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ON GENERALIZED CESÀRO VECTOR VALUED SEQUENCE SPACE

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ABSTRACT: In this paper, we define a generalized Cesàro vector valued sequence space $ces_{(p)}(X, q)$ where p = (p_k) and q = (q_k) are bounded sequences of positive real numbers with $p_k \geq 1$ *and* $q_k \geq 1 \; \forall \; k \in \mathcal{N}$ and consider it equipped with Luxemburg norm. We investigate the space for completeness and also discuss its rotundity. We also extend some results of Petrot and Suantai.

Keywords: Generalized Cesàro vector valued sequence space, Modular, Luxemburg norm, Rotundity

1. INTRODUCTION AND PRELIMINARIES

Throughout this paper $\mathcal{N}$ and **R** stand for set of natural numbers and real numbers respectively, p = (p_k) and q = (q_k) be bounded sequences of positive real numbers with $p_k \geq 1$ *and* $q_k \geq 1 \; \forall \; k \in \mathcal{N}$ and M = $sup_k p_k$ and $M' = sup_k q_k$.

Let l_o be the space of all real sequences. For $1 \leq$ p $< \infty$, the Cesàro sequence space ces_p is defined by

$$ces_p = \left\{ x = (x_k) \in \; l_o \colon \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p \; < \; \infty \right\}$$

equipped with the norm

$$\|x\| = \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p \right)^{\frac{1}{p}}$$

and $ces_{\infty=} \left\{ x = (x_k) \in \; l_o \colon sup_n \frac{1}{n} \sum_{k=1}^{n} |x_k| < \infty \right\}$

equipped with the norm

$\|x\|_\infty = sup_n \frac{1}{n} \sum_{k=1}^{n} |x_k|$.

The Cesàro sequence spaces, $ces_p (1 \leq p < \infty)$ and ces_∞ first appeared in 1968 in connection with the problem of Dutch Mathematical Society to find their duals. Some investigation of these spaces was done by J.S. Shue [1] in 1970. Then G.M. Leibowitz [2] and A.A. Jagers [3] proved that $ces_1 = \{0\}$, ces_p are separable reflexive Banach spaces for 1< p $\leq \infty$ and the l_p spaces are in ces_p

for 1< p $\leq \infty$. Several geometric properties of the Cesàro sequence spaces and generalized Cesàro sequence spaces were studied by many mathematicians [4-8]. Suantai [9] has introduced the generalized Cesàro sequence spaces ces(p).

Let p = (p_k) be a bounded sequence of positive real numbers with $p_k \geq 1 \; \forall \; k \; \in \; \mathcal{N}$, then the generalized Cesàro sequence spaces ces(p) is defined by

$$ces(p) = \{x = (x_k) \in l_o : \rho(\lambda x) < \infty \; for \; some \; \lambda > 0\}$$

where $\rho(x) = \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k|\right)^{p_n}$

equipped with the Luxemburg norm

$$\|x\| = inf \left\{\lambda > 0 : \; \rho\left(\frac{x}{\lambda}\right) \leq 1\right\}.$$

In the case when $p_k = p, 1 \leq p < \infty, \forall \; k \; \in \; \mathcal{N}$, the generalized Cesàro sequence space ces(p) is the Cesàro sequence space ces_p and the Luxemburg norm on ces(p) reduces to the usual norm on ces_p.

Let (X, ‖ ‖) be a Banach space and p = (p_k) be a bounded sequence of positive real numbers with $p_k \geq 1 \; \forall \; k \; \in \; \mathcal{N}$, Chareonsawan [10] defined Cesàro vector valued sequence space ces (X, p) by

$$ces(X, p) = \{x = (x_k) \subseteq X : \rho(\lambda x) < \infty \; for \; some \; \lambda > 0\}$$

where $\rho(x) = \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} \|x_k\|\right)^{p_n}$

equipped with the Luxemburg norm

$\|x\| = inf \left\{\lambda > 0 : \; \rho\left(\frac{x}{\lambda}\right) \leq 1\right\}.$

Khan [11] in 2010 defined the generalized Cesàro sequence space for $1 \leq p < \infty$ and q = (q_k) be a bounded sequence of positive real numbers by

$$ces_p(q) = \left\{ x = (x_k) \in l_o : \|x\| = \left(\sum_{n=1}^{\infty} \left(\frac{1}{Q_n} \sum_{k=1}^{n} |q_k x_k| \right)^p \right)^{\frac{1}{p}} < \infty \right\}$$

where $Q_n = \sum_{k=1}^{n} q_k$,n $\in \mathcal{N}$. If $q_k = 1 \forall \; k \; \in \; \mathcal{N}$, then $ces_p(q)$ reduces to ces_p.

