

Measures of Production Price-Labour Value Deviation and Income Distribution in Actual Economies: Theory and Empirical Evidence

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Many empirical studies indicate that the deviations of actual prices of production from labour values are not too sensitive to the type of measure used for their evaluation. This paper attempts to theorize this rather 'stylized fact' by focusing on the relationships between the traditional and the *numéraire*-free measures of deviation. On the empirical side, it provides an illustration of these relationships using input-output data from the Greek and Japanese economies.

INTRODUCTION

Many empirical studies indicate that the deviations of *actual* prices of production from labour values are not too sensitive to the type of measure used for their evaluation.¹ For example, a recent study on the input-output table of the Chinese economy for the year 1997 (Mariolis and Tsoulfidis, 2009, p. 12), in which the vector of production prices is normalized with the use of Sraffa's (1960, ch. 4) standard commodity, indicates that the absolute error between the actual '*d*-distance' (Steedman and Tomkins, 1998; see also Mariolis and Soklis, 2011) and 'mean absolute deviation' ('mean absolute weighted deviation') is 0.2% (0.5%) and that the relevant relative error is 1.75% (4.39%).²

This paper attempts to theorize this rather 'stylized fact' by focusing on the relationships between the 'traditional' and the *numéraire*-free measures of deviation, where the former include the 'mean absolute deviation' (or MAD), the 'root-mean-square-percent-error' (or RMS%E) and the 'mean absolute weighted deviation' (or MAWD), whilst the latter include the '*d*-distance' and its variants.³ More specifically, the

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main argument is that, for *realistic* values of the ‘relative rate of profit’ (i.e., the ratio of the uniform rate of profit to the maximum rate of profit), a parameter reflecting the *socio-technical* conditions of production, all these measures of deviation tend to be close to each other and, at the same time, follow certain rankings, which we can explore starting from a two-sector economy.

The remainder of the paper is structured as follows: Section 2 deals with the measures of deviation in the case of a two-sector economy. Section 3 generalizes to the n -sector case. Section 4 provides an empirical illustration using input-output data from the Greek and Japanese economies. Section 5 concludes.

THEORETICAL ANALYSIS OF A TWO-SECTOR ECONOMY

Let us suppose a usual linear system of production with two sectors, where prices are normalized by setting $\mathbf{p}^T \mathbf{x} = \mathbf{v}^T \mathbf{x}$ or

$$p_1 x_1 + p_2 x_2 = v_1 x_1 + v_2 x_2 \quad (1)$$

where $\mathbf{p} \equiv [p_i]$, $\mathbf{v} \equiv [v_i]$, are the vectors of prices of production and labour values, respectively, and the semi-positive vector $\mathbf{x} \equiv [x_i]$ represents the standard of value or *numéraire*. Relation (1) can be rewritten as

$$p_2 - v_2 = (v_1 - p_1)x \quad (2)$$

where $x \equiv x_1 / x_2$.

Now, let d_1 show the MAD. Substituting relation (2) in the definition of the MAD, i.e.,

$$d_1 \equiv (1/n) \sum_{j=1}^n |(p_j / v_j) - 1| \quad (3)$$

where n is the number of commodities, we get

$$2d_1 = (|p_1 - v_1| / v_1) + (|v_1 - p_1| / v_2)x \quad (4)$$

In order to simplify our notation we set

$$f \equiv p_1 / p_2 \quad (5)$$

where f is a monotonic function of the rate of profit, and $f = v \equiv v_1 / v_2$ at $r = 0$.⁴ From (2) and (5) we obtain

$$p_1 = (v_1x + v_2)f(1 + fx)^{-1} \quad (6)$$

For the sake of brevity and clarity of presentation, we focus on the case in which f is a strictly increasing function, *i.e.*, $p_1 \geq v_1$. By combining relations (4) and (6) we get

$$2d_1 = (\delta - 1)F_1(x) \quad (7)$$

where $\delta \equiv fv^{-1} (> 1 \text{ for } r > 0)$ represents the ratio of relative prices to relative labour values and $F_1(x) \equiv (1 + vx)(1 + fx)^{-1}$ is a strictly decreasing function reflecting the dependence of d_1 on x . For $x = 0$, we obtain $2d_1(0) = \delta - 1$, whereas at the other extreme, *i.e.*, as $x \rightarrow \infty$, we obtain $2d_1(\infty) = 1 - \delta^{-1}$. Thus, we may write $d_1(0)/d_1(\infty) = \delta$,

$$\Delta_1 \equiv d_1(0) - d_1(\infty) = [(\delta + \delta^{-1})/2] - 1 = 2d_1(0)d_1(\infty) \quad (8)$$

and using the Taylor expansion about $\delta = 1$,

$$\Delta_1 \approx (1/2)[(\delta - 1)^2 - (\delta - 1)^3] = 2(d_1(0))^2(2 - \delta) \quad (8a)$$

where this approximation is most reliable when $\delta < 1.18$.⁵

The next measure of deviation is the RMS%E, d_{II} which is defined as

$$d_{II} \equiv \sqrt{(1/n) \sum_{j=1}^n [(p_j/v_j) - 1]^2} \quad (9)$$

or

$$d_{II} = d_1(\cos \phi)^{-1} \quad (9a)$$

where ϕ represents the angle between the vectors $|\mathbf{p}^T \hat{\mathbf{v}}^{-1} - \mathbf{e}^T|$ and the summation vector $\mathbf{e} \equiv [1, 1, \dots, 1]^T$ ($\hat{\mathbf{v}}$ represents the diagonal matrix formed from the elements of \mathbf{v}). Thus, it holds true that $d_1 \leq d_{II}$. Substituting (2) and (6) in the definition of d_{II} gives

$$d_{II} = (1/2)(\delta - 1)F_{II}(x) = d_1(0)F_{II}(x) \quad (10)$$

where $F_{II}(x) \equiv \sqrt{2[1+(vx)^2]}(1+fx)^{-1}$. From (7), (9a) and (10) we obtain

$$\cos \phi = (1/\sqrt{2})(1+vx)[1+(vx)^2]^{-1/2}$$

which implies

$$\phi(X) = \phi(X^{-1}) \quad (11)$$

where $X \equiv vx$. From the above it follows that:

- (i) At $x^* \equiv v^{-1}$ the absolute percentage deviations of prices from labour values are equal to each other and, therefore, it holds

$$\bar{d} \equiv d_I(x^*) = d_{II}(x^*) = (p_1 - v_1)/v_1 = 2d_1(0)(1+\delta)^{-1}$$

i.e., $\cos \phi = 1$.

