\{x_n\}$ converges weakly to a point $x \in K$ if and only if $\omega_w(x_n) \subseteq K$.

Lemma 2.4: (see, for example, Reich [23]) Let E be a uniformly convex Banach space, $\{x_n\}, \{y_n\}$ in E be two sequences, if there exists a constant $r \geq 0$ such that $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$ and $\lim_{n\to\infty} \|\lambda_n x_n + (1-\lambda_n) y_n\| = r$, where $\{\lambda_n\} \subset [\varepsilon, 1-\varepsilon] \subset (0,1)$ for some $\varepsilon \in (0,1)$, then $\|x_n - y_n\| \to 0$.

Definition 2.3: Let *F*, *K* be two closed and convex nonempty sets in a Hilbert space *H* and $F \subset K$. For any sequence $\{x_n\} \subset K$, if $\{x_n\}$ converges strongly to an element $x \in \partial K \setminus F, x_n \neq x$, implies that $\{x_n\}$ is not Fejer-monotone with respect to the set $F \subset K$, we called that, the pair (*F*, *K*) satisfies *S*-condition.

3. THE MAIN RESULT

Let $T_1, T_2, \ldots T_N : C \to H$ be family of nonself and nonexpansive mappings on a non-empty, closed and convex subset C of a Hilbert space H. It is the purpose of this paper to introduce an iterative method for common fixed point of the family (if it exists) and to prove the convergence of the iterative method.

In lowering the requirement of metric projection calculation, we impose the condition that mappings in the family to be inward, and in line with the proof of lemma 2.1 in [9] we prove the following lemma which will be used to prove our main result:

Lemma 3.1: Let $T_1, T_2, \ldots T_N : C \to H$ be non self, non-expansive and inward mappings. If each $k \in \{1, 2, \ldots, N\}$ we define $h_k : C \to \Re$ by $h_k(x) = \inf \{\lambda \in [0,1] : \lambda x + (1-\lambda) T_k x \in C\}$. Then

(a) for any $x \in C, h_k(x) \in [0,1]$ and $h_k(x) = 0$ if and only if $T_k(x) \in C$;

(b) for any $x \in C$ and $\alpha_k \in [h_k(x), 1], \alpha_k x + (1-\alpha_k) T_k(x) \in C$;

(c) If T_k is inward mapping, then $h_k(x) < 1$ for any $x \in C$;

(d) If $T_k x \notin C$, then $h_k(x)x + (1-h_k(x)) T_k x \in \partial C$

Proof: (a) clearly $h_k(x) \geq 0$, since $\lambda \geq 0$ and,if $\lambda = 1$,we have $x \in C$ is trivial.

Thus $h_k(x) = \inf\{\lambda \geq 0 : \lambda x + (1-\lambda)T_k x \in C\} \in [0,1]$ and if $h_k(x) = 0$ for $\lambda = 0$, then $T_k(x) \in C$.

Suppose $T_k(x) \in C$, then for $\lambda = 0$ $0x + (1-0)T_k x \in C$ hence, $h_k(x) = 0$.

Therefore $h_k(x) = 0$ if and only if $T_k(x) \in C$.

(b) Let $x \in C$, $\alpha_k \in [h_k(x),1]$.

Then $h_k(x) = \inf\{\lambda \geq 0 : \lambda x + (1-\lambda)T_k x \in C\} \leq \alpha_k \leq 1$.

Thus, $\alpha_k x + (1-\alpha_k)T_k(x) \in C$.

(c) Suppose T_k is inward mapping. Then for all $x \in C$ we have $T_k x \in IC(x) = \{x + c(u-x) : c \geq 1 \,\&\, u \in C\}$ which is equivalent to the following $T_k(x) = x + c(u-x) \Leftrightarrow \dfrac{T_k(x)}{c} + \left(1 - \dfrac{1}{c}\right)x = u \in C$.

Thus by definition of in fimum, $h_k(x) \leq 1 - \dfrac{1}{c} < 1$ which gives $h_k(x) < 1$.

(d) Suppose $T_k x \notin C$. Then $h_k(x) > 0$. Let $\eta_{k_n} \in (0, h_k(x))$ be a sequence of real numbers such that $\eta_{k_n} \to h_k(x)$. Then $\eta_{k_n} x + (1-\eta_{k_n})T_k(x) \notin C$, since $\eta_{k_n} \to h_k(x)$ and hence,

$$\|[\eta_{k_n} x + (1-\eta_{k_n})T_k x] - [h_k(x)x + (1-h_k(x))T_k x]\| = |\eta_{k_n} - h_k(x)| \|x - T_k(x)\| \to 0$$

Thus the limit point of the sequence which is in the complement of C must be on the boundary of C. Thus if $T_k x \notin C$, then $h_k(x)x + (1-h_k(x))T_k x \in \partial C$.