In 2012, Mongkolkeha and Kumam [12] defined the generalized Cesàro sequence space for a bounded sequence p = (p_k) and q = (q_k) of positive real numbers with $p_k \geq 1 \; and \; q_k \geq 1 \; \forall \; k \; \in \mathcal{N}$ by

$$ces_{(p)}(q) = \{x = (x_k) \in l_o : \rho(\lambda x) < \infty for \; some \; \lambda > 0\},$$

where $\rho(x) = \sum_{n=1}^{\infty} \left(\frac{1}{Q_n} \sum_{k=1}^{n} |q_k x_k|\right)^{p_n}$

with $Q_n = \sum_{k=1}^{n} q_k$ and consider $ces_{(p)}(q)$ equipped with the Luxemburg norm

$$\|x\| = inf\left\{\lambda > 0 : \rho\left(\frac{x}{\lambda}\right) \leq 1\right\}.$$

For $p_k = p, 1 \leq p < \infty \; \forall \; k \in \mathcal{N}$, then $ces_{(p)}(q)$ reduces to $ces_p(q)$ and if $q_k = 1 \forall \; k \in \mathcal{N}$ then $ces_{(p)}(q)$ reduces to ces(p).

In this paper, we define the generalized Cesàro vector valued sequence space as follows:

Let (X,‖ ‖) be a Banach space and p = (p_k) and q = (q_k) be bounded sequences of positive real numbers with $p_k \geq 1 \; and \; q_k \geq 1 \; \forall \; k \in \mathcal{N}$.

The generalized Cesàro vector valued sequence space, $ces_{(p)}(X, q)$ is defined by

$$ces_{(p)}(X, q) = \{\text{x} = (x_k) \subseteq X : \rho(\lambda x) < \infty for \; some \; \lambda > 0\}$$

where $\rho(x) = \sum_{n=1}^{\infty} \left(\frac{1}{Q_n} \sum_{k=1}^{n} \|q_k x_k\|\right)^{p_n}$

with $Q_n = \sum_{k=1}^{n} q_k$ and consider $ces_{(p)}(X, q)$ equipped with the Luxemburg norm

$$\|x\| = inf\left\{\lambda > 0 : \rho\left(\frac{x}{\lambda}\right) \leq 1\right\}.$$

For X = **R**, $ces_{(p)}(X, q)$ reduces to then $ces_{(p)}(q) \; and \; if \; p_k = p, 1 \leq p < \infty \; \forall \; k \in \mathcal{N}$ then $ces_{(p)}(X, q)$ reduces to $ces_p(q)$.

First we start with some definitions which we will need later on.

For a real vector space X a functional $\rho : X \to [0, \infty]$ is called a **modular** if it satisfies the following conditions:

$(i)\rho(x) = 0$ if and only if x=0;

$(ii)\rho(\alpha x) = \rho(x) \; \forall$ scalars α with $|\alpha|$=1;

(iii)$\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y) \; \forall \; x, y \in X \; and \; \forall \; \alpha, \beta \geq 0 \; with \; \alpha + \beta = 1.$

The modular ρ is called **convex** if

(iv)$\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y) \; \forall \; x, y \in X \; and \; \forall \; \alpha, \beta \geq 0 \; with \; \alpha + \beta = 1.$

For any modular ρ on X, the space

$$X_\rho = \{\text{x} \in X : \rho(\lambda x) < \infty \; for \; some \; \lambda > 0\}$$

is called the **modular space**. Orlicz [13] proved that if ρ is a convex modular, then function

$$\|x\| = inf\left\{\lambda > 0 : \rho\left(\frac{x}{\lambda}\right) \leq 1\right\}$$

is a norm on X_ρ, which is called the **Luxemburg norm**.