- (ii) $d_{II}(x)$ also equals \bar{d} at $x = v^{-1}(\delta^2 + 2\delta - 1)(-\delta^2 + 2\delta + 1)^{-1}$.

- (iii) $F_{II}(x)$ is minimized at $x^{**} \equiv \delta v^{-1}$, where

$$(d_{II})_{\min} \equiv d_{II}(x^{**}) = d_1(0)\sqrt{2(1+\delta^2)^{-1}} = \sqrt{1 - \cos^2 \phi(x^{**})} = \sin \phi(x^{**})$$

or

$$(d_{II})_{\min} = \sqrt{\bar{d}d_I(x^{**})} \quad (12)$$

i.e., the minimum value of $d_{II}(x)$ (a strictly increasing function of δ) equals $\sin \phi(x^{**})$, and constitutes the geometric mean of

$$\bar{d} = \sin \phi(x^{**}) / \cos \phi(x^{**}) = \tan \phi(x^{**})$$

and

$$d_1(x^{**}) = d_1(0)(1+\delta)(1+\delta^2)^{-1}$$

Furthermore, using the Taylor expansion about $\delta = 1$ we get

$$\bar{d} \approx d_1(0)(1-d_1(0)) \quad (12a)$$

and

$$(d_{II})_{\min} \approx d_1(0)(1-d_1(0)) \quad (12b)$$

where these approximations are most reliable when $\delta < 1.35$ and $\delta < 1.42$, respectively.

(iv) Since $d_{II}(\bullet) = \sqrt{2}d_1(\bullet)$, where $\bullet = 0, \infty$ (see (8)), then

$$\Delta_{II} \equiv d_{II}(0) - d_{II}(\infty) = \sqrt{2}\Delta_I \quad (13)$$

Finally, by substituting (2) and (6) in the definition of the MAWD, d_{III} i.e.,

$$d_{III} \equiv \sum_{j=1}^n |(p_j / v_j) - 1| (x_j / \mathbf{e}^T \mathbf{x}) \quad (14)$$

we get

$$d_{III} = d_1(0)F_{III}(x) \quad (15)$$

where $F_{III}(x) \equiv 2(1+v)x[(1+x)(1+fx)]^{-1}$. From the above it follows that:

(i) $F_{III}(0) = F_{III}(\infty) = 0$ and $0 < F_{III}(x) < 2$ for $0 < x < \infty$.

(ii) $F_{III}(x)$ is maximized at $x^{***} \equiv 1/\sqrt{\delta v}$, where

$$2(1+\delta)^{-1} \leq F_{III}(x^{***}) = 2(1+v)(1+\sqrt{\delta v})^{-2} < 2$$

and $F_{III}(x^{***})$ tends to 2 (to $2\delta^{-1}$) as v tends to 0 (to ∞). Moreover,

$F_{III}(x^{***})$ equals $2(1+\delta)^{-1}$ iff $v = \delta$. In that case $d_{III}(x^{***}) = \bar{d}$.

(iii) $d_{III}(x^{***})$ is a strictly increasing function of δ that tends to $2d_1(0)$

(to $2d_1(\infty)$) as v tends to 0 (to ∞).

(iv) $d_1(x) < d_{III}(x)$ when x lies between 1 and x^* ($\equiv v^{-1}$), whilst d_{III}

also equals $d_1(0)(1+v)(1+\delta v)^{-1}$ ($= d_{III}(1)$) at $x = (\delta v)^{-1} = (x^{***})^2$

(see Figure 1a, where $v = 2$ and $\delta = 1.3$, and Figure 1b, where

$v = \delta = 1.3$, which represent the said measures of deviation as functions of x).

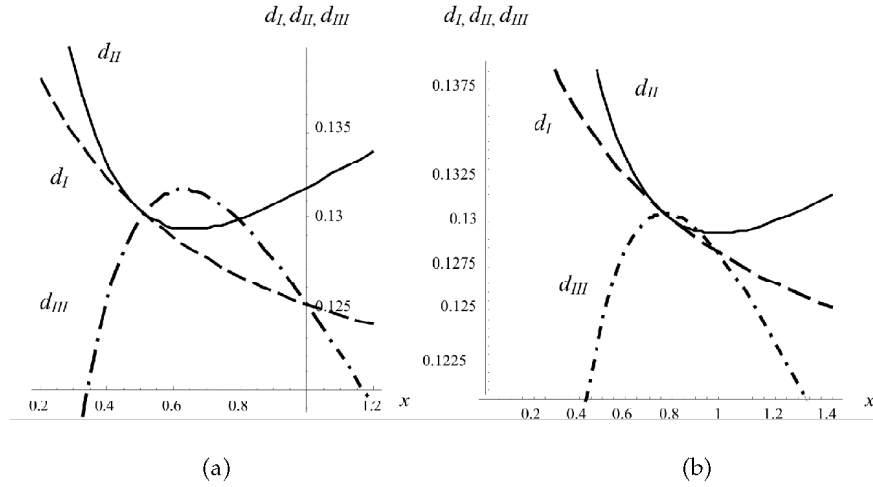


Figure 1: The traditional measures of deviation as functions of the composition of the numéraire; $v = 2$ and $\delta = 1.3$ (a), and $v = \delta = 1.3$ (b)

On the other hand, the *numéraire*-free measure ‘ d - distance’ is defined as $d \equiv \sqrt{2(1 - \cos \theta)}$, where θ is the angle between the vectors $\mathbf{p}^T \hat{\mathbf{v}}^{-1}$ and \mathbf{e} , and d is the Euclidean distance between the unit vectors $(\mathbf{p}^T \hat{\mathbf{v}}^{-1}) / \|\mathbf{p}^T \hat{\mathbf{v}}^{-1}\|$ and $\mathbf{e} / \|\mathbf{e}\|$, where $\|\bullet\|$ represents the Euclidean norm of a vector (Steedman and Tomkins, 1998, pp. 381-382). Given that $\cos \theta$ can be expressed in terms of δ , i.e., $\cos \theta = (1/\sqrt{2})G(\delta)$, where $G(\delta) \equiv (1 + \delta) / \sqrt{1 + \delta^2}$ is maximized at $\delta = 1$ ($\cos \theta = 1$), and $G(\delta) = G(\delta^{-1})$, it follows that