Using the lemma, we will prove the main theorem;

Theorem 3.2: Let $T_1, T_2, \ldots T_N : C \to H$ be a family of nonself, nonexpansive and inward mappings on a non empty, closed and strictly convex subset C of a Hilbert space H with $F = \bigcap_{k=1}^{N} F(T_k)$ is non empty. Let $T_k = T_{k(Mod\,N)}$,

$h_k(x) = \inf\{\lambda \geq 0 : \lambda x + (1-\lambda)T_k x \in C\}$. $\alpha \in (0,1)$ be fixed. Then the sequence $\{x_n\}$ defined by;

$$\begin{cases} x_1 \in C \\ \alpha_1 = \max\{\alpha, h_1(x_1)\} \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)T_n x_n \\ \alpha_{n+1} = \max\{\alpha_n, h_{n+1}(x_{n+1})\} \end{cases}$$

is well-defined and if $\{\alpha_n\} \subset [\varepsilon, 1-\varepsilon] \subset (0,1)$ for some $\varepsilon \in (0,1)$, then $\{x_n\}$ converges weakly to some element p of $F = \bigcap_{k=1}^{N} F(T_k)$.

Proof: By the lemma 3.1 $\{x_n\}$ is well-defined and in C.

Let $p \in F$ .Then for $\{x_n\}$ in C we have the following;

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n x_n + (1-\alpha_n)T_n x_n - p\| \\ &\leq \alpha_n \|x_n - p\| + (1-\alpha_n)\|T_n x_n - T_n p\| \\ &\leq \alpha_n \|x_n - p\| + (1-\alpha_n)\|x_n - p\| = \|x_n - p\|. \end{aligned} \tag{3.1}$$

Thus, $\{x_n\}$ is fejer monotone with respect to F.

Hence, $\{\|x_n - p\|\}$ is decreasing and bounded below, and hence it converges. Furthermore, the following inequalities;

$$\|x_n\| = \|x_n - p + p\| \leq \|x_n - p\| + \|p\| \leq M \tag{3.2}$$

$$\|T_n x_n\| = \|T_n x_n - p + p\| \leq \|T_n x_n - p\| + \|p\| \leq \|x_n - p\| + \|p\| \leq M \tag{3.3}$$

hold for some $M > 0$.

Thus $\{x_n\}$ and $\{Tx_n\}$ are bounded, hence, since any Hilbert space is reflexive Banach space in which the bounded sequence $\{x_n\}$ has a weakly convergent subsequence $\{x_{n_j}\}$ which converges weakly to some $x \in C$ as C is closed.

On the other hand, suppose $x_n \to x$ weakly and for each $l = 1,2,\ldots N$, $x_n - T_l x_n \to 0$, then (by lemma 2.2 in [5]) since T_l is demi-closed; that is, in a uniformly convex Banach space, in particular, in Hilbert space any non self and nonexpansive mapping on nonempty and convex set is demi-closed, $T_l x = x$, hence $x \in F(T_l)$.

By Lemma 2.1 [32] for $\alpha \in [0,1]$, $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2$ $-\alpha(1-\alpha)\|x-y\|$ for all $x, y \in H$ holds. (3.4)

Thus from (3.1),(3.4) and non expansiveness of the mappings we get;

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\alpha_n x_n + (1-\alpha_n)T_n x_n - p\|^2 \\ &= \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|T_n x_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - T_n x_n\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - T_n x_n\|^2 \\ &= \|x_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - T_n x_n\|^2.\end{aligned}$$

This expression with cancellation of terms in the right-hand side with and convergence of the sequence $\{\|x_n - p\|\}$, we have the following;

$$\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n)\|x_n - T_n x_n\|^2 \le \sum_{n=1}^{\infty}(\|x_n - p\| - \|x_{n+1} - p\|) < \infty \tag{3.5}$$

Since, $\{\alpha_n\} \subset [\varepsilon, 1-\varepsilon] \subset (0,1)$ we have

$$\sum_{n=1}^{\infty}\|x_n - T_n x_n\|^2 < \infty \tag{3.6}$$

In fact, $\{\alpha_n\} \subset [\varepsilon, 1-\varepsilon] \subset (0,1)$, since $0 < \delta \le \alpha_n < 1$ and $\{\alpha_n\}$ is nondecreasing there is $\varepsilon > 0$ such that $\varepsilon < \alpha_n < 1-\varepsilon$.

Thus, $W_2(\alpha_n) = \alpha_n(1-\alpha_n) \ge \varepsilon^2 > 0$. Consequently, we have $\sum_{n=1}^{\infty}\|x_n - T_n x_n\|^2 < \infty$.