For a real Banach space$(X, \| \quad \|)$, let B(X) and S(X) be the closed unit ball and the unit sphere of X respectively i.e. B(X) = $\{x \in X: \|x\| \leq 1\}$ and S(X) = $\{x \in X: \|x\| = 1\}$.

A point $x \in S(X)$ is an **extreme point** of B(X) if for any y, z $\in B(X)$, the equality 2x=y+z implies y=z. The set of all extreme points of X is denoted by Ext(X).

A Banach space X is said to be **rotund** (R), if every point of S(X) is an extreme point of B(X).

In other words, if X is **rotund** (R) then S(X)$\subseteq Ext\ B(X)$.

2. MAIN RESULTS

In this section we first study some basic properties of modular on $ces_{(p)}(X, q)$ and its relation with Luxemburg norm. We then show that $ces_{(p)}(X, q)$ is a Banach space. Finally, we discuss rotundity on $ces_{(p)}(X, q)$.

Proposition 2.1: The functional ρ on the space $ces_{(p)}(X, q)$ is a convex modular.

Proof: For x=$(x_k) \in ces_{(p)}(X, q)$, it is obvious that $\rho(x) = 0$ if and only if x=0 and $\rho(\alpha x) = \rho(x)$ for scalar α with $|\alpha|$=1. Let $\alpha \geq 0, \beta \geq 0$ *be such that* $\alpha + \beta = 1$ and x=(x_k), y=$(y_k) \in ces_{(p)}(X, q)$, then,

$$\rho(\alpha x + \beta y) = \sum_{n=1}^{\infty} \left(\frac{1}{Q_n} \sum_{k=1}^{n} \|q_k(\alpha x_k + \beta y_k)\| \right)^{p_n}$$

$$\leq \sum_{n=1}^{\infty} \left(\frac{1}{Q_n} \sum_{k=1}^{n} \|q_k(\alpha x_k)\| + \frac{1}{Q_n} \sum_{k=1}^{n} \|q_k(\beta y_k)\| \right)^{p_n}$$

$$= \sum_{n=1}^{\infty} \left(\alpha \frac{1}{Q_n} \sum_{k=1}^{n} \|q_k x_k\| + \beta \frac{1}{Q_n} \sum_{k=1}^{n} \|q_k y_k\| \right)^{p_n}$$

$$\leq \alpha \sum_{n=1}^{\infty} \left(\frac{1}{Q_n} \sum_{k=1}^{n} \|q_k x_k\| \right)^{p_n} + \beta \sum_{n=1}^{\infty} \left(\frac{1}{Q_n} \sum_{k=1}^{n} \|q_k y_k\| \right)^{p_n}$$

(By the convexity of the function $t \to |t|^{p_k}\ \forall\ k \ \in \boldsymbol{\mathcal{N}}$)

$$= \alpha\rho(x) + \beta\rho(y).$$

This completes the proof.

In next result we study some properties of the modular ρ on $ces_{(p)}(X, q)$.

Proposition 2.2: For x=$(x_k) \in ces_{(p)}(X,q)$, the modular ρ on $ces_{(p)}(X,q)$ satisfies the following properties:

if $0 < a < 1$, then $a^M \rho\left(\frac{x}{a}\right) \leq \rho(x)$.

(i) if $0 < a < 1$, then $\rho(ax) \leq a\rho(x) \leq \rho(x)$.

(ii) if $a \geq 1$, then $\rho(x) \leq a^M \rho\left(\frac{x}{a}\right)$

(iii) if $a \geq 1$, then $\rho(x) \leq a\rho(x) \leq \rho(ax)$.

Proof: (i) Let $0 < a < 1$, then we have

$$\rho(x) = \sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n}\|q_k x_k\|\right)^{p_n}$$

$$= \sum_{n=1}^{\infty}\left(\frac{a}{Q_n}\sum_{k=1}^{n}\left\|q_k \frac{x_k}{a}\right\|\right)^{p_n}$$

$$= \sum_{n=1}^{\infty} a^{p_n}\left(\frac{1}{Q_n}\sum_{k=1}^{n}\left\|q_k \frac{x_k}{a}\right\|\right)^{p_n}$$

$$\geq a^M \sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n}\left\|q_k \frac{x_k}{a}\right\|\right)^{p_n}$$