$$d^2 = 2 - \sqrt{2}G(\delta) \tag{16}$$

or, recalling (12),

$$d^2 = 2D \tag{16a}$$

where $D \equiv 1 - [(d_{II})_{\min} / \bar{d}]$ and, recalling (9a) and (11), $\theta = \phi(x^{**}) = \phi((\delta v)^{-1})$. Thus, for $\delta > 1$, we may write $(d_{II})_{\min} < d < \bar{d}$ or, approximately, $d \approx (d_{II})_{\min}$ and $d \approx \bar{d}$, where

these approximations are most reliable when $\delta < 3.3$ ($\theta^\circ < 28.1$) and $\delta < 1.8$ ($\theta^\circ < 15.9$), respectively. Finally, using the Taylor expansion of (16) about $\delta = 1$ we get $d^2 \approx (d_I(0))^2(2 - \delta)$ or, recalling (8) and (8a),

$$d^2 \approx \Delta_I / 2 = d_I(0)d_I(\infty) \tag{16b}$$

where these approximations are most reliable when $\delta < 1.22$ ($\theta^\circ < 5.7$) and $\delta < 1.30$ ($\theta^\circ < 7.4$), respectively.

From this analysis it follows that (i) $d_I(\infty) < d < d_I(0)$ for $\delta > 1$ (see (7), (16) and Figure 2); (ii) $d < d_{II}(\infty)$ for $1 < \delta < \delta \square 3.732$, and $d_I(0) < d_{II}(\infty)$ for $1 < \delta < \sqrt{2}$; (iii) the absolute errors between d and the *bounds* for the traditional measures, *i.e.*, $\{d_I(\bullet), d_{II}(0), (d_{II})_{\min}, (d_{III})_{\max}\}$, increase with δ ; (iv) the relative errors between d and $\{d_I(\bullet), d_{II}(0), (d_{II})_{\min}\}$ increase with δ (for example, at $\delta = 1.1$ the relative error between d and $d_I(\bullet)$ lies between 4.5% (*i.e.*, $1 - (d_I(\infty)/d) \approx 1 - (1/\sqrt{\delta})$; see (16b)) and 5.1% ($\approx \sqrt{\delta} - 1$),

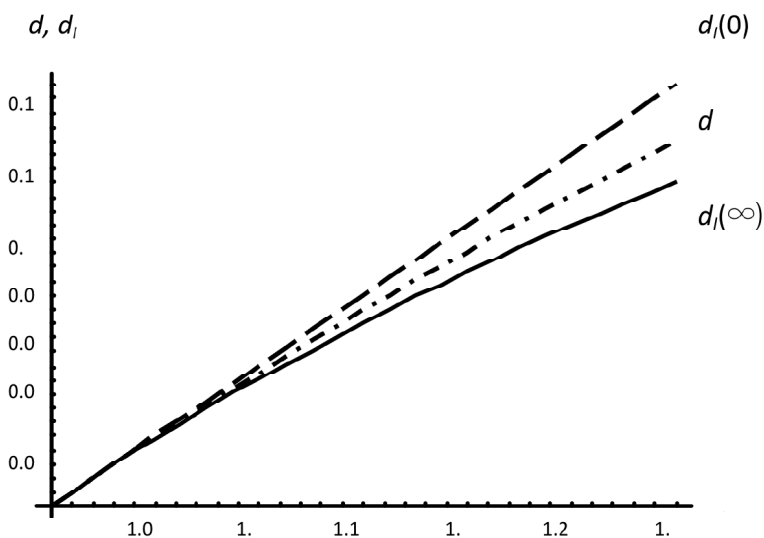


Figure 2: The bounds for MAD and the 'd-distance' as functions of the ratio of relative prices to relative labour values

whilst at $\delta = 2$ it lies between 22.0% and 56.1%; see Table 1);⁶ and (v) the monotonicity of the relative error between d and $(d_{III})_{\max}$ depends on the value of ν (see, e.g., Figure 3, where $\nu = 1$ (monotonic curve) or $\nu = 5$).

Table 1
Measures of Deviation and the Ratio of Relative Prices to Relative Labour Values

δ	1.05	1.10	1.15	1.20	1.25	1.30	1.40	1.50	2.0	3.0
$d_I(0)$	0.025	0.050	0.075	0.100	0.125	0.150	0.200	0.250	0.500	1.000
$d_I(\infty)$	0.024	0.045	0.065	0.084	0.100	0.115	0.143	0.167	0.250	0.333
Δ_I	0.001	0.005	0.010	0.016	0.025	0.035	0.057	0.083	0.250	0.667
$d_{II}(0)$	0.035	0.071	0.106	0.141	0.177	0.212	0.283	0.354	0.707	1.414
$d_{II}(\infty)$	0.034	0.064	0.092	0.119	0.141	0.163	0.202	0.236	0.354	0.471
\bar{d}	0.0244	0.0476	0.0698	0.0909	0.1111	0.1304	0.1667	0.2000	0.3333	0.5000
$(d_{II})_{\min}$	0.0244	0.0476	0.0696	0.0905	0.1104	0.1293	0.1644	0.1961	0.3162	0.4472
d	0.0244	0.0476	0.0696	0.0906	0.1106	0.1296	0.1650	0.1971	0.3204	0.4595
$d) - 1/$	2.46	5.09	7.76	10.38	13.02	15.74	21.21	26.84	56.05	117.63
$d) - 1\%$										
$1 - (d_I(\infty)/d)$ (%)	1.64	4.46	6.61	7.28	9.58	11.27	13.33	15.27	21.97	27.53

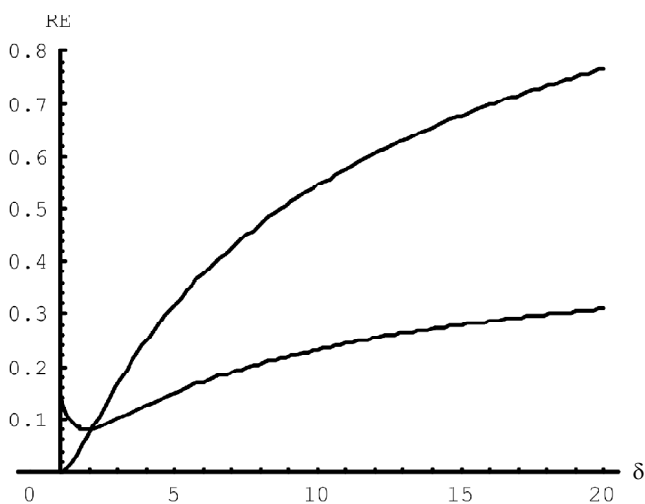


Figure 3: The relative error between the 'd- distance' and the upper bound for MAWD as a function of the ratio of relative prices to relative labour values