Thus, $\|x_n - T_n x_n\| \to 0$, $n \to \infty$ hence for very large n we have the followings;

$$\|x_{n+1} - x_n\| = \|\alpha_n x_n + (1-\alpha_n)T_n x_n - x_n\| = (1-\alpha_n)\|x_n - T_n x_n\| \to 0 \tag{3.7}$$

Since $\|x_n - T_n x_n\| \to 0$ and by induction for, $\|x_n - x_{n+i}\| \to 0$

$$\begin{aligned}\|x_n - T_{n+i}x_n\| &\le \|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i}x_{n+i}\| + \|T_{n+i}x_{n+i} - T_{n+i}x_n\| \\ &\le 2\|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i}x_{n+i}\| \to 0\end{aligned} \tag{3.8}$$

Thus, we have $\|x_n - T_{n+i}x_n\| \to 0$ for $0 \le i \le N$. (3.9)

Suppose $x_n \to p$ weakly; that is, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p$ weakly and let $n_k = j(\mathrm{mod}\ N)$ for some $1 \le j \le N$. Take any $l \in \{1,2,...N\}$, thus there exists $1 \le i \le N$ such that $n_{k+i} = l(\mathrm{mod}\ N)$, thus $\|x_{n_k} - T_l x_{n_k}\| \to 0, k \to \infty$.

Thus $p \in F(T_l)$ .Since l is arbitrary $p \in F = \bigcap_{l=1}^{N} F(T_l)$.

It remains to show $x_n \to p$ weakly;

We claim that $p = q$, suppose not?

Hilbert space satisfies Opial's condition, thus if $x_n \to x$ weakly, then $\limsup_{n\to\infty}\|x_n - x\| < \limsup_{n\to\infty}\|x_n - y\|$ for all $y \in H, y \neq x$ (3.10)

Suppose there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_{jk}} \to q$, similarly $q \in F$

$$\begin{aligned}\lim_{n\to\infty}\|x_n - p\| = \lim_{k\to\infty}\|x_{n_k} - p\| &= \limsup_{k\to\infty}\|x_{n_k} - p\| \\ &< \limsup_{k\to\infty}\|x_{n_k} - q\| \\ &< \lim_{k\to\infty}\|x_{n_k} - p\|\end{aligned} \tag{3.11}$$

is contradiction, thus $p = q$. Thus by *Lemma* 2.2 (Browder [4], Ferreira-Oliveira in [12]).

Therefore, $\{x_n\}$ converges weakly to $p \in F = \bigcap_{l=1}^{N} F(T_l)$.

Remark: We now give an example of two nonself, nonexpansive and inward mappings with the set of common fixed points is nonempty. Let $C = \left[-\frac{1}{2}, 1\right]$ and $X = \Re$ thus, $C \subset X$.

Let $T_1, T_2 : C \to \Re$ be defined by

$$T_1 x = \begin{cases} x \ ,0 \le x \le 1 \\ -x, -\frac{1}{2} \le x < 0 \end{cases}$$

and $$T_2 x = -x, -\frac{1}{2} \le x \le 1 .$$

Then we see that $F(T_1) = [0,1]$, $F(T_2) = \{0\}$ hence $F(T_1) \cap F(T_2) = \{0\}$ which is nonempty.

Now we see that T_1 and T_2 are nonself, non expansive mappings.

Suppose $x, y \in \left[-\frac{1}{2}, 1\right]$, then $\|T_1 x - T_1 y\| = \||x| - |y|\| \le \|x - y\|$ and $\|T_2 x - T_2 y\| = \|x - y\|$, thus T_1 and T_2 are nonexpansive mappings. Now we see the mappings are inward.

Suppose $x \in \left[-\frac{1}{2}, 1\right]$; if $-\frac{1}{2} \le x \le 0$, then $T_1 x = -x = x + 1(-x - x)$ for $c = 1 \,\&\, u = -x \in C$ and if $x \in [0,1]$, then $T_1 x = x = x + 1(x - x), u = x \in C$.

Also, if $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, then $T_2 x = -x = x + 1(-x - x), c = 1 \,\&\, u = -x \in C$ and if $x \in \left[\frac{1}{2}, 1\right]$, then $T_2 x = -x = x + \frac{4}{3}\left(-\frac{x}{2} - x\right), c = \frac{4}{3} \ge 1 \,\&\, u = -\frac{x}{2} \in C$. Thus T_1 and T_2 are nonself, nonexpansive mappings with non empty set of common fixed points.

To make it practical, let $\alpha_n = 1 - \frac{1}{2^n} \in (0,1)$ then with $x_0 = -0.5$ for example the algorithm converges strongly to a common fixed point 0 .We also note that $\sum (1 - \alpha_n) = \sum \frac{1}{2^n} < \infty$.

Moreover, the strong convergence of the algorithm can be shown in the following theorem;

Theorem 3.3: Let $T_1, T_2, ...T_N : C \to H$ be family of, nonself, nonexpansive and inward mappings on a non empty, closed and strictly convex subset C of a real Hilbert space H with $F = \bigcap_{k=1}^{N} F(T_k)$ is non empty. $T_k = T_{k(ModN)}, x_0 \in C$ and if for each we define $h_k : C \to \Re$ by $h_k(x) = \inf\{\lambda \geq 0 : \lambda x + (1-\lambda)T_k x \in C\}$, $\alpha \in (0,1)$ be fixed. Then the sequence $\{x_n\}$ given by

$$\begin{cases} x_1 \in C \\ \alpha_1 = \max\{\alpha, h_1(x_1)\} \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)T_n x_n \\ \alpha_{n+1} = \max\{\alpha_n, h_{n+1}(x_{n+1})\} \end{cases}$$

is well-defined and if $\{\alpha_n\} \subset [\varepsilon, 1-\varepsilon] \subset (0,1)$ for some $\varepsilon \in (0,1)$ or $\sum_{n=1} (1-\alpha_n) < \infty$ and (F,C) satisfies S-condition, then $\{x_n\}$ converges strongly to some element p of $F = \bigcap_{k=1}^{N} F(T_k)$.