$$= a^M \rho\left(\frac{X}{a}\right).$$

(ii) Let $0 < a < 1$, then we have,

$$\rho(ax) = \sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n}\|q_k(ax_k)\|\right)^{p_n}$$

$$= \sum_{n=1}^{\infty} a^{p_n}\left(\frac{1}{Q_n}\sum_{k=1}^{n}\|q_k x_k\|\right)^{p_n}$$

$$\leq a \sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n}\|q_k x_k\|\right)^{p_n}$$

$$= a\,\rho(x) \leq \rho(x).$$

(iii) Let $a \geq 1$, then we have,

$$\rho(x) = \sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n}\|q_k x_k\|\right)^{p_n}$$

$$= \sum_{n=1}^{\infty}\left(\frac{a}{Q_n}\sum_{k=1}^{n}\left\|q_k\frac{x_k}{a}\right\|\right)^{p_n}$$
$$= \sum_{n=1}^{\infty} a^{p_n}\left(\frac{1}{Q_n}\sum_{k=1}^{n}\left\|q_k\frac{x_k}{a}\right\|\right)^{p_n}$$
$$\leq a^M\sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n}\left\|q_k\frac{x_k}{a}\right\|\right)^{p_n}$$
$$=a^M\rho\left(\frac{X}{a}\right).$$

(iv) Let $a \geq 1$, then we have,

$$\rho(ax) = \sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n}\|q_k(ax_k)\|\right)^{p_n}$$
$$= \sum_{n=1}^{\infty} a^{p_n}\left(\frac{1}{Q_n}\sum_{k=1}^{n}\|q_k x_k\|\right)^{p_n}$$
$$\geq a\sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n}\|q_k x_k\|\right)^{p_n}$$
$$= a\rho(x) \ \geq \ \rho(x).$$

This completes the proof.

In the next proposition we discuss some relations between the modular ρ and the Luxemburg norm on $ces_{(p)}(X,q)$.

Proposition 2.3: For any x=$(x_k) \in ces_{(p)}(X,q)$, we have

(i) if $\|x\| < 1$, then $\rho(x) \leq \|x\|$,
(ii) if $\|x\| > 1$, then $\rho(x) \geq \|x\|$,
(iii) $\|x\| = 1$ if and only if $\rho(x) = 1$,
(iv) $\|x\| < 1$ if and only if $\rho(x) < 1$,
(v) $\|x\| > 1$ if and only if $\rho(x) > 1$,
(vi) if $0 < a < 1$ and $\|x\| > a$, then $\rho(x) > a^M$, and
(vii) if $a \geq 1$ and $\|x\| < a$, then $\rho(x) < a^M$.

Proof: (i) Let $\varepsilon > 0$ be such that $0 < \varepsilon < 1 - \|x\|$ so, $\|x\| + \varepsilon < 1$. By definition of $\| \quad \|$, there exists $\lambda > 0$ such that $\|x\| + \varepsilon > \lambda > \|x\|$ and $\rho\left(\frac{x}{\lambda}\right) \leq 1$. Now by (iv) and (ii) of proposition (2.2), we have,

$$\rho(x) \leq \rho\left(\left(\frac{\|x\| + \varepsilon}{\lambda}\right)x\right)$$

$$\leq (\|x\| + \varepsilon)\rho\left(\frac{x}{\lambda}\right)$$

$$\leq \|x\| + \varepsilon.$$

Hence $\rho(x) \leq \|x\| + \varepsilon \;\forall\, \varepsilon \in (0, 1 - \|x\|)$. Put A= $\{\|x\| + \varepsilon : \varepsilon \in (0, 1 - \|x\|)\}$, then $\|x\|$ = inf A. Since $\rho(x)$ is a lower bound of A, $\rho(x) \leq \|x\|$.