Now we shall approach δ as a function of the production technique and the profit rate, *i.e.*, the *socio-technical* conditions of production. Let $\mathbf{A} \equiv [a_{ij}]$ be the irreducible matrix of input-output coefficients, and let $\mathbf{l} \equiv [l_i]$ be the vector of direct labour coefficients. Then, in the case of our economy we may write:

$$f = \{l[1 - a_{22}(1+r)] + a_{21}(1+r)\} / \{la_{12}(1+r) + 1 - a_{11}(1+r)\} \quad (17)$$

where $l \equiv l_1/l_2$. From the definition of δ and (17) it follows that δ is a strictly decreasing function of l (for $r > 0$), and

$$\delta(0) \equiv \lim_{l \rightarrow 0} \delta = (1 + \rho R) / [1 - \rho(R/R_1)] (\geq 1) \quad (18)$$

$$\delta(l^*) \equiv \lim_{l \rightarrow l^*} \delta = 1 \quad (18a)$$

$$\delta(\infty) \equiv \lim_{l \rightarrow \infty} \delta = [1 - \rho(R/R_2)] / (1 + \rho R) (\leq 1) \quad (18b)$$

where l^* denotes the proportion given by the left-hand side Perron-Frobenius eigenvector of \mathbf{A} , $R_i \equiv a_{ii}^{-1} - 1$, $R \equiv \lambda^{-1} - 1$ the maximum rate of profit, λ the Perron-Frobenius eigenvalue of \mathbf{A} , which increases with the elements a_{ij} (therefore $R < R_i$), and $\rho \equiv r/R$ the relative rate of profit, which is less than or equal to the share of profits in net income in Sraffa's standard system (see also Figure 4, where $0 < \rho_1 < \rho_2$).⁷ As $(R/R_1) \rightarrow 0$ we get $\delta(0) \rightarrow 1 + \rho R (= 1/\delta(\infty))$ as $(R/R_2) \rightarrow 0$, whilst as $(R/R_1) \rightarrow 1$ we get $\delta(0) \rightarrow (1 + \rho R)/(1 - \rho) (= 1/\delta(\infty))$ as $(R/R_2) \rightarrow 1$). Consequently, when f increases with r , the values

$$\delta^- \equiv 1 + \rho R \quad (19)$$

$$\delta^+ \equiv (1 + \rho R)/(1 - \rho) = \delta^- (1 + \rho + \rho^2 + \dots) \quad (19a)$$

represent the *theoretically* possible lower and upper bounds for δ , respectively, (whilst when f decreases with r , the values $1/\delta^-$ and $1/\delta^+$ represent the theoretically possible lower and upper bounds for δ , respectively). Thus, we may conclude that $|\delta - 1|$ (and, therefore, the errors between d and, for example, $d_1(\bullet)$; see also Figure 1) is directly related to the deviation of l from l^* , and ρ .

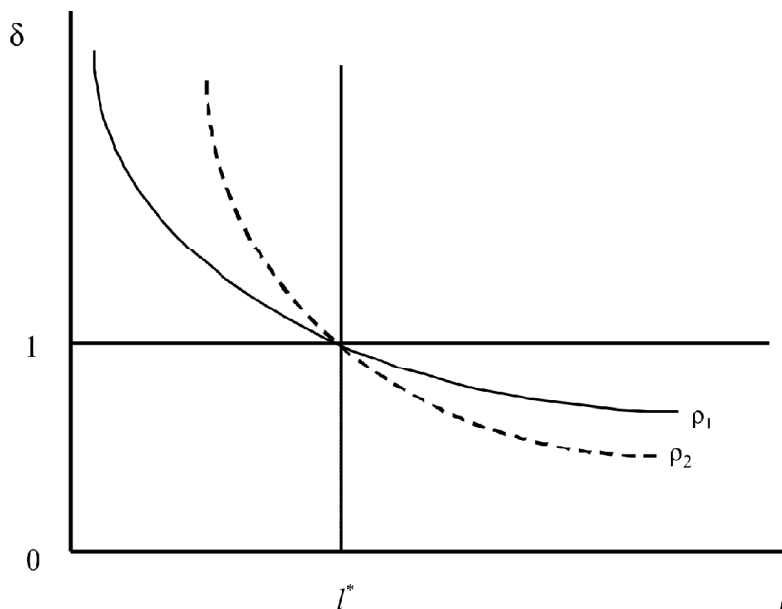


Figure 4: The ratio of relative prices to relative labour values as a function of the relative direct labour inputs at different values of the relative rate of profit

GENERALIZATION

In this section we extend our argument to the n -sector case starting from the following definition of the sectoral ratios of relative prices to relative labour values

$$\delta^T \equiv [\delta_j] \equiv [(p_j / p_n)(v_n / v_j)], j = 1, 2, \dots, n \quad (20)$$

where δ_j are *not* necessarily monotonic functions of the rate of profit (see Sraffa, 1960, ch. 6). Substituting (20) in the definition of d_{II} (see (9)) we get

$$d_{II} \equiv \sqrt{[(1/n) \sum_{j=1}^n (\delta_j b - 1)^2]} \quad (21)$$

where $b \equiv p_n / v_n$, and by invoking the normalization equation, we may write

$$b = \mathbf{v}^T \mathbf{x} / (\delta^T \hat{\mathbf{v}} \mathbf{x}) \quad (21a)$$

Substituting (20) in the definition of $\cos \theta$ we get

$$\cos \theta = (1 + \tan^2 \theta)^{-1/2} = (1/\sqrt{n})G(\boldsymbol{\delta}) \quad (22)$$

where

$$G(\boldsymbol{\delta}) \equiv (\sum_{j=1}^n \delta_j) / \sqrt{\sum_{j=1}^n (\delta_j)^2} \quad (22a)$$