Proof: By the lemma 3.1, $\{x_n\}$ is well-defined.

Thus, first we prove that $\{x_n\}$ is fejer monotone with respect to F, to this end let $p \in F$.

Then we have the following in equality;

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n x_n + (1-\alpha_n)T_n x_n - p\| \\ &\leq \alpha_n \|x_n - p\| + (1-\alpha_n)\|T_n x_n - Tp\| \\ &\leq \alpha_n \|x_n - p\| + (1-\alpha_n)\|x_n - p\| = \|x_n - p\|. \end{aligned} \tag{3.12}$$

Thus $\{x_n\}$ is fejer monotone with respect to F.

Since $\|x_n - p\|$ is decreasing and bounded below it converges. Thus $\{x_n\}$ and hence $\{Tx_n\}$ are bounded.

Also by lemma 2.1 of [32} in Hilbert space, for $\alpha \in [0,1]$ $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2$ for all $x, y \in H$ holds. Thus the following holds;

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\alpha_n(x_n-p)+(1-\alpha_n)(T_nx_n-p)\|^2\\ &\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)\|T_nx_n-p\|^2\\ &\quad -\alpha_n(1-\alpha_n)\|x_n-T_nx_n\|^2.\end{aligned} \tag{3.13}$$

Which implies

$$\sum_{n=1}^{\infty}\alpha_n(1-\alpha_n)\|x_n-T_nx_n\|^2 \le \sum_{n=1}^{\infty}\|x_n-p\|^2-\|x_{n+1}-p\|^2<\infty. \tag{3.14}$$

Suppose $\{\alpha_n\}\subset[\varepsilon,1-\varepsilon]\subset(0,1)$ for some $\varepsilon\in(0,1)$,by cancellation of some terms of the right-hand side, convergence of $\|x_n-p\|$ and,

$W_2(\alpha_n)=\alpha_n(1-\alpha_n)\ \ge\varepsilon^2>0$ we have $\sum_{n=1}^{\infty}\|x_n-T_nx_n\|^2<\infty$;

Thus $\|x_n-T_nx_n\|\to 0$ hence for large n the following holds; (3.15)

$$\begin{aligned}\|x_{n+1}-x_n\| &= \|\alpha_nx_n+(1-\alpha_n)T_nx_n-x_n\|\\ &=(1-\alpha_n)\|x_n-T_nx_n\|\to 0\end{aligned} \tag{3.16}$$

Since $\|x_n-T_nx_n\|\to 0$, $\|x_n-x_{n+i}\|\to 0$ by induction for $0\le i\le N$ and T_{n+i} is non expansive for large n we get the following;

$$\begin{aligned}\|x_n-T_{n+i}x_n\| &\le \|x_n-x_{n+i}\|+\|x_{n+i}-T_{n+i}x_{n+i}\|+\|T_{n+i}x_{n+i}-T_{n+i}x_n\|\\ &\le 2\|x_n-x_{n+i}\|+\|x_{n+i}-T_{n+i}x_{n+i}\|\to 0\end{aligned}$$

Thus $\|x_n-T_{n+i}x_n\|\to 0$ for $0\le i\le N$. (3.17)

Suppose $n>m$, then $\|x_n-x_m\|\le\|x_n-x_{n-1}\|+\|x_{n-1}-x_{n-2}\|.....\|x_{m+1}-x_m\|$ $\to 0$, $n,m\to\infty$. (3.18)

That is $\{x_n\}$ is Cauchy sequence, since H is complete $\{x_n\}$converges in H.

Suppose $x_n\to p$ in H, since C is closed p is in C, thus there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\to p$ and let $n_k=j(\text{mod }N)$ for some $1\le j\le N$. For any $l\in\{1,2,...N\}$ there exists $1\le i\le N$ such that $n_{k+i}=l(\text{mod }N)$ hence $\|x_{n_k}-T_lx_{n_k}\|\to 0,k\to\infty$ (3.19)

In any uniformly convex Banach space, in particular, in Hilbert space, Suppose $x_n \to x$ weakly and $x_n - T_l x_n \to 0$, since T_l is demi-closed, $T_l x = x$.

Thus $p \in F(T_l)$ and since l is arbitrary we have $p \in F = \bigcap_{l=1}^{N} F(T_l)$.

Therefore, $\{x_n\}$ converges strongly to $p \in F = \bigcap_{l=1}^{N} F(T_l)$.

Since strong convergence implies weak convergence, this theorem improves theorem 3.2.