(ii) Let ε be any real number such that $0 < \varepsilon < \frac{\|x\|-1}{\|x\|}$, so, $1 < (1-\varepsilon)\,\|x\| < \|x\|$. Then by definition of $\|\quad\|$, and (ii) of proposition (2.2), we have,

$$1 < \rho\left(\frac{x}{(1-\varepsilon)\|x\|}\right) \leq \frac{1}{(1-\varepsilon)\|x\|}\rho(x).$$

Hence $(1-\varepsilon)\,\|x\| < \rho(x) \;\forall\, \varepsilon \in \left(0, \frac{\|x\|-1}{\|x\|}\right)$. Put A = $\left\{(1-\varepsilon)\|x\| : 0 < \varepsilon < \frac{\|x\|-1}{\|x\|}\right\}$, then $\|x\| = \sup$ A. Since $\rho(x)$ is an upper bound of A, $\|x\| \leq \rho(x)$.

(iii) First suppose that, $\|x\| = 1$. Let $\varepsilon > 0$ be given, then by definition of $\|\quad\|$, there exists $\lambda > 0$ such that $\|x\| + \varepsilon > \lambda > \|x\|$ and $\rho\left(\frac{x}{\lambda}\right) \leq 1$ i.e. $1+\varepsilon > \lambda > \|x\|=1$ and $\rho\left(\frac{x}{\lambda}\right) \leq 1$. Since $\lambda > 1$, by (iv) of proposition (2.2),

$$\rho(x) \leq \lambda^M \rho\left(\frac{x}{\lambda}\right) \leq \lambda^M \leq (1+\varepsilon)^M.$$

Hence $(\rho(x))^{\frac{1}{M}} \leq 1 + \varepsilon \;\forall\, \varepsilon > 0$. This implies $\rho(x) \leq 1$. If $\rho(x) < 1$, we choose a real number $a \in (0,1)$ such that $\rho(x) < a^M < 1$, then by (i) of proposition (2.2), $\rho\left(\frac{x}{a}\right) \leq \frac{1}{a^M}\rho(x) < 1$. By definition of $\|\quad\|$, then, $\|x\| \leq a$. But $a < 1, so\ \|x\| < 1$ which is a contradiction. Therefore, $\rho(x) = 1$.

Conversely, let $\rho(x) = 1$, then by definition of $\|\quad\|$, $\|x\| \leq 1$. If $\|x\| < 1$, then by (i), $\rho(x) \leq \|x\| < 1$, which is a contradiction. Thus $\|x\| = 1$.

(iv) First suppose that $\|x\| < 1$, then by (i), $\rho(x) \leq \|x\| < 1$. Conversely, let $\rho(x) < 1$. Let if possible $\|x\| \geq 1$. If $\|x\| = 1$ then by (iii) $\rho(x) = 1$, which is a contradiction. If $\|x\| > 1$, then by (ii) $\rho(x) \geq \|x\| > 1$, again a contradiction. Hence $\|x\| < 1$.

(v) First suppose that $\|x\| > 1$, then by (ii), $\rho(x) \geq \|x\| > 1$. Conversely, let $\rho(x) > 1$. Let if possible $\|x\| \leq 1$. If $\|x\| = 1$ then by (iii) $\rho(x) = 1$, which is a contradiction and if $\|x\| < 1$, then by (iv) $\rho(x) < 1$, which is again a contradiction. Hence $\|x\| > 1$.

(vi) Suppose $0 < a < 1$ and $\|x\| > a$, then $\left\|\frac{x}{a}\right\| > 1$. Then by (v), $\rho\left(\frac{x}{a}\right) > 1$. This implies $a^M \rho\left(\frac{x}{a}\right) > a^M$. By (i) of proposition (2.2), $\rho(x) > a^M$.

(vii) Suppose $a \geq 1$ and $\|x\| < a$, then $\left\|\frac{x}{a}\right\| < 1$. Then by (iv), $\rho\left(\frac{x}{a}\right) < 1$. This implies $a^M \rho\left(\frac{x}{a}\right) < a^M$. By (iii) of proposition (2.2), $\rho(x) < a^M$.

This completes the proof.

Proposition 2.4: Let (x_n) be a sequence in $ces_{(p)}(X, q)$,

(i) if $\|x_n\| \to 1$ as $n \to \infty$, then $\rho(x_n) \to 1$ as $n \to \infty$,

(ii) if $\rho(x_n) \to 0$ as $n \to \infty$, then $\|x_n\| \to 0$ as $n \to \infty$.