From (22) it follows that $\cos \theta$ is maximized at $\boldsymbol{\delta} = \mathbf{e}$ ($\cos \theta = 1$) and, in contrast to the two-sector case, $G(\boldsymbol{\delta}) \neq G(\mathbf{e}^T \hat{\boldsymbol{\delta}}^{-1})$, where $\hat{\boldsymbol{\delta}}$ represents the diagonal matrix formed from the elements of $\boldsymbol{\delta}$. From (20) to (22a) we obtain

$$(d_{II})^2 = (1 + \tan^2 \theta)(\mu(\mathbf{x}))^2 - 2\mu(\mathbf{x}) + 1 \quad (23)$$

where $\mu(\mathbf{x}) \equiv (1/n)\mathbf{p}^T \hat{\mathbf{v}}^{-1} \mathbf{e}$ is the arithmetic mean of p_j/v_j measured in terms of commodity \mathbf{x} , a magnitude that equals $(1/n)(\sum_{j=1}^n \delta_j)b$ and,

therefore, varies from $(1/n)(\sum_{j=1}^n \delta_j)[\min(1/\delta_j)]$ to $(1/n)(\sum_{j=1}^n \delta_j)[\max(1/\delta_j)]$ (see also Steedman and Tomkins, 1998, pp. 384-385).⁸ By invoking (3), (9a) and (14) we derive the following:

- (i) d_I is a piecewise, linear function of $\mu(\mathbf{x})$.
- (ii) $d_I(\mathbf{x}) = d_{II}(\mathbf{x})$, i.e., $\cos \phi = 1$, iff $\delta_k, k = 1, 2, \dots, n-1$, are equal and $b = 2/(1 + \delta_k)$ (in that case $\mu(\mathbf{x}) = (1/n)[1 + (n-1)\delta_k]b$ and $d_I = d_{III} = |\delta_k - 1|/(1 + \delta_k)$, where $\mu(\mathbf{x}) > 1$ and $d_I > d$ iff $\delta_k > 1$).⁹
- (iii) At $\mathbf{x}^* \equiv x_n v_n \hat{\mathbf{v}}^{-1} \mathbf{e}$ it holds $\mu(\mathbf{x}^*) = 1$ and, therefore, $d_{II}(\mathbf{x}^*) = \tan \theta = \sigma(\mathbf{x}^*) (> d)$, where $\sigma(\mathbf{x}^*)$ is the standard deviation of p_j/v_j measured in terms of commodity \mathbf{x}^* , whilst $d_I(\mathbf{x}^*) = \cos \phi(\mathbf{x}^*)\sigma(\mathbf{x}^*)$.

(iv) $d_{II}(\mathbf{x})$ also equals $\tan \theta$ at $\mu(\mathbf{x}) = \cos^2 \theta - \sin^2 \theta$.

(v) $d_{II}(\mathbf{x})$ is minimized at $\mathbf{x}^{**} \equiv x_n v_n \hat{\delta} \hat{\mathbf{v}}^{-1} \mathbf{e}$, where

$$\mu(\mathbf{x}^{**}) = (1/n)(G(\delta))^2 = \cos^2 \theta \text{ and } (d_{II})_{\min} \equiv d_{II}(\mathbf{x}^{**}) = \sin \theta$$

($< d$) (Figures 5a-b correspond to a four-sector case, where $\delta_1 = 1.1, \delta_2 = 0.9, \delta_3 = 1.3$,¹⁰ and represents d_I, d_{II} , and $\cos \phi$ as functions of $\mu(\mathbf{x})$, respectively).

(vi) Relation (12) must be replaced by

$$(d_{II})_{\min} = \sqrt{d_{II}(\mathbf{x}^*) \sigma(\mathbf{x}^{**})} \tag{24}$$

where $\sigma(\mathbf{x}^{**}) (= \mu(\mathbf{x}^{**}) \tan \theta)$ is the standard deviation of p_j / v_j measured in terms of commodity \mathbf{x}^{**} .

(vii) At a given value of $\mu(\mathbf{x})$, say $\bar{\mu}$, and for strictly positive \mathbf{x} , d_{III} varies from the minimum to the maximum value of

$$\left| [n \bar{\mu} \delta_j / (\sum_{j=1}^n \delta_j)] - 1 \right|.$$

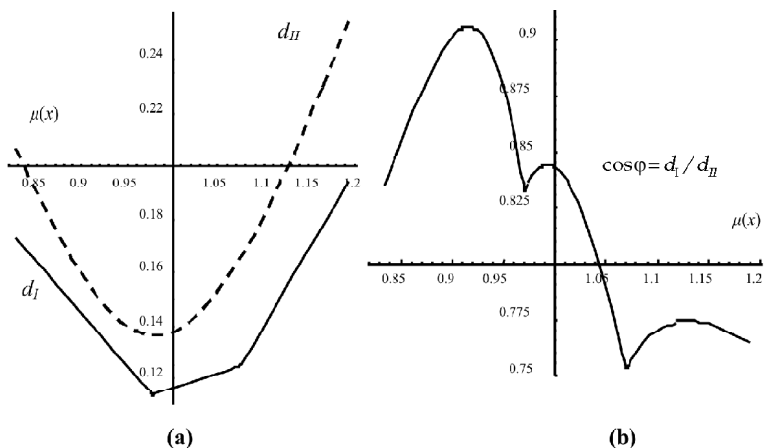


Figure 5: The MAD, the RMS%E and their ratio as functions of the arithmetic mean of the price of production-labour value ratios

Leaving aside the fact that the relationships between the measures of deviation take more complicated forms, the major difference introduced here is that the sectoral ratios of relative prices to relative labour values are not necessarily monotonic functions of the rate of profit. Consequently, the closeness of measures of deviation may occur not only at 'low' but also at 'high' values of ρ . This point can be illustrated with the aid of Sraffa's (*ibid.*, §48) 'wine-oak' numerical example in which $\delta = 20(1+0.25\rho)^8[19+(1+0.25\rho)^{25}]^{-1}$ and $R = 0.25$. As a consequence, δ equals 1 not only at $\rho = 0$ but also at $\rho \approx 0.684$, and, therefore, the ranking of the bounds for MAD and the 'd- distance', associated with 'old wine' and 'oak chest', changes with ρ (see Figures 6a-b, which represent δ_k and $d_l(0), d_l(\infty)$ and d as functions of ρ , respectively, and compare with Figure 2).