Moreover, if $\sum_{n=1} (1-\alpha_n) < \infty$ and (F,C) satisfies S-condition, then by (3.1) and (3.2) we have

$$\begin{aligned} \|x_{n+1} - x_n\| &\le \|\alpha_n x_n + (1-\alpha_n)T_n x_n - x_n\| = (1-\alpha_n)\|x_n - T_n x_n\| \\ &\le 2(1-\alpha_n)M \end{aligned} \tag{3.20}$$

Thus, we have the following;

$$\begin{aligned} \sum \|x_{n+1} - x_n\| &\le \sum \|\alpha_n x_n + (1-\alpha_n)T_n x_n - x_n\| = (1-\alpha_n)\|x_n - T_n x_n\| \\ &\le \sum 2(1-\alpha_n)M < \infty \end{aligned} \tag{3.21}$$

$\|x_{n+1} - x_n\| \to 0$, as $n \to \infty$, thus $\{x_n\}$ is Cauchy sequence.

$\{x_n\}$ converges in norm to some $x \in C$.

It suffices to show that $x \in F$.

For each n, T_n is inward implies that $h_n(x) < 1$, thus for $\beta_n \in [h_n(x), 1)$ we have $\beta_n x + (1-\beta_n)T_n x \in C$.

Since $\lim_{n\to\infty} \alpha_n = 1$ and $\alpha_{n+1} = \max\{\alpha_n, h_{n+1}(x_{n+1)}\}$, there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\lim_{j\to\infty} h_{n_j}(x_{n_j}) = 1$.

Since $\frac{j}{j+1} h_{n_j}(x_{n_j}) < h_{n_j}(x_{n_j})$, $\frac{j}{j+1} h_{n_j}(x_{n_j}) x_{n_j} + \left(1 - \frac{j}{j+1} h_{n_j}(x_{n_j})\right)$ $T_{n_j} x_{n_j} \notin C$ and $\lim_{j\to\infty}(\frac{j}{j+1} h_{n_j}(x_{n_j}) x_{n_j} + (1 - \frac{j}{j+1} h_{n_j}(x_{n_j}))T_{n_j} x) = x$, thus by lemma 3.1 $x \in \partial C$

Since $F \subset C$, the sequence $\{x_n\}$ in C is fejer monotone with respect to F and (F,C) satisfies S-condition, $x \in F$.

Thus, $x_n \to x \in F$ strongly, which completes the proof

Therefore $\{x_n\}$ converges strongly to some element $p \in F = \bigcap_{l=1}^{N} F(T_l)$

Applications: as we mentioned earlier in our introduction part, finding a fixed point is equivalent to finding the zero of some nonlinear operator, hence we should have the following theorem;

Theorem 3.4: Let $A_1, A_1, \ldots A_N : C \to H$ be a family of nonself mappings such that for each $i = 1,2,\ldots,N$, $I - A_i$ is nonexpansive and inward mapping on a non empty, closed and strictly convex subset C of a Hilbert space H with $N = \bigcap_{k=1}^{N} N(A_k)$ is non empty, where $N(A) = \{x \in C : Ax = 0\}$.Let $A_k = A_{k(ModN)}$, $h_k(x) = \inf\{\lambda \geq 0 : \lambda x + (1-\lambda)(I - A_k)x \in C\}$. $\alpha \in (0,1)$. Then the sequence $\{x_n\}$ defined by;

$$\begin{cases} x_1 \in C \\ \alpha_1 = \max\{\alpha, h_1(x_1)\} \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)(I - A_n)x_n \\ \alpha_{n+1} = \max\{\alpha_n, h_{n+1}(x_{n+1})\} \end{cases}$$

is well-defined and if $\{\alpha_n\} \subset [\varepsilon, 1-\varepsilon] \subset (0,1)$ for some $\varepsilon \in (0,1)$ or $\sum_{n=1} (1-\alpha_n) < \infty$ and (F,C) satisfies S-condition, then the sequence $\{x_n\}$ converges strongly to some element p of $N = \bigcap_{k=1}^{N} N(A_k)$.

The proof can be done in similar way as the proof of the above theorem.

For example if we consider the two nonself mappings on the domain $C = \left[-\frac{1}{2}, 1\right]$ as follows; let $A_1, A_2 : C \to \Re$ be defined by

$$A_1 x = \begin{cases} 0 \ , 0 \leq x \leq 1 \\ 2x, -\frac{1}{2} \leq x < 0 \end{cases}$$

and $$A_2 x = 2x, -\frac{1}{2} \le x \le 1.$$

Thus the mappings satisfy the conditions of the theorem with non empty set of common zeros. Hence the algorithm given by the theorem converges to the number 0, which is a common zero of the mappings.

Since there are quasi nonexpansive mappings, which are not nonexpansive for example, let $T : \Re \to \Re$ be defined by

$$T(x) = \begin{cases} \dfrac{x \sin\left(\dfrac{1}{x}\right)}{2}, x \ne 0. \\ 0, \; x = 0 \end{cases} \tag{3.22}$$

Then T is quasi nonexpansive mapping, which is not nonexpansive. Thus the class of quasi-nonexpanisive mappings is more general than the class of nonexpansive mappings.