Proof: (i) Suppose $\|x_n\| \to 1$ as $n \to \infty$. Let $\varepsilon \in (0,1)$ then there exists N $\in \mathcal{N}$ such that $1 - \varepsilon < \|x_n\| < 1 + \varepsilon \; \forall \, n \geq N$. By (vi) and (vii) of proposition (2.3), we have, $(1 - \varepsilon)^M < \rho(x_n) < (1 + \varepsilon)^M \; \forall \, n \geq N$. This implies $(\rho(x_n))^{\frac{1}{M}} \to 1$ as $n \to \infty$ i.e. $\rho(x_n) \to 1$ as $n \to \infty$.

(ii) Suppose $\rho(x_n) \to 0$ as $n \to \infty$. Let if possible $\|x_n\| \nrightarrow 0$ as $n \to \infty$. Then there exists some $\varepsilon \in (0,1)$ such that $\|x_n\| > \varepsilon \;\; \forall \, n \in \mathcal{N}$. By (vi) of proposition (2.3), $\rho(x_n) > \varepsilon^M \forall \, n \in \mathcal{N}$. This implies $\rho(x_n) \nrightarrow 0$ as $n \to \infty$ which is a contradiction. Hence $\|x_n\| \to 0$ as $n \to \infty$.

Next we show that $ces_{(p)}(X, q)$ is a Banach space under the Luxemburg norm.

Theorem 2.5: $ces_{(p)}(X, q)$ is a Banach space.

Proof: Let $(x_m)_{m=1}^{\infty}$ be any Cauchy sequence in $ces_{(p)}(X, q)$ where $x_m = \left(x_k^{(m)}\right)_{k=1}^{\infty}$, then for each $\varepsilon > 0$ there exists a positive integer N such that $\|x_m - x_r\| < \left(\frac{\varepsilon}{M'}\right)^M \; \forall \, m, r \geq N$.

This implies by (i) of proposition (2.3)

$$\rho(x_m - x_r) < \left(\frac{\varepsilon}{M'}\right)^M \;\; \forall \, m, r \geq N$$

i.e.

$$\sum_{n=1}^{\infty} \left(\frac{1}{Q_n} \sum_{k=1}^{n} \left\| q_k x_k^{(m)} - q_k x_k^{(r)} \right\| \right)^{p_n} < \left(\frac{\varepsilon}{M'}\right)^M \; \forall \, m, r \geq N \; \ldots (1)$$

Let us write $\eta(x)$ for the sequence of Cesàro means defined by the given sequence x = $(x_m)_{m=1}^{\infty}$ then $\eta(x)_m = \frac{1}{m} \sum_{i=1}^{m} x_i$.

Now for m, r $\geq N$ and n $\in \mathcal{N}$, consider

$$\|\eta(x_m)_n-\eta(x_r)_n\|^{p_n}=\left\|\frac{1}{n}\sum_{k=1}^{n}x_k^{(m)}-\frac{1}{n}\sum_{k=1}^{n}x_k^{(r)}\right\|^{p_n}$$

$$=\left\|\frac{1}{n}\sum_{k=1}^{n}(x_k^{(m)}-x_k^{(r)})\right\|^{p_n}$$

$$\leq (\frac{1}{n}\sum_{k=1}^{n}\left\|x_k^{(m)}-x_k^{(r)}\right\|)^{p_n}$$

$$\leq (\frac{1}{n}\sum_{k=1}^{n}q_k\left\|x_k^{(m)}-x_k^{(r)}\right\|)^{p_n}$$

$$=(\frac{1}{n}\sum_{k=1}^{n}\left\|q_kx_k^{(m)}-q_kx_k^{(r)}\right\|)^{p_n}$$

$$\leq (\frac{M'}{Q_n}\sum_{k=1}^{n}\left\|q_kx_k^{(m)}-q_kx_k^{(r)}\right\|)^{p_n}$$

$$=(M')^{p_n}(\frac{1}{Q_n}\sum_{k=1}^{n}\left\|q_kx_k^{(m)}-q_kx_k^{(r)}\right\|)^{p_n}$$

$$\leq (M')^{M}(\frac{1}{Q_n}\sum_{k=1}^{n}\left\|q_kx_k^{(m)}-q_kx_k^{(r)}\right\|)^{p_n}\ldots②$$