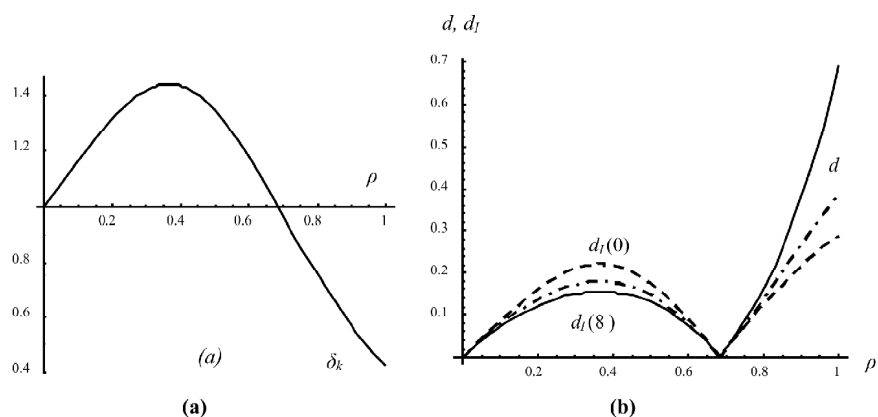


Figure 6: The ratio of relative prices to relative labour values, the bounds for MAD and the 'd- distance' as functions of the relative rate of profit

EMPIRICAL ILLUSTRATION

In order to get a realistic view of the trajectories of price of production-value deviations for alternative measures and for different ρ in actual economies, we use input-output data from the Greek and Japanese economies for the year 1990 (where $n = 19$ and $n = 33$, respectively).

The results are summarized in Tables 2, 3, 4 and 5. Tables 2 and 3 present estimates of $\delta_k \equiv [(p_k / p_n)(v_n / v_k)]$, $k = 1, 2, \dots, n - 1$, at

different, hypothetical values of the relative rate of profit, for each of the 19 sectors of the Greek economy and for each of the 33 sectors of the Japanese economy, respectively (the last columns in both tables give the arithmetic mean of $|\varepsilon_k| \equiv |\delta_k - 1|$). From the analysis of the associated numerical results and these estimates, we may derive the following: (i) with one exception (*i.e.*, the ratio δ_{32} of the Japanese economy), δ_k are monotonic functions of ρ (however, in terms of others commodities, there are production prices that are not monotonic functions of ρ); (ii) the arithmetic means of $|\varepsilon_k|$ increase with ρ ; (iii) in the Greek (Japanese) economy the Euclidean angle, measured in degrees, between the vector of direct labour coefficients and the left-hand-side Perron-Frobenius eigenvector of the matrix of input-output coefficients is almost 47.18° (56.19°), and their 'd-distance' is almost 71% (91%); (iv) in the Greek (Japanese) economy the arithmetic mean of $|\varepsilon_k|$ is greater than 40% for $\rho > 0.5$ ($\rho > 0.4$); and (v) given that the *actual* value of ρ in the Greek (Japanese) economy is approximately equal to 0.249 (to 0.331), it follows that the actual arithmetic mean of $|\varepsilon_k|$ is less than 19.4% (than 31.2%) (and this is consistent with that expected on theoretical grounds; see Steedman, 1999, pp. 315-316).¹¹

Table 2
The Sectoral Ratios of Relative Prices to Relative Labour Values and
the Relative Rate of Profit; Greece 1990

Sectors	1	2	3	4	5	6	7	8	9	10	11
ρ											
0.0	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.1	1.01	1.02	1.00	0.99	1.10	1.06	1.05	1.05	1.08	1.10	1.06
0.2	1.03	1.03	1.00	0.98	1.20	1.13	1.10	1.11	1.17	1.21	1.13
0.3	1.04	1.05	1.00	0.97	1.31	1.20	1.15	1.18	1.26	1.35	1.20
0.4	1.06	1.07	1.00	0.95	1.44	1.29	1.22	1.26	1.35	1.51	1.28
0.5	1.07	1.09	1.01	0.94	1.57	1.39	1.29	1.36	1.46	1.72	1.37
0.6	1.09	1.12	1.02	0.93	1.73	1.51	1.38	1.48	1.57	1.97	1.48
0.7	1.11	1.14	1.04	0.91	1.90	1.65	1.48	1.63	1.71	2.31	1.61
0.8	1.13	1.17	1.07	0.90	2.10	1.82	1.61	1.83	1.87	2.76	1.78
0.9	1.15	1.20	1.12	0.88	2.34	2.04	1.78	2.11	2.09	3.41	2.01
1.0	1.16	1.24	1.21	0.87	2.62	2.30	2.01	2.49	2.39	4.31	2.33

contd.

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Sectors	12	13	14	15	16	17	18	19	$(\sum \varepsilon_k)/(n-1)$ (%)
0.0	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.0
0.1	1.13	1.08	1.06	1.02	1.08	1.03	1.07	1.00	6.1
0.2	1.28	1.17	1.13	1.04	1.17	1.06	1.15	1.00	11.1
0.3	1.46	1.28	1.21	1.07	1.27	1.10	1.23	1.00	19.4
0.4	1.69	1.43	1.31	1.09	1.40	1.14	1.33	1.00	27.2
0.5	1.98	1.62	1.44	1.12	1.56	1.19	1.44	1.00	38.3
0.6	2.35	1.87	1.60	1.15	1.75	1.24	1.57	1.00	50.6
0.7	2.85	2.21	1.82	1.19	2.01	1.32	1.71	1.00	63.9
0.8	3.57	2.70	2.13	1.23	2.37	1.41	1.88	1.00	86.7
0.9	4.68	3.48	2.60	1.30	2.90	1.54	2.07	1.00	115.6
1.0	6.35	4.66	3.29	1.38	3.68	1.71	2.29	1.00	158.5

Table 3
The Sectoral Ratios of Relative Prices to Relative Labour Values and
the Relative Rate of Profit; Japan 1990

Sectors	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	ρ
0.0	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.1	1.01	1.02	1.07	1.06	1.05	1.05	1.11	1.06	1.10	1.08	1.05	1.14	1.11	1.06	1.07	1.10	1.08	1.08
0.2	1.01	1.05	1.15	1.12	1.11	1.11	1.22	1.13	1.20	1.17	1.10	1.31	1.24	1.14	1.15	1.21	1.17	1.17
0.3	1.02	1.08	1.23	1.18	1.16	1.17	1.36	1.20	1.31	1.28	1.16	1.53	1.39	1.24	1.24	1.33	1.27	1.27
0.4	1.02	1.11	1.32	1.26	1.23	1.24	1.52	1.28	1.43	1.40	1.23	1.81	1.57	1.36	1.36	1.48	1.40	1.40
0.5	1.03	1.14	1.41	1.35	1.30	1.32	1.71	1.38	1.56	1.56	1.30	2.19	1.78	1.53	1.51	1.65	1.55	1.55
0.6	1.04	1.18	1.52	1.45	1.38	1.41	1.94	1.48	1.70	1.74	1.39	2.71	2.04	1.76	1.72	1.86	1.75	1.75
0.7	1.05	1.23	1.64	1.57	1.47	1.52	2.22	1.61	1.86	1.98	1.50	3.47	2.37	2.09	2.00	2.13	2.02	2.02
0.8	1.06	1.30	1.78	1.72	1.59	1.64	2.59	1.77	2.07	2.29	1.65	4.71	2.82	2.62	2.45	2.48	2.40	2.40
0.9	1.07	1.42	1.96	1.91	1.76	1.80	3.11	2.00	2.36	2.73	1.89	7.02	3.47	3.60	3.26	3.02	3.02	3.02
1.0	1.09	1.64	2.23	2.16	2.02	2.01	3.90	2.34	2.84	3.42	2.32	11.86	4.47	5.62	4.89	3.91	4.16	4.16