We can also extend our theorem for the family of quasi -non expansive mappings with additional assumptions;

Definition 3.1: Let $T_1, T_2, \ldots T_N : C \to H$ be finite family of mappings. Let $T_k = T_k (\bmod N)$. Then if for a sequence $\{x_n\} \subseteq C$ such that for any $0 \le r \le N$, $\lim_{n\to\infty} \|x_n - x_{n+r}\| = 0$, then $\lim_{n\to\infty} \|T_{n+i}(x_n - x_{n+r})\| = 0$, then we say the family $\{T_1, T_2, \ldots, T_N\}$ satisfy weakly closed (W,C) condition.

Definition 3.2: *A* mapping $T : C \to H$ is said to be fixed point closed, if for $\{x_n\}$ in C, $x_n \to x$ weakly and $x_n - Tx_n \to 0$ strongly, then $x \in F(T)$.

Theorem 3.5: Let $T_1, T_2, \ldots T_N : C \to H$ be family of, nonself, quasi-nonexpansive, fixed point closed and inward mappings, satisfying (W, C) condition on a non-empty closed strictly convex subset K of a real Hilbert space H, with $F = \bigcap_{k=1}^{N} F(T_k)$ is non empty. Let $T_k = T_{k(ModN)}$, let $x_0 \in C$ and we define $h_k(x) = \inf\{\lambda \ge 0 : \lambda x + (1-\lambda) T_k x \in C\}$, let $\alpha \in (0,1)$ be fixed. Then the sequence $\{x_n\}$ given by

$$\begin{cases} x_1 \in K \\ \alpha_1 = \max\{\alpha, h_1(x_1)\} \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)T_n x_n \\ \alpha_{n+1} = \max\{\alpha_n, h_n(x_{n+1})\} \end{cases}$$

is well-defined and if $\{\alpha_n\} \subset [\varepsilon, 1-\varepsilon] \subset (0,1)$ for some $\varepsilon \in (0,1)$ or $\sum_{n=1} (1-\alpha_n) < \infty$ and (F,C) satisfies S- condition, then the sequence $\{x_n\}$ converges strongly to some element p of $F = \bigcap_{k=1}^{N} F(T_k)$.

Proof: Let $p \in F$, then we have the following in equality.

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n x_n + (1-\alpha_n)T_n x_n - p\| \\ &\le \alpha_n \|x_n - p\| + (1-\alpha_n)\|T_n x_n - p\| \qquad (3.23) \\ &\le \alpha_n \|x_n - p\| + (1-\alpha_n)\|x_n - p\| = \|x_n - p\|. \end{aligned}$$

Thus $\{x_n\}$ is fejer monotone with respect to F.

Since $\|x_n - p\|$ is decreasing and bounded below, thus it converges. Thus $\{x_n\}$ and hence $\{T_n x_n\}$ are bounded. In a uniformly convex Banach space, Suppose $x_n \to x$ weakly and $x_n - T_l x_n \to 0$ strongly, since T_l is Fixed point closed, x is a fixed point of T_l.

By lemma 2.1 of [32] in Hilbert space, for $\alpha \in [0,1]$ $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2$ for all $x, y \in H$ holds. (3.24)

Thus

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\alpha_n(x_n - p) + (1-\alpha_n)(T_n x_n - p)\|^2 \\ &\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|T_n x_n - p\|^2 \qquad (3.25) \\ &\quad - \alpha_n(1-\alpha_n)\|x_n - T_n x_n\|^2. \end{aligned}$$

Which implies that

$$\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n)\|x_n - T_n x_n\|^2 \le \sum_{n=1}^{\infty} (\|x_n - p\|^2 - \|x_{n+1} - p\|^2) < \infty.$$

By cancellation of some terms of the right-hand side, convergence of $\|x_n - p\|$, $0 < \alpha_n < 1$ and $W_2(\alpha_n) = \alpha_n(1-\alpha_n) \geq \varepsilon^2 > 0$, we have $\sum_{n=1}^{\infty} \|x_n - T_n x_n\|^2 < \infty$, hence $\|x_n - T_n x_n\| \to 0$ for large n and hence the following holds;

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|\alpha_n x_n + (1-\alpha_n)T_n x_n - x_n\| \\ &= (1-\alpha_n)\|x_n - T_n x_n\| \to 0 \end{aligned} \tag{3.26}$$

Since $\|x_n - T_n x_n\| \to 0$, $\|x_n - x_{n+i}\| \to 0$ by induction for $0 \leq i \leq N$ and the family of mappings satisfies (W,C) condition for large n we have the following;

$$\begin{aligned} \|x_n - T_{n+i}x_n\| &\leq \|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i}x_{n+i}\| + \|T_{n+i}x_{n+i} - T_{n+i}x_n\| \\ &\leq \|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i}x_{n+i}\| + \|T_{n+i}(x_{n+i} - x_n)\| \to 0 \end{aligned} \tag{3.27}$$

$$\begin{aligned} \|x_n - T_{n+i}x_n\| &\leq \|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i}x_{n+i}\| + \|T_{n+i}x_{n+i} - T_{n+i}x_n\| \\ &\leq \|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i}x_{n+i}\| \to 0 \end{aligned} \tag{3.28}$$

The remaining part of the proof is similar to the proof of Theorem 3.3.