By inequalities ① and ②, we have

$$\sum_{n=1}^{\infty}\|\eta(x_m)_n-\eta(x_r)_n\|^{p_n}\leq (M')^M\left(\frac{\varepsilon}{M'}\right)^M\ \forall\, m,r\geq N.$$

Hence

$$\sum_{n=1}^{\infty}\|\eta(x_m)_n-\eta(x_r)_n\|^{p_n}\leq \varepsilon^M\ \forall\, m,r\geq N.$$

By proposition (3.2.5) of [10],we then have

$$\|\eta(x_m) - \eta(x_r)\|_{l(X,p)} < \varepsilon \ \forall\, m, r \geq N.$$

Hence $(\eta(x_m))$ is a Cauchy sequence in $l(X,p)$, the Nakano vector valued sequence space. Since $l(X,p)$ is complete, there is an element x = (x_k) in $l(X,p)$ such that

$\|\eta(x_m) - x\|_{l(X,p)} \to 0$ as m$\to \infty$.

Further convergence in $l(X,p)$ is component wise, so that,

$x_k = \lim_{m\to\infty}\eta(x_m)_k = \lim_{m\to\infty}\frac{1}{k}\sum_{i=1}^{k} x_i^{(m)}$ for each k.

For k = 1, $\lim_{m\to\infty}x_1^{(m)} = x_1$. Similarly, for k = 2, $x_2 = \lim_{m\to\infty}\frac{1}{2}(x_1^{(m)} + x_2^{(m)})$. This implies $\lim_{m\to\infty}x_2^{(m)} = 2x_2 - x_1$. Continuing in this way, for each k, $\lim_{m\to\infty}x_k^{(m)}$ exists and is equal to k$x_k - (k-1)x_{k-1}$ where, we set $x_0 = 0$.

Define a sequence,

y = (y_k) such that $y_k =$ k$x_k - (k-1)x_{k-1}$.

Then,

$\lim_{m\to\infty}x_k^{(m)} = y_k$ for each k.

Now for any t ∈ $\boldsymbol{\mathcal{N}}$,

$\sum_{n=1}^{t}\left(\frac{1}{Q_n}\sum_{k=1}^{n}\left\|q_k x_k^{(m)}\right\|\right)^{p_n} \leq \sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n}\left\|q_k x_k^{(m)}\right\|\right)^{p_n}$ = S (say)$<\infty$ as $x_m \in ces_{(p)}(X,q)$.

Hence

$\sum_{n=1}^{t}\left(\frac{1}{Q_n}\sum_{k=1}^{n}\left\|q_k x_k^{(m)}\right\|\right)^{p_n} \leq$ S; t = 1, 2, 3,…

Letting $m \to \infty$, we have

$\sum_{n=1}^{t}\left(\frac{1}{Q_n}\sum_{k=1}^{n}\|q_k y_k\|\right)^{p_n} \leq$ S; t = 1, 2, 3,…

Since t is arbitrary, $\sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n}\|q_k y_k\|\right)^{p_n} \leq$S$<\infty$. This implies $y = (y_k) \in ces_{(p)}(X,q)$.

Now from ①, we have,

$$\sum_{n=1}^{t}\left(\frac{1}{Q_n}\sum_{k=1}^{n}\left\|q_k x_k^{(m)} - q_k x_k^{(r)}\right\|\right)^{p_n} < \left(\frac{\varepsilon}{M'}\right)^{M} \forall\, m, r \geq N \text{ and } t = 1, 2, 3, \ldots$$

Letting $r \to \infty$, we get

$\sum_{n=1}^{t}\left(\frac{1}{Q_n}\sum_{k=1}^{n}\left\|q_k x_k^{(m)} - q_k y_k\right\|\right)^{p_n} < \left(\frac{\varepsilon}{M'}\right)^M \forall\, m \geq N \; and \; t = 1, 2, 3, \ldots$

Since above inequality is true for each $t \in \mathcal{N}$,

$\sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n}\left\|q_k x_k^{(m)} - q_k y_k\right\|\right)^{p_n} < \left(\frac{\varepsilon}{M'}\right)^M \forall\, m \geq N.$

This implies $\rho(x_m - y) \leq \left(\frac{\varepsilon}{M'}\right)^M \forall\, m \geq N.$

By (vi) of proposition (2.3), we have $\|x_m - y\| < \frac{\varepsilon}{M'} \forall\, m \geq N$. This implies $x_m \to y$ in $ces_{(p)}(X, q)$.