Sectors	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	$(\sum \varepsilon_k)/(n-1)$ (%)
0.0	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.0
0.1	1.09	1.07	1.11	1.14	1.07	1.05	1.07	1.05	1.03	0.98	1.02	1.02	0.98	0.99	1.00	1.00	6.9
0.2	1.19	1.16	1.25	1.31	1.16	1.11	1.14	1.11	1.07	0.96	1.05	1.04	0.95	0.97	1.01	1.00	12.2
0.3	1.30	1.26	1.42	1.53	1.26	1.17	1.22	1.17	1.12	0.94	1.07	1.06	0.93	0.96	1.01	1.00	22.2
0.4	1.43	1.40	1.64	1.81	1.37	1.25	1.32	1.23	1.18	0.91	1.10	1.09	0.90	0.94	1.01	1.00	31.2
0.5	1.59	1.56	1.94	2.19	1.51	1.34	1.44	1.30	1.25	0.88	1.13	1.13	0.86	0.91	1.01	1.00	42.8
0.6	1.79	1.79	2.34	2.71	1.68	1.45	1.58	1.39	1.34	0.85	1.16	1.19	0.82	0.89	1.01	1.00	59.1
0.7	2.03	2.11	2.93	3.47	1.90	1.59	1.75	1.48	1.46	0.81	1.20	1.27	0.78	0.85	1.01	1.00	80.3
0.8	2.37	2.62	3.86	4.67	2.18	1.79	1.99	1.60	1.65	0.77	1.24	1.40	0.72	0.80	1.00	1.00	110.0
0.9	2.88	3.52	5.55	6.83	2.60	2.10	2.35	1.78	1.98	0.70	1.28	1.63	0.63	0.73	1.00	1.00	159.3
1.0	3.75	5.36	8.93	11.1	3.25	2.64	2.95	2.09	2.64	0.61	1.33	2.09	0.51	0.61	0.99	1.00	250.6

Finally, Tables 4 and 5 present estimates of (i) the measures of deviation (the prices of production are normalized with the use of Sraffa's standard commodity and the actual gross output vector);¹² and (ii) the mean absolute error (MAE), the relative errors (e.g., $(RE)_I \equiv |d_I - d|/d$) and the mean relative error (MRE) associated with the traditional measures, at different, hypothetical values of ρ . Thus, it is observed that (i) not quite unexpected, all the measures increase with ρ ; (ii) setting aside d_v , the ranking of the measures changes with ρ (for example, the Greek economy is characterized by $d_{II} < d < d_{III}$ for $0.1 \leq \rho \leq 0.4$, $d_{II} < d_{III} < d$ for $0.4 < \rho \leq 0.5$, and $d_{III} < d_{II} < d$ for $0.5 < \rho \leq 1$, whilst the Japanese economy is characterized by $d_{II} < d < d_{III}$ for $0.1 \leq \rho \leq 0.2$ and $d < d_{II} < d_{III}$ for $0.3 \leq \rho \leq 1$); (iii) both the absolute and relative errors between d and the traditional measures may decrease with ρ ; and (iv) in the Greek economy the actual mean absolute (relative) error of the traditional measures of deviation is less than 0.80% (lies between 7.07%-7.10%), and in the Japanese economy it is less than 2.29% (12.91%).

Table 4
The Measures of Deviation and the Relative Rate of Profit; Greece, 1990

ρ	d_I (%)	d_{II} (%)	d_{III} (%)	d (%)	MAE (%)	RE_I (%)	RE_{II} (%)	RE_{III} (%)	MRE (%)
0.0	0.00	0.00	0.00	0.00	0.00	-	-	-	-
0.1	3.00	3.56	3.71	3.64	0.26	17.58	2.20	1.92	7.23
0.2	6.08	7.28	7.52	7.42	0.53	18.06	1.90	1.35	7.10
0.3	9.22	11.18	11.45	11.36	0.80	18.84	1.58	0.79	7.07
0.4	12.49	15.30	15.56	15.55	1.11	19.68	1.61	0.06	7.12
0.5	16.05	19.72	19.92	20.04	1.48	19.91	1.60	0.60	7.37
0.6	20.11	24.62	24.60	24.94	1.83	19.37	1.28	1.36	7.34
0.7	24.58	29.96	29.72	30.40	2.31	19.14	1.45	2.24	7.61
0.8	29.85	35.86	35.43	36.62	2.91	18.49	2.08	3.25	7.94
0.9	36.25	42.48	41.98	43.88	3.64	17.39	3.19	4.33	8.30
1.0	43.01	49.19	48.90	51.66	4.63	16.74	4.78	5.34	8.95

Table 5
The Measures of Deviation and the Relative Rate of Profit; Japan, 1990

ρ	d_I (%)	d_{II} (%)	d_{III} (%)	d (%)	MAE (%)	RE_I (%)	RE_{II} (%)	RE_{III} (%)	MRE (%)
0.0	0.00	0.00	0.00	0.00	0.00	-	-	-	-
0.1	3.20	4.02	4.60	4.04	0.47	20.79	0.50	13.86	11.72
0.2	6.58	8.29	9.55	8.30	0.99	20.72	0.12	15.06	11.97
0.3	10.18	12.88	14.93	12.85	1.59	20.78	0.23	16.19	12.40
0.4	14.06	17.89	20.81	17.76	2.29	20.83	0.73	17.17	12.91
0.5	18.32	23.47	27.32	23.16	3.10	20.84	1.34	17.96	13.38
0.6	23.10	29.80	34.67	29.24	4.04	21.00	1.92	18.57	13.83
0.7	28.60	37.25	43.17	36.24	5.19	21.08	2.79	19.12	14.33
0.8	35.12	46.36	53.20	44.60	6.61	21.26	3.95	19.28	14.83
0.9	43.46	58.08	66.21	54.95	8.63	20.90	5.70	20.49	15.70
1.0	57.10	72.45	81.58	66.87	10.02	14.61	8.34	21.00	14.65

To our knowledge, there is *no* relevant empirical study where the actual value of ρ is greater than 0.40 (and less than 0.17).¹³ Thus, it is reasonable to *expect* that, in the 'real' world, all the considered measures of deviation are not far from each other: in fact we have experimented with the input-output tables of China (1997), Greece (1988-1997) and Japan (for the years 1970, 1975, 1980, and 1985), and the results were quite similar.¹⁴

CONCLUDING REMARKS

It has been argued that for realistic values of the relative rate of profit, which is no greater than the share of profits in Sraffa's standard system, the traditional measures of production price-labour value deviations (*i.e.*, the MAD, RMS%E and MAWD), which depend on the choice of *numéraire*, and the '*d*-distance', which is a *numéraire*-free measure, tend to be close to each other. This does not imply, of course, that there is basis for not preferring the latter measure, but rather that future research efforts should be focused on the socio-technical conditions that determine the level of the relative rate of profit in actual economies.