Corollary 3.6: Suppose $T : C \to H$ is non self, nonexpansive and inward mapping on a non-empty closed convex subset C of a Hilbert space H. with $F = F(T)$ is non empty.

Let $x_0 \in C$, $h(x) = \inf\{\lambda \geq 0 : \lambda x + (1-\lambda)Tx \in C\}$ and $\alpha \in (0,1)$ be fixed. Then the sequence $\{x_n\}$ given by;

$$\begin{cases} x_1 \in C \\ \alpha_1 = \max\{\alpha, h(x_1)\} \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n \\ \alpha_{n+1} = \max\{\alpha_n, h(x_{n+1})\} \end{cases}$$

is well-defined and converges weakly to $F = F(T)$.

The corollary agrees with Krasnoselskii-Mann type algorithm for $\lambda-$strictly Pseudo contractive mappings in a 2-uniformly smooth Banach space.

3. CONCLUSION

Suppose $T_1, T_2, ...T_N : C \to H$ be family of nonself, (nonexpansive, quasi-nonexpansive with fixed point closed and satisfies (W,C) condition) respectively and inward mappings on a non-empty, closed and strictly convex subset C of a real Hilbert space H, with $F = \bigcap_{k=1}^{N} F(T_k)$ is non empty. Let $T_k = T_{k(ModN)}$, $x_0 \in C$, for each $k \in \{1,2,...,N\}$ let $h_k : C \to \Re$ be defined by $h_k(x) = \inf\{\lambda \geq 0 : \lambda x + (1-\lambda)T_k x \in C\}$.

Then the sequence $\{x_n\}$ given by

$$\begin{cases} x_1 \in C \\ \alpha_1 = \max\{\alpha, h_1(x_1)\} \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)T_n x_n \\ \alpha_{n+1} = \max\{\alpha_n, h_{n+1}(x_{n+1})\} \end{cases}$$

is well-defined and if $\{\alpha_n\} \subset [\varepsilon, 1-\varepsilon] \subset (0,1)$ for some $\varepsilon \in (0,1)$ or $\sum_{n=1}^{\infty}(1-\alpha_n) < \infty$ and (F,C) satisfies S-condition, $\{x_n\}$ converges strongly to some element p of $F = \bigcap_{k=1}^{N} F(T_k)$.

As a result, our theorems generalize many results for approximating a common fixed point of finite family of nonself mappings without the computation for projection for sunny nonexpansive retraction.

Remark: If each $T_1, T_2, ...T_N : C \to H$ are self-mappings, then, for all $x, h_i(x) = 0, \forall i$ .Thus $\alpha_n = \alpha$, which is a constant, thus the above iterative method reduces to Mann's iterative method.

Meanwhile, we raise open questions;

Question 1: Is it possible to extend our theorems in more general spaces such as? Uniformly convex Banach space, uniformly smooth Banach space, reflexive Banach space and in Banach space? If so under what conditions?

Question 2: Can we extend the above results to a more general Banach space with condition (H) and other nailed restrictions?

Question 3: What can we conclude about the convergence of the sequence generated in Theorem 3.1, if $\sum_{n=1}^{\infty}(1-\alpha_n)=\infty$ and $\lim_{n\to\infty}\alpha_n=1$?

Question 4: In the above results, can we remove the requirement for strictly convexity of C?

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References

[1] BAUSCHKE, HH, COMBETTES, PL. A weak-to-strong convergence principle for Fejér-monotone methods in Hilbert spaces. *Math. Oper. Res.* **26**(2), 248-264 (2001).

[2] BAUSCHKE.HH. The approximation of fixed points of compositions of nonexpansive mappings in Hilbert spaces. *J. Math. Anal. Appl.* 202,150-159 (1996).

[3] BERINDE.V. Iterative Approximation of Fixed Points. *Springer.* 1912(2007).

[4] BROWDER.F.E. Convergence theorems for sequences of nonlinear operators in Banach spaces. *Math. Zeitschr.*100, 201-225 (1967).

[5] BROWDER.F.E. Semi contractive and semiaccretive nonlinear mappings in Banach spaces. *Bull. Amer. Math. Soc.*74, 660–665(1968).

[6] CENG.LC, CUBIOTTI.P, YAO.JC. Strong convergence theorems for finitely many nonexpansive mappings and applications. *Nonlinear Anal.* 67, 1464-1473 (2007).

[7] CHIDUME.C.E, ZEGEYE.H AND SHAHZAD.N. Convergence theorems for a Common fixed point of a finite family of nonself nonexpansive mappings. *Fixed Point Theory and Applications*. 2 (2005) 233–241(2005).

[8] CIESIELSKI, K. On Stefan Banach and some of his results. *Banach J. Math. Anal.* 1(1), 1-10(2007).