This completes the proof.

Corollary 2.6: For p = (p_k) and q = (q_k) of positive real numbers with $p_k \geq 1 \; and \; q_k \geq 1 \; \forall\, k \; \in \mathcal{N} \; ces_{(p)}(q)$ is a Banach space.

Next we prove that $ces_{(p)}(X, q)$ is rotund.

Theorem 2.7: $ces_{(p)}(X, q)$ is rotund(R).

Proof: We show that S($ces_{(p)}(X, q)$)⊆ Ext B($ces_{(p)}(X, q)$). Let $x = (x_k) \in S\left(ces_{(p)}(X, q)\right)$, $y = (y_k)$ and $z = (z_k) \in B\left(ces_{(p)}(X, q)\right)$ such that 2x = y+z. We prove that y=z.

Since $= (x_k) \in S\left(ces_{(p)}(X, q)\right)$, $\|x\| = 1$. By (iii) of proposition (2.3), $\rho(x) = 1$.

Now

$$1 = \rho(x) = \sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n}\|q_k x_k\|\right)^{p_n}$$

$$= \sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n}\left\|q_k\left(\frac{y_k + z_k}{2}\right)\right\|\right)^{p_n}$$

$$= \sum_{n=1}^{\infty}\left(\frac{\frac{1}{Q_n}\sum_{k=1}^{n}\|q_k y_k + q_k z_k\|}{2}\right)^{p_n}$$

$$\leq \sum_{n=1}^{\infty}\left(\frac{\frac{1}{Q_n}\sum_{k=1}^{n}\|q_k y_k\| + \frac{1}{Q_n}\sum_{k=1}^{n}\|q_k z_k\|}{2}\right)^{p_n}$$

$$\leq \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n}\|q_k y_k\|\right)^{p_n} + \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n}\|q_k z_k\|\right)^{p_n}$$

$$= \frac{1}{2}\rho(y) + \frac{1}{2}\rho(z)$$

$$\leq \frac{1}{2}.1 + \frac{1}{2}.1 = 1.$$

Therefore, $\rho(x) = \rho\left(\frac{y+z}{2}\right) = \frac{1}{2}\rho(y) + \frac{1}{2}\rho(z)$.

Suppose y$\neq z$.

Let i$\in \mathcal{N}$ be the smallest positive integer such that $\frac{1}{Q_i}\sum_{k=1}^{n}\|q_k y_k\| \neq \frac{1}{Q_i}\sum_{k=1}^{n}\|q_k z_k\|$.

Since the function t$\rightarrow |t|^{p_n}$ is strictly convex for n $\in \mathcal{N}$, we have,

$$\left(\frac{\frac{1}{Q_i}\sum_{k=1}^{i}\|q_k y_k\|+\frac{1}{Q_n}\sum_{k=1}^{n}\|q_k z_k\|}{2}\right)^{p_i} < \frac{1}{2}\left(\frac{1}{Q_i}\sum_{k=1}^{i}\|q_k y_k\|\right)^{p_i}+\frac{1}{2}\left(\frac{1}{Q_i}\sum_{k=1}^{i}\|q_k z_k\|\right)^{p_i}.$$

So, it follows,

$$1=\sum_{n=1}^{\infty}\left(\frac{\frac{1}{Q_n}\sum_{k=1}^{n}\|q_k y_k\|+\frac{1}{Q_n}\sum_{k=1}^{n}\|q_k z_k\|}{2}\right)^{p_n}$$

$$<\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n}\|q_k y_k\|\right)^{p_n}+\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n}\|q_k z_k\|\right)^{p_n}$$

$$=\frac{1}{2}\rho(y)+\frac{1}{2}\rho(z)\leq \frac{1}{2}.1+\frac{1}{2}.1=1$$

which is a contradiction. Hence y=z.

This completes the proof.

Corollary 2.8: For p = (p_k) and q = (q_k) of positive real numbers with $p_k \geq 1$ *and* $q_k \geq 1\ \forall\ k\ \in \mathcal{N}$ $ces_{(p)}(q)$ is a rotund(R).

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