Acknowledgements

This is an enlarged version of Mariolis and Tsoulfidis (2010): We are indebted to two anonymous referees and the Editors of *Metroeconomica* for extremely helpful comments and advice on earlier versions of the papers. The usual disclaimer applies.

Notes

1. See Ochoa (1984, chs 6-8; 1989, pp. 418-422), Petrović (1987, pp. 206-208), Chilcote (1997, chs 6-7), Shaikh (1998, p. 233), Tsoulfidis and Mariolis (2007, p. 428), Tsoulfidis (2008, p. 715), *inter alia*.
2. Throughout the paper we use the term 'error' because we hypothesize that the '*d*-distance' represents the 'true or accepted' value of the deviation under study.
3. For an alternative measure, i.e. the 'mean absolute eigen-deviation of labour commanded prices from labour values', see Mariolis (2011, 2013).
4. As is well known, $f = v$ for each r iff the capital-intensity is equal across sectors or, equivalently, the vector of direct labour coefficients is the left-hand side Perron-Frobenius eigenvector of the matrix of input-output coefficients
5. Throughout the paper, 'most reliable' means that the relative error is less than 3%. It may also be noted that $\Delta_f(\delta) = \Delta_f(\delta^{-1})$. Moreover, when f decreases with r , (i) equation (7) holds with $(1-\delta)$; and, therefore, (ii) $d_f(\infty) - d_f(0)$ equals $2d_f(0)d_f(\infty)$
6. It may be noted that the relative error associated with $d_f(\infty)$ and $(d_{II})_{\min}$ tend to $1 - (2\sqrt{2} - \sqrt{2})^{-1} \approx 34.7\%$ and $1 - (\sqrt{4} - 2\sqrt{2})^{-1} \approx 7.6\%$, respectively.
7. If wages are paid *ex post*, the rate of profit in the standard system is $r = R(1-w)$, where w denotes the money wage rate and, at the same time, the share of wages in the standard system. Thus, $\rho = 1 - w$. If wages are paid *ex ante*, then

$$r = R(1 - w) (1 + wR)^{-1}$$
 or

$$\rho = (1 - w) [R^{-1}/(R^{-1} + w)] \leq 1 - w$$
 where the square bracket represents the ratio of the means of production to the total capital in the standard system (see also Kurz and Salvadori, 1998, pp. 136-138).
8. It should be noted that to any given \mathbf{x}/x_n there corresponds a unique b , whilst the converse does not hold true.
9. Setting aside the equal capital-intensity case, the *entire* price vector cannot be proportional to that of labour values at a positive level of the rate of profit (see Mainwaring, 1976). Consequently, the case $\delta_k = 1$, $k = 1, 2, \dots, n - 1$, does not really exist.
10. Consequently, $\cos \theta \approx 0.9906$, $\theta \approx 7.8$, $d \approx 0.137$, $d_{II}(\mathbf{x}^*) \approx 0.138$, $(d_{II})_{\min} \approx 0.136$, $43/52 \leq \mu(\mathbf{x}) \leq 43/36$, and $\mu(x^*) = \cos^2 \theta \approx 0.981 > 43/44$.
11. For the estimation of the actual values of ρ in the economies under consideration, see Tsoulfidis and Mariolis (2007) and Tsoulfidis (2008), respectively. It should be noted that in the Greek economy (1988-1997) the actual value of ρ lies between 0.230 (1993) and 0.270 (1997), and in the Japanese

economy (1970, 1975, 1980, 1985, 1990) it lies between 0.298 (1975) and 0.371 (1985) (*ibid.*).

12. That is, $\mathbf{p}^T \bar{\mathbf{s}} = \bar{\mathbf{v}}^T \bar{\mathbf{s}}$, where $\bar{\mathbf{s}} \equiv [(\bar{\mathbf{v}}^T \bar{\mathbf{x}})/(\bar{\mathbf{v}}^T \mathbf{q})]\mathbf{q}$, $\bar{\mathbf{x}}$ denotes the actual gross output vector, \mathbf{q} the right-hand side Perron-Frobenius eigenvector of \mathbf{A} , $\bar{\mathbf{v}} \equiv [(\mathbf{e}^T \bar{\mathbf{x}})/(\bar{\mathbf{v}}^T \bar{\mathbf{x}})]\mathbf{v}$, $\mathbf{v}^T \equiv \mathbf{I}^T[\mathbf{I} - \mathbf{A}]^{-1}$ the vector of labour values, and \mathbf{e} represents the vector of market prices (*i.e.*, the physical unit of measurement of the output of each sector is that unit which is worth of a monetary unit; see, *e.g.*, Miller and Blair, 1985, p. 356). These normalizations imply that $\mathbf{p}^T \bar{\mathbf{s}} = \bar{\mathbf{v}}^T \bar{\mathbf{s}} = \bar{\mathbf{v}}^T \bar{\mathbf{x}} = \mathbf{e}^T \bar{\mathbf{x}}$ (see also Ochoa, 1984, ch. 4; Shaikh, 1998, pp. 227-229).
13. See, *e.g.*, the empirical studies mentioned in note 1. Taking into account that ρ is no greater than the share of profits in the standard system (see note 7), this seems to be in accordance with many well-known estimations of the share of profits (approximated by the net operating surplus) in actual economies. For example, Ellis and Smith (2007) find that the share of profits in a sample of 20 OECD countries (for the period 1960-2005) only in a few years and a few countries has slightly exceeded the 40 percent, and, typically, fluctuates a few percentage points around an average of 30 percent (see also Harvie, 2000).
14. The results are available on request.

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