[9] COLAO.V, MARINO.G.Krasnoselskii-Mann method for nonself mappings. *Fixed Point Theory and Appl.* 39(2015).

[10] COLAO.V, MARINO.G,XU, HK. An iterative method for finding common solutions of equilibrium and fixed point problems. *J. Math. Anal. Appl.* 344, 340-352 (2008).

[11] DAMAN.OA, ZEGEYE.H. Strong convergence theorems for a common fixed point of a finite family of pseudo-contractive mappings. *Int. J. Math. Math. Sci.* 2012, Article ID 405315 dol.10.1155/2012/405315(2012).

[12] FERREIR.O.P, OLIVEIRA.P.R.Proximal point algorithm on Riemannianmanifolds. *Optimization.* 51, no.2, 257-270(2002).

[13] HALPERN.B. Fixed points of nonexpansive maps, *Bull. Am. Math. Soc.* 73, 957-961(1967).

[14] HUKMI. K, MURAT .O AND SEZGIN.A. On Common Fixed Points of two non-self nonexpansive mappings in Banach Spaces. *Chiang Mai J. Sci.*34(3), 281-288(2007).

[15] ISHIKAWA.S. Fixed points by a new iteration method. Proc. Amer. Math. Soc.44(1), 147-150 (1974).

[16] KHAN.AR.Ishikawa Iteration Process for Non-self and non-expansive Maps in Banach Spaces. *International Journal of Applied Mathematics.* 38 (3), 03(2008).

[17] KRASNOSELSKII, M.A.Two remarks on the method of successive approximations. *(Russian) Uspehi Mat. Nauk.* 10, 123–127 (1955).

[18] KIZILTUNC.H, YILDIRIM.I. On Common Fixed Point of nonself, nonexpansive mappings for Multistep Iteration in Banach Spaces. *Thai Journal of Mathematics.* 6 (2), 343-349(2008).

[19] MANN, W.R. Mean value method in iteration. *Proc. Am. Math. Soc. 4*, 506–510 (1953).

[20] MARINO.G, XU.H.K. Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces. *J. Math. Anal. Appl.* **329**(1), 336-346 (2007).

[21] OSILIKE.M.O, SHEHU.Y. Cyclic algorithm for common fixed points of finite family of strictlypseudo contractivemappings of Browder-Petryshyn type. *Nonlinear Anal.* 70, 3575-3583 (2009).

[22] PLUBTIENG.S, PUNPAENG.R. A general iterative method for equilibrium problems and fixed point problems in Hilbertspaces. *J.Math. Anal. Appl.* 336, 455-469 (2007).

[23] REICH.S. Weak convergence theorems for nonexpansive mappings in Banach spaces. *J Math Anal Appl.* 67(2):274–276(1979).

[24] SUZUKI,T.Strong convergence of Krasnoselskii and Mann type sequences for one-Parameter nonexpansive semigroups without Bochner integrals. *J. Math. Anal. Appl.* ***305***(1), 227-239 (2005).

[25] TAKAHASHI.S, TAKAHASHI.W. Viscosity approximation methods for equilibrium problems and fixed point problems inHilbert spaces. *J. Math. Anal. Appl.* 331, 506-515 (2007).

[26] YAO, Y. A general iterative method for a finite family of nonexpansive mappings. *Nonlinear Anal.* 66, 2676-2687 (2007).

[27] YAO.Y, ZHOU.H, LIOU.Y.C.Strong convergence of a modified Krasnoselskii–Mann iterative algorithm for nonexpansive mappings. *J. Appl. Math. Comput.* 29, 383–389 (2009).

[28] ZEGEYE.H, SHAHZAD.N. An algorithm for a common fixed point of a family of pseudo contractive mappings. *Fixed Point Theory and Application.* 234(1), 234(2013).

[29] ZEGEYE. H, SHAHZAD.N.Strong convergence theorems for a common zero of a countable infinite family of λ-inversestrongly accretive mappings. *Nonlinear Anal.* 71, 531-538 (2009).

[30] ZEGEYE.H, SHAHZAD.N. Strong convergence theorems for a common zero of a finite family of *m*-accretive mappings. *Nonlinear Anal.* 66, 1161-1169 (2007).

[31] ZEGEYE.H,SHAHZAD.N,ALGHAMDI.B, MA. Convergence of Ishikawa's iteration method for pseudo contractive mappings. *Nonlinear Anal.* 74, 7304-7311 (2011).

[32] ZEGEYE. H, SHAHZAD.N. Convergence of Mann's type iteration method for generalized asymptotically nonexpansive mappings. *Comput. Math. Appl*. 62, 4007-4014 (2011).

[33] ZHANG.Y, GUO.Y. Weak convergence theorems of three iterative methods for strictly pseudo contractive mappings of Browder-Petryshyn type. *Fixed Point Theory Appl.* 2008, Article ID 672301(2008).

[34] ZHOU.H. Convergence theorems of fixed points for a finite family of Lipschitz pseudo-contractions in Banach spaces. *Nonlinear Anal.* 68, 2977-2983 (2008).